GENERALISED SUM CONSTRUCTION OF AUTOMORPHIC BIBDS AND THEIR APPLICATIONS IN EXPERIMENTAL DESIGNS AND TECHNOLOGY

G. S. DUGGAL¹ and N. B. OKELO*

¹SCHOOL OF STATISTICS, UNIVERSITY OF GUJARAT, INDIA

*SCHOOL OF MATHEMATICS AND ACTUARIAL SCIENCE,
 JARAMOGI OGINGA ODINGA UNIVERSITY OF SCIENCE AND TECHNOLOGY,
 P. O. BOX 210-40601, BONDO-KENYA

Email: bnyaare@yahoo.com

ABSTRACT

We give more constructions equivalent to “sum construction”, of BIBDs where BIBD is added to a BIBD that is automorphic to it is presented. Moreover we present their applications in technology

INTRODUCTION

Wilson (1975) came up with block spreading method for $t = 2$ and for prime power index. Let v be a positive integer $v \geq 2$ and let q be a prime power. Suppose that there exists a $S_q(2, k, v)$ design satisfying $q \geq v + 1$. Then there exists a group divisible design (GDD) of group type $(q^d)^v$ with block size k and index one, whenever $d \geq \binom{v}{2}$. This method has application in the construction of Steiner 2 – designs. Blanchard (1995a, 1995b, 1995c) generalizes Wilson’s results for $t \geq 2$ as follows: (The “block spreading” method for $t \geq 2$ and for prime power index). Let v and t be positive integers, $2 \leq t \leq v$, and let q be a prime power. Then there exists a number $q_0 = q_0(t, v)$ such that for any $S_q(t, k, v)$ design satisfying $q \geq q_0$, there is a t – GDD of group type $(q^d)^v$ with block size k and index one whenever $d \geq \binom{v}{t}$. More so, Mohacsy and Chaudhuri (2001, 2003) generalizes Blanchard’s construction for general index. (The “block spreading” method for $t \geq 2$ and general index). Let v, t and λ be positive integers $2 \leq t \leq v$. Then there exists a number $q_0 = q_0(t, v)$ such that for any $S_q(t, k, v)$ design with prime power decomposition $\lambda = q_1 q_2 q_3 \cdots q_n$ satisfying $q_i \geq q_0$, $1 \leq i \leq n$; there is a t – GDD of group type $(\lambda^d)^v$ with block size k and index one whenever $d \geq \binom{v}{t}$. This generalized “block spreading” construction has several applications such as constructing new Steiner 3 – designs and new group divisible t – designs with index one. Limitation of this method is that the bounds on d are too large.

Table 3.3.3.12. Case 2; for $x = 3$ and $c = 6$ the possible cases of $3 - (v, k, c)$ design

λ_1	k	λ_1	v	b
15	6	33	12	66
24	9	87	30	290
51	18	411	138	3151

Table 3.3.3.13. Case 2; for $x = 6$ and $c = 6$ the possible cases of $3 - (v, k, c)$ design

λ_2	k	λ_1	v	b
18	4	42	8	84
30	6	126	22	462
42	8	258	44	1419

Also for $c = 7$ and $x = 7$ we get the following table:

Table 3.3.3.14. Case 2; for $x = 7$ and $c = 7$ the possible cases of $3 - (v, k, c)$ design

λ_2	k	λ_1	v	b
21	4	49	8	98
35	6	147	22	539
42	7	217	32	992

Lastly, for $c = 8, x = 4$ and $x = 8$ we also get the following tables respectively.

Table 3.3.3.15. Case 2; for $x = 4$ and $c = 8$ the possible cases of $3 - (v, k, c)$ design

λ_2	k	λ_1	v	b
12	4	16	5	20
20	6	44	12	88
28	8	88	23	252
44	12	224	57	1064

Table 3.3.3.16. Case 2; for $x = 8$ and $c = 8$ the possible cases of $3 - (v, k, c)$ design

λ_2	k	λ_1	v	b
24	4	56	8	112
40	6	168	22	616
56	8	344	32	1892
88	12	888	112	8288

3.3.4. CASE III, $x \neq y$ and $x > 1, y > 1$

In this case (3.3.1.3) is

$$\lambda_1 = \frac{x}{y}(v-1) \Rightarrow v = \frac{y\lambda_1 + x}{x}$$

$$\lambda_2 = \frac{x}{y}(k-1) \Rightarrow k = \frac{y\lambda_2 + x}{x}$$

Where x and y are positive integers and λ_2 and y are divisible by x .

Now substituting these values of v and k in (3.3.1.2) we get

$$\lambda_2 = \frac{c(v-2)}{k-2}$$

We get λ_1, v , and k as follows:

$$\lambda_1 = \frac{y\lambda_2^2 - x\lambda_2 + cx}{cy}$$

$$v = \frac{y\lambda_2^2 - x\lambda_2 + 2cx}{cx}$$

And

$$k = \frac{y\lambda_2 + x}{x}$$

Using $b = \frac{\lambda_1 v}{k}$ for this to be $3 - (v, k, 1)$ design then

$$\frac{(y\lambda_2^2 - x\lambda_2 + cx)}{cy} \left(\frac{y\lambda_2^2 - x\lambda_2 + 2cx}{cx} \right) \equiv 0 \pmod{\left(\frac{y\lambda_2 + x}{x} \right)}$$

That is

$$\frac{(y\lambda_2^2 - x\lambda_2 + cx)(y\lambda_2^2 - x\lambda_2 + 2cx)}{c^2 y (y\lambda_2 + x)}$$

is positive integer.

Expanding and dividing we will obtain

$$y\lambda_2^3 - 3x\lambda_2^2 + \frac{\lambda_2^2(3cxy + 4x^2)}{y} - \frac{(6cx^2y + 4x^3)}{y^2} \text{ rem } \frac{2c^2x^2y^2 + 6cx^3y + 4x^4}{y^2}$$

That is:

$$y\lambda_2^3 - 3x\lambda_2^2 + \frac{\lambda_2^2(3cxy + 4x^2)}{y} - \frac{(6cx^2y + 4x^3)}{y^2} + \frac{2c^2x^2y^2 + 6cx^3y + 4x^4}{c^2y^3(y\lambda_2 + x)} \quad (3.3.4.1)$$

Under this case and using similar method we find $3 - (v, k, c)$ exists if:

$$\frac{2c^2x^2y^2 + 6cx^3y + 4x^4}{c^2y^3(y\lambda_2 + x)} = \frac{2x^2(c^2 + \frac{3cx}{y} + \frac{2x^2}{y^2})}{c^2y(y\lambda_2 + x)} \quad (3.3.4.2)$$

is an integer.

Take $c = 5, x = 5$ and $y = 2$ in this case there is only one non-trivial $3 - (v, k, c)$ which we get as follows:

Equation (3.3.4.2) in this case is

$$\frac{75}{y\lambda_2 + x}$$

And λ_2 would take the values 5 or 35 with corresponding values of k, λ_1, v and b as in the table below.

Table 3.3.4.1. Case 2; for $x = 5, y = 2$ and $c = 5$ the possible cases of $3 - (v, k, c)$ design

λ_2	k	λ_1	v	b
5	3	5	3	5
35	15	230	93	1426

Also for $c = 7, x = 7$ and $y = 2$, we get the following table:

Table 3.3.4.2. Case 2; for $x = 7, y = 2$ and $c = 7$ the possible cases of $3 - (v, k, c)$ design

λ_2	k	λ_1	v	b
7	3	7	3	7
21	7	56	17	136

3.4 CONSTRUCTION OF $3 - (v, k, 1)$

In this section, the same technique that has been used to construct $3 - (v, k, 1)$ is applied.

Here $t = 4$ and $\lambda_4 = 1$

From;

$$\lambda_t(v - (t - 1)) = \lambda_{t-1}(k - (t - 1))$$

We have;

$$\lambda_3 = \frac{v - 3}{k - 3} \quad (3.4.1)$$

$$\frac{\lambda_2}{\lambda_3} = \frac{v-2}{k-2} \quad (3.4.2)$$

And ;

$$\frac{\lambda_1}{\lambda_2} = \frac{v-1}{k-1} \quad (3.4.3)$$

Now

$$\frac{\lambda_2}{\lambda_3} = \frac{v-2}{k-2}$$

This can be written as;

$$\lambda_2 = \alpha (v-2)$$

And

$$\lambda_3 = \alpha (k-2) \quad (3.4.4)$$

Given $\lambda_1, \lambda_2, \lambda_3, v-2, v-1, k-2$, and $k-1$ are all integers. α is a rational number which we will represent by $\frac{x}{y}$ where x and y are positive integers.

3.4.1. Case 1 $x = 1$

Then (3.4.4) becomes ;

$$y\lambda_2 = v-2, \Rightarrow v = y\lambda_2 + 2$$

And

$$y\lambda_3 = k-2, \Rightarrow k = y\lambda_3 + 2$$

From (3.4.1) we have now

$$\lambda_3 = \frac{y\lambda_2 - 1}{y\lambda_3 - 1}$$

$$\lambda_2 = \frac{y\lambda_3^2 - \lambda_3 - 1}{y}$$

Similarly from (3.4.3) we obtain

$$\lambda_1 = \frac{y\lambda_2^2 - \lambda_2}{y\lambda_3 + 1}$$

$$\lambda_1 = \frac{y^2\lambda_3^4 - 2y\lambda_3^3 + 3y\lambda_3^2 + \lambda_3^2 - 3\lambda_3 + 2}{y\lambda_3 + 1}$$

Which implies

$$v = y\lambda_2^2 - \lambda_3 + 3$$

$$k = y\lambda_3 + 2$$

For this design to be $4 - (v, k, 1)$ and from $bk = v\lambda_1$ it means

$$\frac{(y^2\lambda_3^4 - 2y\lambda_3^3 + 3y\lambda_3^2 + \lambda_3^2 - 3\lambda_3 + 2)(y\lambda_3^2 - \lambda_3 + 3)}{(y\lambda_3 + 1)} \equiv 0 \pmod{(y\lambda_3 + 2)}$$

That is

$$\frac{(y^2\lambda_3^4 - 2y\lambda_3^3 + 3y\lambda_3^2 + \lambda_3^2 - 3\lambda_3 + 2)(y\lambda_3^2 - \lambda_3 + 3)}{(y\lambda_3 + 1)(y\lambda_3 + 2)}$$

Is a positive integer

(3.4.1.1)

Expanding and dividing we obtain

$$\frac{y^2\lambda_3^5}{(y\lambda_3 + 1)} - \frac{5y\lambda_3^4}{(y\lambda_3 + 1)} + \frac{\lambda_3^3(6y^2 + 13y)}{y(y\lambda_3 + 1)} - \frac{\lambda_3^2(24y^2 + 27y)}{y^2(y\lambda_3 + 1)} + \frac{\lambda_3(11y^3 + 54y^2 + 4y)}{y^3(y\lambda_3 + 1)}$$

$$- \frac{(33y^3 + 108y^2 + 108)}{y^4(y\lambda_3 + 1)} + \frac{6y^4 + 66y^3 + 216y^2 + 216y}{y^4(y\lambda_3 + 1)(y\lambda_3 + 2)}$$

(3.4.1.2)

Now (3.4.1.2) will be integer when

$$\frac{6y^4 + 66y^3 + 216y^2 + 216y}{y^4(y\lambda_3 + 1)(y\lambda_3 + 2)}$$

(3.4.1.3)

Is an integer.

Now (3.4.1.3) will be integer only when y is 1 and 2

For $y = 2$

$$\frac{6y^4 + 66y^3 + 216y^2 + 216y}{y^4(y\lambda_3 + 1)(y\lambda_3 + 2)} = \frac{504}{\lambda_3^2 + 3\lambda_3 + 2}$$

In this case (3.4.1.3) is not an integer.

For $y = 1$ (3.4.1.3) is

$$\frac{504}{\lambda_3^2 + 3\lambda_3 + 2}$$

Thus (3.4.1.3) will be integer if λ_3 takes any of the following values 2 and 5

The table below gives corresponding values of $\lambda_1, \lambda_2, k, v$ and b .

Table 3.4.1.1. Case 1; for $y = 1$ the possible cases of $4 - (v, k, 1)$ design

λ_3	λ_2	λ_1	k	v	b
2	3	4	4	5	5
5	21	77	7	23	253

The first $4 - (5, 4, 1)$ design is trivial. The $4 - (23, 7, 1)$ is the only non trivial. This $4 - (v, k, 1)$ is then identified with the following BIB designs ; $2 - (5, 4, 3)$ and $2 - (23, 7, 21)$

3.4.1. Case 2, $y = 1$

In this case (3.4.4) is

$$\lambda_2 = x(v - 2), \quad \Rightarrow v = \frac{\lambda_2 + 2x}{x}$$

And

$$\lambda_3 = x(k-2), \Rightarrow k = \frac{\lambda_3 + 2x}{x}$$

From (3.4.1) we have now

$$\lambda_3 = \frac{\lambda_2 - x}{\lambda_3 - x}$$

$$\lambda_2 = \lambda_3^2 - x\lambda_3 + x$$

Similarly from (3.4.3) we obtain

$$\lambda_1 = \frac{\lambda_2^2 + x\lambda_2}{\lambda_3 + x}$$

$$\lambda_1 = \frac{\lambda_3^4 - 2x\lambda_3^3 + 3x\lambda_3^2 + x^2\lambda_3^2 - 3x^2\lambda_3 + 2x^2}{\lambda_3 + x}$$

This implies

$$v = \frac{\lambda_3^2 - x\lambda_3 + 3x}{x}$$

$$k = \frac{\lambda_3 + 2x}{x}$$

For this design to be $4 - (v, k, 1)$ and from $bk = v\lambda_1$ it means

$$\frac{\lambda_3^4 - 2x\lambda_3^3 + 3x\lambda_3^2 + x^2\lambda_3^2 - 3x^2\lambda_3 + 2x^2}{(\lambda_3 + x)} (\lambda_3^2 - x\lambda_3 + 3x) \equiv 0 \pmod{(\lambda_3 + 2x)}$$

That is

$$\frac{(\lambda_3^4 - 2x\lambda_3^3 + 3x\lambda_3^2 + x^2\lambda_3^2 - 3x^2\lambda_3 + 2x^2)(\lambda_3^2 - x\lambda_3 + 3x)}{(\lambda_3 + x)(\lambda_3 + 2x)} \quad (3.4.2.1)$$

Is a positive integer

Using similar argument as before (3.4.2.1) will be integer if

$$\frac{6x^3 + 216x^5 + 216x^6}{(\lambda_3 + 2)(\lambda_3 + 2x)} \quad (3.4.2.2)$$

Is an integer.

If $x = 1$, we get the result of case 1 for $x = 2$, (3.4.2.2) becomes

$$\frac{18768}{(\lambda_3 + 2)(\lambda_3 + 4)}$$

Thus λ_3 takes any of the following values, 2 and 4. But λ_3 must be greater than 2, hence 2 is not a possibility. We give corresponding values of $\lambda_1, \lambda_2, v, k$ and b in the table below.

Table 3.4.2.1. Case 2; for $x = 1$ the possible cases of $4 - (v, k, 1)$ design

λ_3	λ_2	λ_1	k	v	b
4	10	20	4	7	35

This $4 - (7, 4, 1)$ design is trivial. Hence, for $x = 2$ there is no nontrivial $4 - (v, k, 1)$ design.

For $x = 3$, (3.4.2.2) is

$$184356$$

$$(\lambda_3 + 3)(\lambda_3 + 6)$$

And λ_3 takes only 6 as its value. The corresponding values of $\lambda_1, \lambda_2, v, k$ and b are given in the table below.

Table 3.4.2.2. Case 2; for $x = 3$ the possible cases of $4 - (v, k, 1)$ design

λ_3	λ_2	λ_1	k	v	b
6	21	56	4	9	126

This $4 - (9, 4, 1)$ design is trivial. Hence, also for $x = 3$ there is no nontrivial $4 - (v, k, 1)$ design.

For $x = 4$, (3.4.2.2) is

$$975744$$

$$(\lambda_3 + 4)(\lambda_3 + 8)$$

And λ_3 takes the following values 8 and 28. The corresponding values of $\lambda_1, \lambda_2, v, k$ and b are given in the table below.

Table 3.4.2.3. Case 2; for $x = 4$ the possible cases of $4 - (v, k, 1)$ design

λ_3	λ_2	λ_1	k	v	b
8	36	120	4	11	330
28	676	14365	9	171	272935

Again we obtain the desired $4 - (v, k, 1)$ designs from BIB designs below:

$$2 - (11, 4, 36) \text{ and } (171, 676)$$

For $x = 5$ and using similar arguments as before, the possible values of λ_3 are as follows:

10, 15 and 20, which give the values of $\lambda_1, \lambda_2, v, k$ and b as in the table below.

Table 3.4.2.4. Case 2; for $x = 5$ the possible cases of $4 - (v, k, 1)$ design

λ_3	λ_2	λ_1	k	v	b
10	55	220	4	13	715
15	155	1240	5	33	8184
20	305	3782	6	63	39711

3.4.3.Case 3 , $x \neq y$ $x > 1, y > 1$

In this case (3.4.4) is

$$y\lambda_2 = x(v - 2), \quad \Rightarrow v = \frac{y\lambda_2 + 2x}{x}$$

And

$$y\lambda_3 = x(k - 2), \quad \Rightarrow k = \frac{y\lambda_3 + 2x}{x}$$

From (3.4.1) we have now

$$\lambda_3 = \frac{y\lambda_2 - x}{y\lambda_3 - x}$$

$$\lambda_2 = \frac{y\lambda_3^2 - x\lambda_3 + x}{y}$$

Similarly from (3.4.3) we obtain

$$\lambda_1 = \frac{y\lambda_2^2 + x\lambda_2}{y\lambda_3 + x}$$

$$\lambda_1 = \frac{y^2\lambda_3^4 - 2xy\lambda_3^3 + 3xy\lambda_3^2 + x^2\lambda_3^2 - 3x^2\lambda_3 + 2x^2}{y^2\lambda_3 + xy}$$

Which implies

$$v = \frac{y\lambda_3^2 - x\lambda_3 + 3x}{x}$$

$$k = \frac{y\lambda_3 + 2x}{x}$$

For this design to be $(v, k) = 1$ and from $bk = v\lambda_1$ it means

$$\frac{y^2\lambda_3^4 - 2xy\lambda_3^3 + 3xy\lambda_3^2 + x^2\lambda_3^2 - 3x^2\lambda_3 + 2x^2}{(y^2\lambda_3 + xy)}(y\lambda_3^2 - x\lambda_3 + 3x) \equiv 0 \pmod{(y\lambda_3 + 2x)}$$

That is

$$\frac{(y^2\lambda_3^4 - 2xy\lambda_3^3 + 3xy\lambda_3^2 + x^2\lambda_3^2 - 3x^2\lambda_3 + 2x^2)(y\lambda_3^2 - x\lambda_3 + 3x)}{(y^2\lambda_3 + x)(y\lambda_3 + 2x)} \quad (3.4.3.1)$$

Is a positive integer

Using similar argument as before (3.4.3.1) will be integer if

$$\frac{6x^3y^4 + 216x^5y^2 + 66x^4y^3 + 216x^6y}{y^5(y\lambda_3 + x)(y\lambda_3 + 2x)}$$

(3.4.3.2)

Is an integer

For $x = 3$ and $y = 2$, (3.4.3.2) becomes

$$\frac{17820}{(2\lambda_3 + 3)(2\lambda_3 + 6)}$$

In this case there is no non-trivial $4 - (v, k, 1)$. We give corresponding values of $\lambda_1, \lambda_2, v, k$ and b in the table below.

Table 3.4.3.1. Case 3; for $x = 3$ and $y = 2$ the possible cases of $4 - (v, k, 1)$ design

λ_3	λ_2	λ_1	k	v	b
3	6	10	4	6	15

This $4 - (6, 4, 1)$ design is trivial. Hence, for $x = 3$ and $y = 2$ there is no non-trivial $4 - (v, k, 1)$ design

Also for $x = 5$ and $y = 2$ there is no non-trivial $4 - (v, k, 1)$, the following table gives the corresponding values of $\lambda_1, \lambda_2, v, k$ and b .

Table 3.4.3.2. Case 3; for $x = 5$ and $y = 2$ the possible cases of $4 - (v, k, 1)$ design

λ_3	λ_2	λ_1	k	v	b
5	6	35	5	8	70

For $x = 4$ and $y = 3$ equation (3.4.3.2) becomes

$$\frac{21120}{(3\lambda_3 + 4)(3\lambda_3 + 8)}$$

In this case there is only one non-trivial $4 - (v, k, 1)$. The following table gives the corresponding values of $\lambda_1, \lambda_2, v, k$ and b .

Table 3.4.3.3. Case 3; for $x = 4$ and $y = 3$ the possible cases of $4 - (v, k, 1)$ design

λ_3	λ_2	λ_1	k	v	b
4	12	30	5	11	66

For $x = 6$ and $y = 3$ equation (3.4.3.2) becomes

$$\frac{196560}{(3\lambda_3 + 6)(3\lambda_3 + 12)}$$

And the corresponding values of $\lambda_1, \lambda_2, v, k$ and b are as shown below.

Table 3.4.3.4. Case 3; for $x = 6$ and $y = 3$ the possible cases of $4 - (v, k, 1)$ design

λ_3	λ_2	λ_1	k	v	b
4	10	20	4	7	35
6	26	91	5	15	273
8	50	260	6	27	1170
10	82	574	7	43	3526
22	442	8177	13	223	140267

For $x = 7$ and $y = 3$ equation (3.4.3.2) becomes

$$466480$$

$$(3\lambda_3 + 7)(3\lambda_3 + 14)$$

In this case there is only one non-trivial $4 - (v, k, 1)$. The following table gives the corresponding values of $\lambda_1, \lambda_2, v, k$ and b .

Table 3.4.3.5. Case 3; for $x = 7$ and $y = 3$ the possible cases of $4 - (v, k, 1)$ design

λ_3	λ_2	λ_1	k	v	b
7	35	140	5	17	476

For $x = 8$ and $y = 3$ equation (3.4.3.2) becomes

$$992256$$

$$(3\lambda_3 + 8)(3\lambda_3 + 16)$$

Also in this case, there is only one non-trivial $4 - (v, k, 1)$. The following table gives the corresponding values of $\lambda_1, \lambda_2, v, k$ and b .

Table 3.4.3.6. Case 3; for $x = 8$ and $y = 3$ the possible cases of $4 - (v, k, 1)$ design

λ_3	λ_2	λ_1	k	v	b
40	183	52547	17	563	1740233

For $x = 9$ and $y = 3$ equation (3.4.3.2) becomes

$$139140$$

$$(3\lambda_3 + 9)(3\lambda_3 + 16)$$

And the corresponding values of $\lambda_1, \lambda_2, v, k$ and b are as shown below.

Table 3.4.3.7. Case 3; for $x = 9$ and $y = 3$ the possible cases of $4 - (v, k, 1)$ design

λ_3	λ_2	λ_1	k	v	b
6	21	56	4	9	126
9	57	285	5	21	1197
15	183	1891	7	63	17019

Conclusions

In note a new recursive technique has been developed for the construction of $t - (v, k, \lambda_t)$ designs. It has also clearly shown that every t design is also a BIB (v, k, λ) design. Therefore, this construction technique also generates BIBDs. Thus, the study has presented an alternative method that is simpler and unified for the construction of BIBDs that are very important in the experimental designs. As it provides designs for different values of k , unlike many methods that provide designs for a single value of k . Moreso, it provides both Steiner and non-Steiner designs.

Recommendation

Although this study has provided a technique for the construction of t designs, it is still not clear that construction method of t designs is somehow known in general. In fact, it is not clear how one might construct t designs with arbitrary block size. We therefore invite researchers to come up with "additive theorems" for this construction to make it general for any value of t as this may bring in new techniques and ideas. There is also need for obtaining a theorem which would give values of x and y for the case three in this construction in order to see new Steiner designs. Lastly, if there is a computer package that could be incorporated in the method to aid calculations.

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