

An Investigation on the Beta Function IV: An approximation of the Error Function

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Abstract. I discovered an approximation of the error function.

1. INTRODUCTION

In this paper, I discovered an approximation of the error function:

$$\operatorname{erf}(z) \cong \frac{8z \tan^{-1}(z) - 4 \ln(z^2 + 1) - 2z^2}{z\sqrt{\pi}},$$

for $0 < z \leq 1$.

2. THEOREMS

Theorem 1. For $\Re(x) > 0$ and $\Re(y) > 0$, then

$$\frac{\Gamma(x)\Gamma(y)\Gamma\left(x+y+\frac{1}{2}\right)}{\Gamma(x+y)\Gamma\left(y+\frac{1}{2}\right)} = \int_0^\infty e^{-t} {}_1F_1\left(\frac{1}{2}; x+y+\frac{1}{2}; t\right) t^{x-1} dt,$$

where ${}_1F_1(a; b; z)$ denotes the hypergeometric function.

Proof. In [1, page 883], I have the Euler's definition of gamma function

$$\Gamma(z) \stackrel{\text{def}}{=} \int_0^\infty e^{-t} t^{z-1} dt, \tag{1}$$

for $\Re(x) > 0$. I substitute (1) into Theorem 1 of previous paper [2]

$$\begin{aligned} G_2(x, y) &= \sum_{k=0}^{\infty} \frac{(2k)!}{4^k k!^2 \Gamma\left(x+y+k+\frac{1}{2}\right)} \int_0^\infty e^{-t} t^{x+k-1} dt \\ &= \int_0^\infty e^{-t} \sum_{k=0}^{\infty} \frac{(2k)! t^k}{4^k k!^2 \Gamma\left(x+y+k+\frac{1}{2}\right)} t^{x-1} dt \\ &= \int_0^\infty e^{-t} \frac{{}_1F_1\left(\frac{1}{2}; x+y+\frac{1}{2}; t\right)}{\Gamma\left(x+y+\frac{1}{2}\right)} t^{x-1} dt \\ &= \frac{1}{\Gamma\left(x+y+\frac{1}{2}\right)} \int_0^\infty e^{-t} {}_1F_1\left(\frac{1}{2}; x+y+\frac{1}{2}; t\right) t^{x-1} dt, \end{aligned}$$

so,

$$\Gamma\left(x+y+\frac{1}{2}\right) G_2(x, y) = \int_0^\infty e^{-t} {}_1F_1\left(\frac{1}{2}; x+y+\frac{1}{2}; t\right) t^{x-1} dt,$$

in other words,

$$\frac{\Gamma(x)\Gamma(y)\Gamma\left(x+y+\frac{1}{2}\right)}{\Gamma(x+y)\Gamma\left(y+\frac{1}{2}\right)} = \int_0^\infty e^{-t} {}_1F_1\left(\frac{1}{2}; x+y+\frac{1}{2}; t\right) t^{x-1} dt. \square$$

Theorem 2. For $\Re(x) > 0$ and $\Re(y) > 0$, then

$$\frac{\Gamma(x)\Gamma(y)\Gamma\left(x+y+\frac{1}{2}\right)}{\Gamma(x+y)\Gamma\left(y+\frac{1}{2}\right)} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \Gamma(x+n)}{\left(x+y+\frac{1}{2}\right)_n n!}.$$

Proof. By Theorem 1, I have

$$\begin{aligned} \frac{\Gamma(x)\Gamma(y)\Gamma\left(x+y+\frac{1}{2}\right)}{\Gamma(x+y)\Gamma\left(y+\frac{1}{2}\right)} &= \int_0^\infty e^{-t} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n t^n}{\left(x+y+\frac{1}{2}\right)_n n!} t^{x-1} dt \\ &= \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{\left(x+y+\frac{1}{2}\right)_n n!} \int_0^\infty e^{-t} t^{x+n-1} dt \\ &= \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \Gamma(x+n)}{\left(x+y+\frac{1}{2}\right)_n n!}. \end{aligned}$$

Corollary 1. Let n and k are positive integers and $n > k$, then

$$\binom{n}{k} = \frac{\Gamma\left(n+k+\frac{3}{2}\right)}{\Gamma(n-k+1)\Gamma\left(k+\frac{1}{2}\right)\Gamma(n+k+2)} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r \Gamma(n+r+1)}{(n+k+2)_r r!}.$$

Proof. Let $x \rightarrow x+1$ and $y \rightarrow y+\frac{1}{2}$ in Theorem 2

$$\frac{\Gamma(x+1)\Gamma\left(y+\frac{1}{2}\right)\Gamma(x+y+2)}{\Gamma\left(x+y+\frac{3}{2}\right)\Gamma(y+1)} = \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r \Gamma(x+r+1)}{(x+y+2)_r r!}$$

Multiply by $\frac{1}{\Gamma(x-y+1)}$

$$\frac{\Gamma(x+1)\Gamma\left(y+\frac{1}{2}\right)\Gamma(x+y+2)}{\Gamma(x-y+1)\Gamma\left(x+y+\frac{3}{2}\right)\Gamma(y+1)} = \frac{1}{\Gamma(x-y+1)} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r \Gamma(x+r+1)}{(x+y+2)_r r!} \Rightarrow$$

$$\frac{\Gamma(x+1)}{\Gamma(x-y+1)\Gamma(y+1)} = \frac{\Gamma\left(x+y+\frac{3}{2}\right)}{\Gamma(x-y+1)\Gamma\left(y+\frac{1}{2}\right)\Gamma(x+y+2)} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r \Gamma(x+r+1)}{(x+y+2)_r r!} \Rightarrow$$

$$\binom{x}{y} = \frac{\Gamma\left(x + y + \frac{3}{2}\right)}{\Gamma(x - y + 1)\Gamma\left(y + \frac{1}{2}\right)\Gamma(x + y + 2)} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r \Gamma(x + r + 1)}{(x + y + 2)_r r!}.$$

Letting $x = n$ and $y = k$, I complete the proof. \square

Theorem 3. For $0 > z$, then

$$\frac{\sqrt{\pi}\operatorname{erfi}(z)}{2z} = \frac{z^{2l+2}}{(2l+3)\Gamma(l+2)} {}_2F_2\left(\begin{matrix} 1, l + \frac{3}{2} \\ l + 2, l + \frac{5}{2} \end{matrix} \middle| z^2\right) + \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r}{r!} \left[\sum_{n=0}^l \frac{\Gamma\left(3n + \frac{3}{2}\right)\Gamma(2n + r + 1)}{\Gamma(n + 1)\Gamma\left(n + \frac{1}{2}\right)\Gamma(3n + 2)(3n + 2)_r} \frac{z^{2n}}{\left(\frac{3}{2}\right)_n 2^{2n}} \right],$$

where $\operatorname{erfi}(x)$ denotes the imaginary error function and ${}_2F_2\left(\begin{matrix} a, b \\ c, d \end{matrix} \middle| z\right)$ denotes the hypergeometric function.

Proof. Put $n \rightarrow 2n$ and $k \rightarrow n$ in Corollary 1

$$\binom{2n}{n} = \frac{\Gamma\left(3n + \frac{3}{2}\right)}{\Gamma(n + 1)\Gamma\left(n + \frac{1}{2}\right)\Gamma(3n + 2)} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r \Gamma(2n + r + 1)}{(3n + 2)_r r!}. \tag{2}$$

Multiply by $\frac{x^{2n}}{\left(\frac{3}{2}\right)_n 2^{2n}}$ and sum from 0 at l in n , then

$$\begin{aligned} \sum_{n=0}^l \binom{2n}{n} \frac{x^{2n}}{\left(\frac{3}{2}\right)_n 2^{2n}} &= \sum_{n=0}^l \frac{\Gamma\left(3n + \frac{3}{2}\right)}{\Gamma(n + 1)\Gamma\left(n + \frac{1}{2}\right)\Gamma(3n + 2)} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r \Gamma(2n + r + 1)}{(3n + 2)_r r!} \frac{x^{2n}}{\left(\frac{3}{2}\right)_n 2^{2n}} \\ &= \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r}{r!} \sum_{n=0}^l \frac{\Gamma\left(3n + \frac{3}{2}\right)\Gamma(2n + r + 1)}{\Gamma(n + 1)\Gamma\left(n + \frac{1}{2}\right)\Gamma(3n + 2)(3n + 2)_r} \frac{x^{2n}}{\left(\frac{3}{2}\right)_n 2^{2n}}. \end{aligned} \tag{3}$$

I calculate the left hand side of (3)

$$\sum_{n=0}^l \binom{2n}{n} \frac{x^{2n}}{\left(\frac{3}{2}\right)_n 2^{2n}} = \frac{\sqrt{\pi}\operatorname{erfi}(x)}{2x} - \frac{x^{2l+2}}{(2l+3)\Gamma(l+2)} {}_2F_2\left(\begin{matrix} 1, l + \frac{3}{2} \\ l + 2, l + \frac{5}{2} \end{matrix} \middle| x^2\right). \tag{4}$$

I substitute (4) in the left hand side of (3), put z by x and after of some algebraic manipulation the proof this complete. \square

Corollary 2. For $z > 0$, then

$$\frac{\sqrt{\pi}\operatorname{erfi}(z)}{2z} = 1 + \frac{z^2}{3} {}_2F_2\left(\begin{matrix} 1, \frac{3}{2} \\ 2, \frac{5}{2} \end{matrix} \middle| z^2\right),$$

where $\operatorname{erfi}(x)$ denotes the imaginary error function and ${}_2F_2\left(\begin{matrix} a, b \\ c, d \end{matrix} \middle| z\right)$ denotes the hypergeometric function.

Proof. I replace 0 by l in Theorem 3. \square

Corollary 3. For $z > 0$, then

$$\frac{\sqrt{\pi}\operatorname{erfi}(z)}{2z} = 1 + \frac{z^2}{3} + \frac{z^4}{10} {}_2F_2\left(\begin{matrix} 1, \frac{5}{2} \\ 3, \frac{7}{2} \end{matrix} \middle| z^2\right),$$

where $\operatorname{erfi}(x)$ denotes the imaginary error function and ${}_2F_2\left(\begin{matrix} a, b \\ c, d \end{matrix} \middle| z\right)$ denotes the hypergeometric function.

Proof. I replace 1 by l in Theorem 3. \square

Theorem 4. For $0 < z \leq 1$, then

$$\operatorname{erf}(z) \cong \frac{8z \tan^{-1}(z) - 4 \ln(z^2 + 1) - 2z^2}{z\sqrt{\pi}},$$

where $\operatorname{erf}(x)$ denotes the error function $\tan^{-1}(z)$ denotes the inverse tangent function and $\ln(z)$ denotes the natural logarithm function.

Proof. Put $x \rightarrow n + 1$ and $y \rightarrow 1$ in Theorem 2

$$\frac{1}{n!} = \frac{(n+1)\sqrt{\pi}}{2\Gamma(n+1)\Gamma\left(n+\frac{5}{2}\right)} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r \Gamma(n+r+1)}{\left(n+\frac{5}{2}\right)_r r!}. \quad (5)$$

Multiply by $\frac{2(-1)^n z^{2n+1}}{\sqrt{\pi}(2n+1)}$ and sum from 0 at ∞ in n , then

$$\begin{aligned} \operatorname{erf}(z) &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1} (n+1)}{(2n+1)\Gamma(n+1)\Gamma\left(n+\frac{5}{2}\right)} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r \Gamma(n+r+1)}{\left(n+\frac{5}{2}\right)_r r!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1} n}{(2n+1)\Gamma(n+1)\Gamma\left(n+\frac{5}{2}\right)} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r \Gamma(n+r+1)}{\left(n+\frac{5}{2}\right)_r r!} + \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)\Gamma(n+1)\Gamma\left(n+\frac{5}{2}\right)} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r \Gamma(n+r+1)}{\left(n+\frac{5}{2}\right)_r r!}, \quad (6) \end{aligned}$$

since $0 < z \leq 1$, I have

$$\begin{aligned} &\cong \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r}{r!} \sum_{n=0}^{\infty} \frac{(-1)^n n \Gamma(n+r+1) z^{2n+1}}{(2n+1)\Gamma(n+1)\Gamma\left(n+\frac{5}{2}\right)\left(n+\frac{5}{2}\right)_r} + \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r}{r!} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n+r+1) z^{2n+1}}{(2n+1)\Gamma(n+1)\Gamma\left(n+\frac{5}{2}\right)\left(n+\frac{5}{2}\right)_r} \\ &= -\frac{8z^3}{45\sqrt{\pi}} \sum_{r=0}^{\infty} \frac{(r+1)! \left(\frac{1}{2}\right)_r}{r! \left(\frac{7}{2}\right)_r} {}_2F_2\left(\begin{matrix} \frac{3}{2}, r+2 \\ 5, r+\frac{7}{2} \end{matrix} \middle| -z^2\right) + \frac{4z}{3\sqrt{\pi}} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r}{\left(\frac{5}{2}\right)_r} {}_2F_2\left(\begin{matrix} \frac{1}{2}, r+1 \\ 3, r+\frac{5}{2} \end{matrix} \middle| -z^2\right) \\ &= -\frac{8z^3}{45\sqrt{\pi}} \sum_{r=0}^{\infty} \frac{(r+1)! \left(\frac{1}{2}\right)_r}{r! \left(\frac{7}{2}\right)_r} \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{3}{2}\right)_n (r+2)_n}{\left(\frac{5}{2}\right)_n \left(r+\frac{7}{2}\right)_n} z^{2n} + \frac{4z}{3\sqrt{\pi}} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r}{\left(\frac{5}{2}\right)_r} \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{2}\right)_n (r+1)_n}{\left(\frac{3}{2}\right)_n \left(r+\frac{5}{2}\right)_n} z^{2n} \end{aligned}$$

$$\begin{aligned}
 &= -\frac{8z^3}{45\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{3}{2}\right)_n}{\left(\frac{5}{2}\right)_n} \sum_{r=0}^{\infty} \frac{(r+1)! \left(\frac{1}{2}\right)_r (r+2)_n}{r! \left(\frac{7}{2}\right)_r (r+\frac{7}{2})_n} z^{2n} + \frac{4z}{3\sqrt{\pi}} \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{2}\right)_n}{\left(\frac{3}{2}\right)_n} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r (r+1)_n}{\left(\frac{5}{2}\right)_r (r+\frac{5}{2})_n} z^{2n} \\
 &= -\frac{16z^3}{45\pi} \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{3}{2}\right)_n (2)_n \Gamma\left(\frac{2n+7}{2}\right)}{\left(\frac{5}{2}\right)_n \left(\frac{7}{2}\right)_n \Gamma(n+3)} z^{2n} + \frac{8z}{3\pi} \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{2}\right)_n (1)_n \Gamma\left(\frac{2n+5}{2}\right)}{\left(\frac{3}{2}\right)_n \left(\frac{5}{2}\right)_n \Gamma(n+2)} z^{2n} \\
 &= \frac{2(2z \tan^{-1}(z) - \ln(z^2 + 1) - z^2)}{z\sqrt{\pi}} + \frac{2(2z \tan^{-1}(z) - \ln(z^2 + 1))}{z\sqrt{\pi}} \\
 &= \frac{4z \tan^{-1}(z) - 2 \ln(z^2 + 1) - 2z^2 + 4z \tan^{-1}(z) - 2 \ln(z^2 + 1)}{z\sqrt{\pi}} \\
 &= \frac{8z \tan^{-1}(z) - 4 \ln(z^2 + 1) - 2z^2}{z\sqrt{\pi}}. \square
 \end{aligned}$$

Special Values. Let $z = 1$ in Theorem 4, then

$$\operatorname{erf}(1) \cong \frac{2\pi - 2 - 4\ln(2)}{\sqrt{\pi}},$$

with one place decimal correct.

Let $z = \frac{1}{2}$ in Theorem 4, then

$$\operatorname{erf}\left(\frac{1}{2}\right) \cong \frac{8 \tan^{-1}\left(\frac{1}{2}\right) - 1 - 8 \ln\left(\frac{5}{4}\right)}{\sqrt{\pi}},$$

with two place decimal correct.

Let $z = \frac{1}{3}$ in Theorem 4, then

$$\operatorname{erf}\left(\frac{1}{3}\right) \cong \frac{8 \tan^{-1}\left(\frac{1}{3}\right) - \frac{2}{3} - 8 \ln\left(\frac{10}{9}\right)}{\sqrt{\pi}},$$

with three place decimal correct.

Let $z = \frac{1}{4}$ in Theorem 4, then

$$\operatorname{erf}\left(\frac{1}{4}\right) \cong \frac{16 \tan^{-1}\left(\frac{1}{4}\right) - 1 - 32 \ln\left(\frac{17}{16}\right)}{\sqrt{\pi}},$$

with four place decimal correct; and so on. In Appendix, I leave to the reader the Figure 1.

Note. I leave to reader: Prove that, for $0 < z \leq 1$, then

$$\frac{1}{2} + \frac{\sqrt{\pi} \operatorname{erf}(z)}{4z} \cong \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k+1)(k+1)},$$

where $\operatorname{erf}(x)$ denotes the error function $\tan^{-1}(z)$ denotes the inverse tangent function and $\ln(z)$ denotes the natural logarithm function.

Theorem 5. For $0 > z$, then

$$\operatorname{erf}(z) = \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r}{r!} \sum_{n=0}^l \frac{(-1)^n (n+1) \Gamma(n+r+1) z^{2n+1}}{(2n+1) \Gamma(n+1) \Gamma\left(n+\frac{5}{2}\right) \left(n+\frac{5}{2}\right)_r} - \frac{2(-1)^l z^{2l+3}}{\sqrt{\pi}(2l+3)\Gamma(l+2)} {}_2F_2\left(\begin{matrix} 1, l+\frac{3}{2} \\ l+2, l+\frac{5}{2} \end{matrix} \middle| -z^2\right),$$

where $\operatorname{erf}(z)$ denotes the error function and ${}_2F_2\left(\begin{matrix} a, b \\ c, d \end{matrix} \middle| z\right)$ denotes the hypergeometric function.

Proof. Put $x \rightarrow n+1$ and $y \rightarrow 1$ in Theorem 2

$$\frac{1}{n!} = \frac{(n+1)\sqrt{\pi}}{2\Gamma(n+1)\Gamma\left(n+\frac{5}{2}\right)} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r \Gamma(n+r+1)}{\left(n+\frac{5}{2}\right)_r r!}. \quad (7)$$

Multiply by $\frac{2(-1)^n x^{2n+1}}{\sqrt{\pi}(2n+1)}$ and sum from 0 at l in n , then

$$\begin{aligned} \frac{2}{\sqrt{\pi}} \sum_{n=0}^l \frac{(-1)^n x^{2n+1}}{(2n+1)n!} &= \sum_{n=0}^l \frac{(-1)^n x^{2n+1} (n+1)}{(2n+1)\Gamma(n+1)\Gamma\left(n+\frac{5}{2}\right)} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r \Gamma(n+r+1)}{\left(n+\frac{5}{2}\right)_r r!} \\ &= \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r}{r!} \sum_{n=0}^l \frac{(-1)^n x^{2n+1} (n+1) \Gamma(n+r+1)}{(2n+1)\Gamma(n+1)\Gamma\left(n+\frac{5}{2}\right) \left(n+\frac{5}{2}\right)_r}. \end{aligned} \quad (8)$$

I calculate the left hand side of (8)

$$\frac{2}{\sqrt{\pi}} \sum_{n=0}^l \frac{(-1)^n x^{2n+1}}{(2n+1)n!} = \operatorname{erf}(x) + \frac{2(-1)^l x^{2l+3}}{\sqrt{\pi}(2l+3)\Gamma(l+2)} {}_2F_2\left(\begin{matrix} 1, l+\frac{3}{2} \\ l+2, l+\frac{5}{2} \end{matrix} \middle| -x^2\right). \quad (9)$$

I substitute (9) in the left hand side of (8), put z by x and after of some algebraic manipulation the proof this complete. \square

Corollary 4. For $z > 0$, then

$$\operatorname{erf}(z) = \frac{2z}{\sqrt{\pi}} - \frac{2z^3}{3\sqrt{\pi}} {}_2F_2\left(\begin{matrix} 1, \frac{3}{2} \\ 2, \frac{5}{2} \end{matrix} \middle| -z^2\right),$$

where $\operatorname{erf}(z)$ denotes the error function and ${}_2F_2\left(\begin{matrix} a, b \\ c, d \end{matrix} \middle| z\right)$ denotes the hypergeometric function.

Proof. I replace 0 by l in Theorem 4. \square

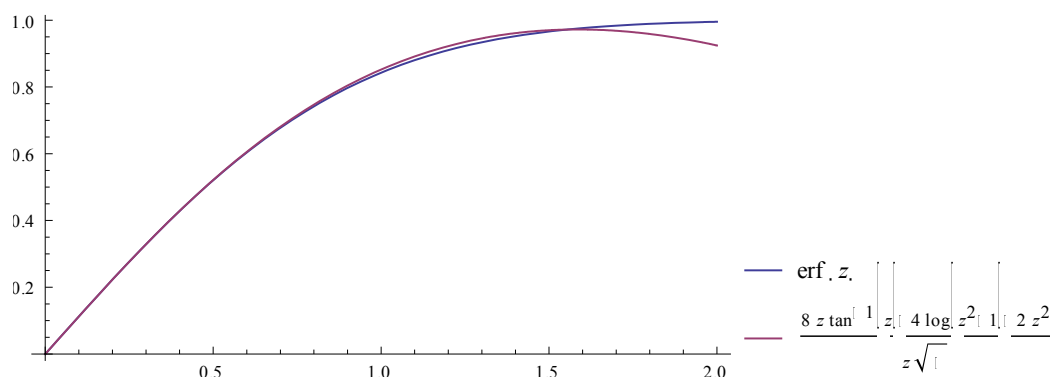
Corollary 5. For $z > 0$, then

$$\operatorname{erf}(z) = \frac{2z}{\sqrt{\pi}} - \frac{2z^3}{3\sqrt{\pi}} + \frac{z^5}{5\sqrt{\pi}} {}_2F_2\left(\begin{matrix} 1, \frac{5}{2} \\ 3, \frac{7}{2} \end{matrix} \middle| -z^2\right),$$

where $\operatorname{erf}(z)$ denotes the error function and ${}_2F_2\left(\begin{matrix} a, b \\ c, d \end{matrix} \middle| z\right)$ denotes the hypergeometric function.

Proof. I replace 1 by l in Theorem 4. \square

APPENDIX



REFERENCES

[1] Gradshteyn, I. S. and Ryzhik, I. M., *Table of Integrals, Series and Products*, Academic Press, 2000.

[2] Guedes, Edigles, *An Investigation on the Beta Function I: New Versions of the Euler Beta Function*, 2013.