The Bulletin of Society for Mathematical Services and Standards
ISSN: 2277-8020, Vol. 9, pp 3-9
doi:10.18052/www.scipress.com/BSMaSS.9.3

Online: 2014-03-03

An Investigation on the Beta Function IV: An approximation of the Error Function

Edigles Guedes[1] and Prof. Dr. K. Raja Rama Gandhi[2]

Number Theorist, Brazil[1]
Resource perosn in Mathematics for Oxford University Press and Professor at BITS-Vizag[2]

Abstract. I discovered an approximation of the error function.

1. INTRODUCTION

In this paper, I discovered an approximation of the error function:

$$\operatorname{erf}(z) \cong \frac{8z\tan^{-1}(z) - 4\ln(z^2+1) - 2z^2}{z\sqrt{\pi}},$$

for $0 < z \leq 1$.

2. THEOREMS

Theorem 1. *For* $\Re(x) > 0$ *and* $\Re(y) > 0$, *then*

$$\frac{\Gamma(x)\Gamma(y)\Gamma\left(x+y+\frac{1}{2}\right)}{\Gamma(x+y)\Gamma\left(y+\frac{1}{2}\right)} = \int_0^\infty e^{-t}\, {}_1F_1\left(\frac{1}{2}; x+y+\frac{1}{2}; t\right) t^{x-1} \mathrm{d}t,$$

where ${}_1F_1(a; b; z)$ *denotes the hypergeometric function.*

Proof. In [1, page 883], I have the Euler's definition of gamma function

$$\Gamma(z) \stackrel{\text{def}}{=} \int_0^\infty e^{-t} t^{z-1}\, \mathrm{d}t, \tag{1}$$

for $\Re(x) > 0$. I substitute (1) into Theorem 1 of previous paper [2]

$$G_2(x,y) = \sum_{k=0}^{\infty} \frac{(2k)!}{4^k k!^2\, \Gamma\left(x+y+k+\frac{1}{2}\right)} \int_0^\infty e^{-t} t^{x+k-1}\, \mathrm{d}t$$

$$= \int_0^\infty e^{-t} \sum_{k=0}^{\infty} \frac{(2k)!\, t^k}{4^k k!^2\, \Gamma\left(x+y+k+\frac{1}{2}\right)} t^{x-1} \mathrm{d}t$$

$$= \int_0^\infty e^{-t} \frac{{}_1F_1\left(\frac{1}{2}; x+y+\frac{1}{2}; t\right)}{\Gamma\left(x+y+\frac{1}{2}\right)} t^{x-1} \mathrm{d}t$$

$$= \frac{1}{\Gamma\left(x+y+\frac{1}{2}\right)} \int_0^\infty e^{-t}\, {}_1F_1\left(\frac{1}{2}; x+y+\frac{1}{2}; t\right) t^{x-1} \mathrm{d}t,$$

so,

$$\Gamma\left(x+y+\frac{1}{2}\right) G_2(x,y) = \int_0^\infty e^{-t}\, {}_1F_1\left(\frac{1}{2}; x+y+\frac{1}{2}; t\right) t^{x-1} \mathrm{d}t,$$

in other words,

$$\frac{\Gamma(x)\Gamma(y)\Gamma\left(x+y+\frac{1}{2}\right)}{\Gamma(x+y)\Gamma\left(y+\frac{1}{2}\right)}=\int_0^\infty e^{-t}\,{}_1F_1\left(\frac{1}{2};x+y+\frac{1}{2};t\right)t^{x-1}\mathrm{d}t.\square$$

Theorem 2. *For* $\Re(x)>0$ *and* $\Re(y)>0$, *then*

$$\frac{\Gamma(x)\Gamma(y)\Gamma\left(x+y+\frac{1}{2}\right)}{\Gamma(x+y)\Gamma\left(y+\frac{1}{2}\right)}=\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\Gamma(x+n)}{\left(x+y+\frac{1}{2}\right)_n n!}.$$

Proof. By Theorem 1, I have

$$\frac{\Gamma(x)\Gamma(y)\Gamma\left(x+y+\frac{1}{2}\right)}{\Gamma(x+y)\Gamma\left(y+\frac{1}{2}\right)}=\int_0^\infty e^{-t}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n t^n}{\left(x+y+\frac{1}{2}\right)_n n!}t^{x-1}\mathrm{d}t$$

$$=\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n}{\left(x+y+\frac{1}{2}\right)_n n!}\int_0^\infty e^{-t}\,t^{x+n-1}\mathrm{d}t$$

$$=\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\Gamma(x+n)}{\left(x+y+\frac{1}{2}\right)_n n!}.$$

Corollary 1. *Let* n *and* k *are positive integers and* $n>k$, *then*

$$\binom{n}{k}=\frac{\Gamma\left(n+k+\frac{3}{2}\right)}{\Gamma(n-k+1)\Gamma\left(k+\frac{1}{2}\right)\Gamma(n+k+2)}\sum_{r=0}^{\infty}\frac{\left(\frac{1}{2}\right)_r\Gamma(n+r+1)}{(n+k+2)_r r!}.$$

Proof. Let $x\to x+1$ and $y\to y+\frac{1}{2}$ in Theorem 2

$$\frac{\Gamma(x+1)\Gamma\left(y+\frac{1}{2}\right)\Gamma(x+y+2)}{\Gamma\left(x+y+\frac{3}{2}\right)\Gamma(y+1)}=\sum_{r=0}^{\infty}\frac{\left(\frac{1}{2}\right)_r\Gamma(x+r+1)}{(x+y+2)_r r!}$$

Multiply by $\frac{1}{\Gamma(x-y+1)}$

$$\frac{\Gamma(x+1)\Gamma\left(y+\frac{1}{2}\right)\Gamma(x+y+2)}{\Gamma(x-y+1)\Gamma\left(x+y+\frac{3}{2}\right)\Gamma(y+1)}=\frac{1}{\Gamma(x-y+1)}\sum_{r=0}^{\infty}\frac{\left(\frac{1}{2}\right)_r\Gamma(x+r+1)}{(x+y+2)_r r!}\Rightarrow$$

$$\frac{\Gamma(x+1)}{\Gamma(x-y+1)\Gamma(y+1)}=\frac{\Gamma\left(x+y+\frac{3}{2}\right)}{\Gamma(x-y+1)\Gamma\left(y+\frac{1}{2}\right)\Gamma(x+y+2)}\sum_{r=0}^{\infty}\frac{\left(\frac{1}{2}\right)_r\Gamma(x+r+1)}{(x+y+2)_r r!}\Rightarrow$$

$$\binom{x}{y} = \frac{\Gamma\left(x+y+\frac{3}{2}\right)}{\Gamma(x-y+1)\Gamma\left(y+\frac{1}{2}\right)\Gamma(x+y+2)} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r \Gamma(x+r+1)}{(x+y+2)_r r!}.$$

Letting $x = n$ and $y = k$, I complete the proof. □

Theorem 3. *For* $0 > z$, *then*

$$\frac{\sqrt{\pi}\operatorname{erfi}(z)}{2z} = \frac{z^{2l+2}}{(2l+3)\Gamma(l+2)} {}_2F_2\left(\left.\begin{matrix} 1, l+\frac{3}{2} \\ l+2, l+\frac{5}{2} \end{matrix}\right| z^2\right)$$
$$+ \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r}{r!} \left[\sum_{n=0}^{l} \frac{\Gamma\left(3n+\frac{3}{2}\right)\Gamma(2n+r+1)}{\Gamma(n+1)\Gamma\left(n+\frac{1}{2}\right)\Gamma(3n+2)(3n+2)_r} \frac{z^{2n}}{\left(\frac{3}{2}\right)_n 2^{2n}}\right],$$

where $\operatorname{erfi}(x)$ *denotes the imaginary error function and* ${}_2F_2\left(\left.\begin{matrix} a, b \\ c, d \end{matrix}\right| z\right)$ *denotes the hypergeometric function.*

Proof. Put $n \to 2n$ and $k \to n$ in Corollary 1

$$\binom{2n}{n} = \frac{\Gamma\left(3n+\frac{3}{2}\right)}{\Gamma(n+1)\Gamma\left(n+\frac{1}{2}\right)\Gamma(3n+2)} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r \Gamma(2n+r+1)}{(3n+2)_r r!}. \tag{2}$$

Multiply by $\frac{x^{2n}}{\left(\frac{3}{2}\right)_n 2^{2n}}$ and sum from 0 at l in n, then

$$\sum_{n=0}^{l} \binom{2n}{n} \frac{x^{2n}}{\left(\frac{3}{2}\right)_n 2^{2n}} = \sum_{n=0}^{l} \frac{\Gamma\left(3n+\frac{3}{2}\right)}{\Gamma(n+1)\Gamma\left(n+\frac{1}{2}\right)\Gamma(3n+2)} \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r \Gamma(2n+r+1)}{(3n+2)_r r!} \frac{x^{2n}}{\left(\frac{3}{2}\right)_n 2^{2n}}$$
$$= \sum_{r=0}^{\infty} \frac{\left(\frac{1}{2}\right)_r}{r!} \sum_{n=0}^{l} \frac{\Gamma\left(3n+\frac{3}{2}\right)\Gamma(2n+r+1)}{\Gamma(n+1)\Gamma\left(n+\frac{1}{2}\right)\Gamma(3n+2)(3n+2)_r} \frac{x^{2n}}{\left(\frac{3}{2}\right)_n 2^{2n}}. \tag{3}$$

I calculate the left hand side of (3)

$$\sum_{n=0}^{l} \binom{2n}{n} \frac{x^{2n}}{\left(\frac{3}{2}\right)_n 2^{2n}} = \frac{\sqrt{\pi}\operatorname{erfi}(x)}{2x} - \frac{x^{2l+2}}{(2l+3)\Gamma(l+2)} {}_2F_2\left(\left.\begin{matrix} 1, l+\frac{3}{2} \\ l+2, l+\frac{5}{2} \end{matrix}\right| x^2\right). \tag{4}$$

I substitute (4) in the left hand side of (3), put z by x and after of some algebraic manipulation the proof this complete. □

Corollary 2. *For* $z > 0$, *then*

$$\frac{\sqrt{\pi}\operatorname{erfi}(z)}{2z} = 1 + \frac{z^2}{3} {}_2F_2\left(\left.\begin{matrix} 1, \frac{3}{2} \\ 2, \frac{5}{2} \end{matrix}\right| z^2\right),$$

where $\operatorname{erfi}(x)$ *denotes the imaginary error function and* ${}_2F_2\left(\left.\begin{matrix} a, b \\ c, d \end{matrix}\right| z\right)$ *denotes the hypergeometric function.*

Proof. I replace 0 by l in Theorem 3. □

Corollary 3. *For* $z > 0$, *then*

$$\frac{\sqrt{\pi}\operatorname{erfi}(z)}{2z} = 1 + \frac{z^2}{3} + \frac{z^4}{10}\,{}_2F_2\left(\left.\begin{matrix}1, \frac{5}{2}\\ 3, \frac{7}{2}\end{matrix}\right| z^2\right),$$

where $\operatorname{erfi}(x)$ *denotes the imaginary error function and* ${}_2F_2\left(\left.\begin{matrix}a, b\\ c, d\end{matrix}\right| z\right)$ *denotes the hypergeometric function.*

Proof. I replace 1 by l in Theorem 3. □

Theorem 4. *For* $0 < z \leq 1$, *then*

$$\operatorname{erf}(z) \cong \frac{8z\tan^{-1}(z) - 4\ln(z^2+1) - 2z^2}{z\sqrt{\pi}},$$

where $\operatorname{erf}(x)$ *denotes the error function* $\tan^{-1}(z)$*denotes the inverse tangent function and* $\ln(z)$ *denotes the natural logarithm function.*

Proof. Put $x \to n+1$ and $y \to 1$ in Theorem 2

$$\frac{1}{n!} = \frac{(n+1)\sqrt{\pi}}{2\Gamma(n+1)\Gamma\left(n+\frac{5}{2}\right)}\sum_{r=0}^{\infty}\frac{\left(\frac{1}{2}\right)_r\Gamma(n+r+1)}{\left(n+\frac{5}{2}\right)_r r!}. \tag{5}$$

Multiply by $\frac{2(-1)^n z^{2n+1}}{\sqrt{\pi}(2n+1)}$ and sum from 0 at ∞ in n, then

$$\operatorname{erf}(z) = \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}(n+1)}{(2n+1)\Gamma(n+1)\Gamma\left(n+\frac{5}{2}\right)}\sum_{r=0}^{\infty}\frac{\left(\frac{1}{2}\right)_r\Gamma(n+r+1)}{\left(n+\frac{5}{2}\right)_r r!}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1} n}{(2n+1)\Gamma(n+1)\Gamma\left(n+\frac{5}{2}\right)}\sum_{r=0}^{\infty}\frac{\left(\frac{1}{2}\right)_r\Gamma(n+r+1)}{\left(n+\frac{5}{2}\right)_r r!} + \sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)\Gamma(n+1)\Gamma\left(n+\frac{5}{2}\right)}\sum_{r=0}^{\infty}\frac{\left(\frac{1}{2}\right)_r\Gamma(n+r+1)}{\left(n+\frac{5}{2}\right)_r r!}, \tag{6}$$

since $0 < z \leq 1$, I have

$$\cong \sum_{r=0}^{\infty}\frac{\left(\frac{1}{2}\right)_r}{r!}\sum_{n=0}^{\infty}\frac{(-1)^n n\Gamma(n+r+1)z^{2n+1}}{(2n+1)\Gamma(n+1)\Gamma\left(n+\frac{5}{2}\right)\left(n+\frac{5}{2}\right)_r} + \sum_{r=0}^{\infty}\frac{\left(\frac{1}{2}\right)_r}{r!}\sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(n+r+1)z^{2n+1}}{(2n+1)\Gamma(n+1)\Gamma\left(n+\frac{5}{2}\right)\left(n+\frac{5}{2}\right)_r}$$

$$= -\frac{8z^3}{45\sqrt{\pi}}\sum_{r=0}^{\infty}\frac{(r+1)!\left(\frac{1}{2}\right)_r}{r!\left(\frac{7}{2}\right)_r}\,{}_2F_2\left(\left.\begin{matrix}\frac{3}{2}, r+2\\ \frac{5}{2}, r+\frac{7}{2}\end{matrix}\right| -z^2\right) + \frac{4z}{3\sqrt{\pi}}\sum_{r=0}^{\infty}\frac{\left(\frac{1}{2}\right)_r}{\left(\frac{5}{2}\right)_r}\,{}_2F_2\left(\left.\begin{matrix}\frac{1}{2}, r+1\\ \frac{3}{2}, r+\frac{5}{2}\end{matrix}\right| -z^2\right)$$

$$= -\frac{8z^3}{45\sqrt{\pi}}\sum_{r=0}^{\infty}\frac{(r+1)!\left(\frac{1}{2}\right)_r}{r!\left(\frac{7}{2}\right)_r}\sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{3}{2}\right)_{\boldsymbol{n}}(r+2)_{\boldsymbol{n}}}{\left(\frac{5}{2}\right)_n\left(r+\frac{7}{2}\right)_n}z^{2n} + \frac{4z}{3\sqrt{\pi}}\sum_{r=0}^{\infty}\frac{\left(\frac{1}{2}\right)_r}{\left(\frac{5}{2}\right)_r}\sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{1}{2}\right)_{\boldsymbol{n}}(r+1)_{\boldsymbol{n}}}{\left(\frac{3}{2}\right)_n\left(r+\frac{5}{2}\right)_n}z^{2n}$$

$$= -\frac{8z^3}{45\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{3}{2}\right)_{\boldsymbol{n}}}{\left(\frac{5}{2}\right)_n}\sum_{r=0}^{\infty}\frac{(r+1)!\left(\frac{1}{2}\right)_r(r+2)_{\boldsymbol{n}}}{r!\left(\frac{7}{2}\right)_r\left(r+\frac{7}{2}\right)_n}z^{2n}+\frac{4z}{3\sqrt{\pi}}\sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{1}{2}\right)_{\boldsymbol{n}}}{\left(\frac{3}{2}\right)_n}\sum_{r=0}^{\infty}\frac{\left(\frac{1}{2}\right)_r(r+1)_{\boldsymbol{n}}}{\left(\frac{5}{2}\right)_r\left(r+\frac{5}{2}\right)_n}z^{2n}$$

$$= -\frac{16z^3}{45\pi}\sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{3}{2}\right)_{\boldsymbol{n}}(2)_n}{\left(\frac{5}{2}\right)_n\left(\frac{7}{2}\right)_n}\frac{\Gamma\left(\frac{2n+7}{2}\right)}{\Gamma(n+3)}z^{2n}+\frac{8z}{3\pi}\sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{1}{2}\right)_{\boldsymbol{n}}(1)_n}{\left(\frac{3}{2}\right)_n\left(\frac{5}{2}\right)_n}\frac{\Gamma\left(\frac{2n+5}{2}\right)}{\Gamma(n+2)}z^{2n}$$

$$= \frac{2(2z\tan^{-1}(z)-\ln(z^2+1)-z^2)}{z\sqrt{\pi}}+\frac{2(2z\tan^{-1}(z)-\ln(z^2+1))}{z\sqrt{\pi}}$$

$$= \frac{4z\tan^{-1}(z)-2\ln(z^2+1)-2z^2+4z\tan^{-1}(z)-2\ln(z^2+1)}{z\sqrt{\pi}}$$

$$= \frac{8z\tan^{-1}(z)-4\ln(z^2+1)-2z^2}{z\sqrt{\pi}}. \square$$

Special Values. Let $z = 1$ in Theorem 4, then

$$\operatorname{erf}(1) \cong \frac{2\pi-2-4\ln(2)}{\sqrt{\pi}},$$

with one place decimal correct.

Let $z = \frac{1}{2}$ in Theorem 4, then

$$\operatorname{erf}\left(\frac{1}{2}\right) \cong \frac{8\tan^{-1}\left(\frac{1}{2}\right)-1-8\ln\left(\frac{5}{4}\right)}{\sqrt{\pi}},$$

with two place decimal correct.

Let $z = \frac{1}{3}$ in Theorem 4, then

$$\operatorname{erf}\left(\frac{1}{3}\right) \cong \frac{8\tan^{-1}\left(\frac{1}{3}\right)-\frac{2}{3}-8\ln\left(\frac{10}{9}\right)}{\sqrt{\pi}},$$

with three place decimal correct.

Let $z = \frac{1}{4}$ in Theorem 4, then

$$\operatorname{erf}\left(\frac{1}{4}\right) \cong \frac{16\tan^{-1}\left(\frac{1}{4}\right)-1-32\ln\left(\frac{17}{16}\right)}{\sqrt{\pi}},$$

with four place decimal correct; and so on. In Appendix, I leave to the reader the Figure 1.

Note. I leave to reader: Prove that, for $0 < z \leq 1$, then

$$\frac{1}{2}+\frac{\sqrt{\pi}\operatorname{erf}(z)}{4z} \cong \sum_{k=0}^{\infty}\frac{(-1)^k z^{2k}}{(2k+1)(k+1)},$$

where $\operatorname{erf}(x)$ denotes the error function $\tan^{-1}$(z)denotes the inverse tangent function and ln(z) denotes the natural logarithm function.

Theorem 5. *For* $0 > z$, *then*

$$\operatorname{erf}(z)=\sum_{r=0}^{\infty}\frac{\left(\frac{1}{2}\right)_r}{r!}\sum_{n=0}^{l}\frac{(-1)^n(n+1)\Gamma(n+r+1)z^{2n+1}}{(2n+1)\Gamma(n+1)\Gamma\left(n+\frac{5}{2}\right)\left(n+\frac{5}{2}\right)_r}$$
$$-\frac{2(-1)^l z^{2l+3}}{\sqrt{\pi}(2l+3)\Gamma(l+2)}\,{}_2F_2\left(\left.\begin{matrix}1,l+\frac{3}{2}\\ l+2,l+\frac{5}{2}\end{matrix}\right|-z^2\right),$$

where $\operatorname{erf}(z)$ *denotes the error function and* ${}_2F_2\left(\left.\begin{matrix}a,b\\c,d\end{matrix}\right|z\right)$ *denotes the hypergeometric function.*

Proof. Put $x \to n+1$ and $y \to 1$ in Theorem 2

$$\frac{1}{n!}=\frac{(n+1)\sqrt{\pi}}{2\Gamma(n+1)\Gamma\left(n+\frac{5}{2}\right)}\sum_{r=0}^{\infty}\frac{\left(\frac{1}{2}\right)_r\Gamma(n+r+1)}{\left(n+\frac{5}{2}\right)_r r!}. \tag{7}$$

Multiply by $\frac{2(-1)^n x^{2n+1}}{\sqrt{\pi}(2n+1)}$ and sum from 0 at l in n, then

$$\frac{2}{\sqrt{\pi}}\sum_{n=0}^{l}\frac{(-1)^n x^{2n+1}}{(2n+1)n!}=\sum_{n=0}^{l}\frac{(-1)^n x^{2n+1}(n+1)}{(2n+1)\Gamma(n+1)\Gamma\left(n+\frac{5}{2}\right)}\sum_{r=0}^{\infty}\frac{\left(\frac{1}{2}\right)_r\Gamma(n+r+1)}{\left(n+\frac{5}{2}\right)_r r!}$$

$$=\sum_{r=0}^{\infty}\frac{\left(\frac{1}{2}\right)_r}{r!}\sum_{n=0}^{l}\frac{(-1)^n x^{2n+1}(n+1)\Gamma(n+r+1)}{(2n+1)\Gamma(n+1)\Gamma\left(n+\frac{5}{2}\right)\left(n+\frac{5}{2}\right)_r}. \tag{8}$$

I calculate the left hand side of (8)

$$\frac{2}{\sqrt{\pi}}\sum_{n=0}^{l}\frac{(-1)^n x^{2n+1}}{(2n+1)n!}=\operatorname{erf}(x)+\frac{2(-1)^l x^{2l+3}}{\sqrt{\pi}(2l+3)\Gamma(l+2)}\,{}_2F_2\left(\left.\begin{matrix}1,l+\frac{3}{2}\\ l+2,l+\frac{5}{2}\end{matrix}\right|-x^2\right). \tag{9}$$

I substitute (9) in the left hand side of (8), put z by x and after of some algebraic manipulation the proof this complete. □

Corollary 4. *For* $z > 0$, *then*

$$\operatorname{erf}(z)=\frac{2z}{\sqrt{\pi}}-\frac{2z^3}{3\sqrt{\pi}}\,{}_2F_2\left(\left.\begin{matrix}1,\frac{3}{2}\\ 2,\frac{5}{2}\end{matrix}\right|-z^2\right),$$

where $\operatorname{erf}(z)$ *denotes the error function and* ${}_2F_2\left(\left.\begin{matrix}a,b\\c,d\end{matrix}\right|z\right)$ *denotes the hypergeometric function.*

Proof. I replace 0 by l in Theorem 4. □

Corollary 5. *For* $z > 0$, *then*

$$\operatorname{erf}(z) = \frac{2z}{\sqrt{\pi}} - \frac{2z^3}{3\sqrt{\pi}} + \frac{z^5}{5\sqrt{\pi}}\, {}_2F_2\left(\begin{matrix} 1, \frac{5}{2} \\ 3, \frac{7}{2} \end{matrix}\middle| -z^2\right),$$

where $\operatorname{erf}(z)$ *denotes the error function and* ${}_2F_2\left(\begin{matrix} a, b \\ c, d \end{matrix}\middle| z\right)$ *denotes the hypergeometric function.*

Proof. I replace 1 by l in Theorem 4. □

APPENDIX

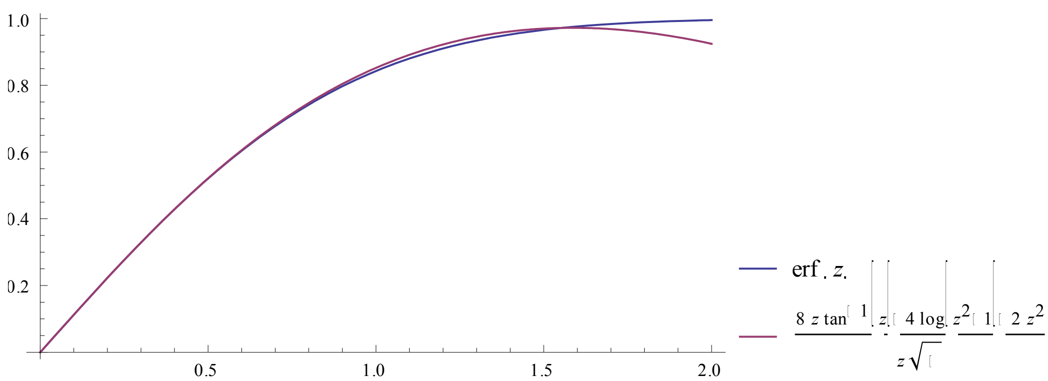

Figure 1

REFERENCES

[1] Gradshteyn, I. S. and Ryzhik, I. M., *Table of Integrals, Series and Products*, Academic Press, 2000.

[2] Guedes, Edigles, *An Investigation on the Beta Function I: New Versions of the Euler Beta Function,* 2013.