

Oscillation Criteria for a Class of Discrete Nonlinear Fractional Equations

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ABSTRACT. In this paper, we study the oscillatory behavior of the fractional difference equations of the following form

$$\Delta(p(t)\Delta(r(t)\Delta^\alpha x(t))^\eta) + F\left(t, \sum_{s=t_0}^{t-1+\alpha} (t-s-1)^{(-\alpha)} x(s)\right) = 0 \text{ for } t \in N_{t_0+1-\alpha},$$

for $0 < \alpha \leq 1$, $t_0 \geq t > 0$, Δ^α denotes the Riemann-Liouville difference operator and $\eta > 0$ is a quotient of odd positive integers. We establish some oscillation criteria for the equation by using Riccati transformation technique and some inequalities. An example is shown to illustrate our main results.

1. INTRODUCTION

The theory of fractional derivatives goes back to Leibniz's note to L'Hospital dated 30 September 1695 about the meaning of the derivative of non integer order and this led to the appearance of the theory of derivatives and integrals of arbitrary order. Fractional differential equations are generalizations of classical differential equations of integer order. In recent days, oscillatory behavior of fractional differential/difference equations has been investigated by authors, see papers [2]-[12]. Formal treatment on the subject of fractional derivatives and fractional integrals are presented in the books, see [16]-[19]. In the last few years, many authors found that fractional derivatives and fractional integrals were applied in widespread fields of science and engineering, especially in mathematical modeling real world phenomenon and simulation of systems and processes and control systems. Nowadays, many authors have investigated some qualitative aspects of fractional differential equations, such as the existence, the uniqueness and stability of solutions. But the discrete analog of fractional difference equations are studied by very few authors, see [13]-[15]. Now we study the oscillatory behavior of the following fractional difference equation of the form

$$\Delta(p(t)\Delta(r(t)\Delta^\alpha x(t))^\eta) + F\left(t, \sum_{s=t_0}^{t-1+\alpha} (t-s-1)^{(-\alpha)} x(s)\right) = 0 \text{ for } t \in N_{t_0+1-\alpha}, \quad (9)$$

$t \geq t > 0, \eta$ is the ratio of two odd positive integers and $\alpha \Delta$ denotes the Riemann-Liouville difference operator of order $0 < \alpha \leq 1$.

(H₁) $p(t), r(t)$ are positive sequences $\sum_{s=t_0}^{\infty} \frac{1}{p(s)} = \infty, \sum_{s=t_0}^{\infty} \frac{1}{r(s)} = \infty$, and $f: R \rightarrow R$ is a

continuous function such that $f(x)/(x^\eta) \geq k$ for a certain constant $k > 0$ and for all $x \neq 0$.

(H₂) $F(t, G) \in C([t_0, \infty) \times R; R)$ and there exists a real sequence $q(t)$ such that $F(t, G)/G^\eta \geq q(t)$ for $G \neq 0$ and $x \neq 0, t \geq t_0$.

A solution $x(t)$ of (1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is nonoscillatory. Equation (1) is said to be oscillatory if all its solutions are oscillatory.

2. Preliminaries and Basic Lemmas

In this section, we provide preliminary results of discrete fractional calculus, which will be used throughout this paper.

Definition 2.1. (see [13]) Let $\nu > 0$. The ν -th fractional sum of f is defined by

$$\Delta^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)} f(s),$$

where f is defined for $s \equiv a \pmod{1}$ and $\Delta^{-\nu}$ is defined for $t \equiv (a + \nu) \pmod{1}$ and $t^{(\nu)} = \frac{\Gamma(t+1)}{\Gamma(t-\nu+1)}$. The fractional sum $\Delta^{-\nu} f$ maps functions defined on N_a to functions defined on $N_{a+\nu}$.

Definition 2.2. (see [13]) Let $\mu > 0$ and $m-1 < \mu < m$, where m denotes a positive integer, $m = \lceil \mu \rceil$. Set $\nu = m - \mu$. The μ -th fractional Riemann-Liouville difference is defined as

$$\Delta^{\mu} f(t) = \Delta^{m-\nu} f(t) = \Delta^m \Delta^{-\nu} f(t)$$

In order to discuss our results in Section 3, now we state the following lemma [1].

Lemma 2.3. Let A be a positive real number, B a positive arbitrary number and let γ be a quotient of odd positive integers. Then

$$Bu - Au^{\gamma+1/\lambda} \leq \frac{\gamma^{\gamma}}{(\gamma+1)^{\gamma+1}} \frac{B^{\gamma+1}}{A^{\gamma}} \quad (2)$$

Lemma 3.1. Let $x(t)$ be a solution of (1) and let

$$G(t) = \sum_{s=t_0}^{t-1+\alpha} (t-s-1)^{(-\alpha)} x(s) \quad (3)$$

then

$$\Delta(G(t)) = \Gamma(1-\alpha) \Delta^{\alpha} x(t). \quad (4)$$

Proof.

$$\begin{aligned} G(t) &= \sum_{s=t_0}^{t-1+\alpha} (t-s-1)^{(-\alpha)} x(s) = \sum_{s=t_0}^{t-1+\alpha} (t-s-1)^{(1-\alpha)-1} x(s) \\ &= \Gamma(1-\alpha) \Delta^{-(1-\alpha)} x(t), \end{aligned}$$

which implies

$$\Delta(G(t)) = \Gamma(1-\alpha) \Delta \Delta^{-(1-\alpha)} x(t) = \Gamma(1-\alpha) \Delta^{\alpha} x(t).$$

Lemma 3.2. Assume that $x(t)$ is an eventually positive solution of equation (1). If $x(t) > 0$, $\Delta^{\alpha} x(t) > 0$, and $G(t) > 0$, then $\Delta(r(t) \Delta^{\alpha} x(t))^{\eta} > 0$ for $t \geq t_0$.

Proof. Let $x(t)$ be an eventually positive solution of (1) for $t_1 \geq t_0$. From (1), $\Delta(p(t)\Delta(r(t)\Delta^\alpha x(t))^\eta) \leq 0$ for $t \geq t_1$. We claim that there is a $t_2 \geq t_1$ such that $\Delta(r(t)\Delta^\alpha x(t))^\eta > 0$ for $t \geq t_2$. Suppose to the contrary that $\Delta(r(t)\Delta^\alpha x(t))^\eta \leq 0$ for $t \geq t_2$. Since $p(t) > 0$ and $p(t)\Delta(r(t)\Delta^\alpha x(t))^\eta$ is non increasing, there exists a negative constant δ and $t_3 \geq t_2$ such that $p(t)\Delta(r(t)\Delta^\alpha x(t))^\eta \leq \delta$ for $t \geq t_3$. Dividing by $p(t)$ and summing from t_3 to $t-1$, we obtain

$$(r(t)\Delta^\alpha x(t))^\eta \leq (r(t_3)\Delta^\alpha x(t_3))^\eta + \sum_{s=t_3}^{t-1} \frac{\delta}{p(s)}.$$

Letting $t \rightarrow \infty$, we obtain $x(t) \alpha \Delta \rightarrow -\infty$ which is a contradiction, since $\Delta^\alpha x(t) > 0$. This completes the proof.

Lemma 3.3. Assume that $x(t)$ is an eventually positive solution of equation (1) and suppose that Lemma (3.2) holds. Then there exists $t_1 > t_0$ such that $\Delta^\alpha x(t) > 0, \Delta(r(t)\Delta^\alpha x(t))^\eta > 0$. Then we have

$$(\Delta G(t))^\eta \geq \Gamma(1-\alpha)^\eta \Delta(r(t)\Delta^\alpha x(t))^\eta \frac{1}{r(t)^\eta} \sum_{s=t_1}^{t-1} \frac{1}{p(s)}. \tag{5}$$

Proof. From equation (1), we see that $\Delta(p(t)\Delta(r(t)\Delta^\alpha x(t))^\eta) \leq 0$ and hence $(r(t)\Delta^\alpha x(t))^\eta$ is non increasing. Then we obtain

$$\begin{aligned} (r(t)\Delta^\alpha x(t))^\eta &= (r(t_1)\Delta^\alpha x(t_1))^\eta + \sum_{s=t_1}^{t-1} \frac{p(s)\Delta(r(s)\Delta^\alpha x(t))^\eta}{p(s)} \\ &\geq p(t)\Delta(r(t)\Delta^\alpha x(t))^\eta \sum_{s=t_1}^{t-1} \frac{1}{p(s)} \\ (\Delta G(t))^\eta &\geq \Gamma(1-\alpha)^\eta p(t)\Delta(r(t)\Delta^\alpha x(t))^\eta \frac{1}{r(t)^\eta} \sum_{s=t_1}^{t-1} \frac{1}{p(s)}. \end{aligned}$$

Theorem 3.4. Suppose that conditions H_1 and H_2 hold. If there exists a positive sequence $\rho(s)$ such that

$$\limsup_{t \rightarrow \infty} \sum_{s=t_0}^{t-1} \left(\rho(s)q(s) - \frac{(\Delta \rho_+ s)^2}{4\rho^2(s+1)R(s)} \right) = \infty, \tag{5}$$

$$\Delta \rho_+(s) = \max(\rho(s), 0) \quad \text{and} \quad R(t) = \frac{2^{1-\eta} \rho(t) \Gamma(1-\alpha)^\eta \frac{1}{r(t)^\eta} \sum_{s=t_0}^{t-1} \frac{1}{p(s)}}{\rho^2(t+1)p(t)}.$$

Where Then every solution of equation (1) is oscillatory.

Proof. Suppose to the contrary that equation (1) has a nonoscillatory solution $x(t)$. Without loss of generality, we may assume that $x(t) > 0$ on $1 \leq t \leq \infty$. We define the function $w(t)$ by Riccati substitution

$$w(t) = \rho(t) \frac{p(t) \Delta(r(t) \Delta^\alpha x(t))^\eta}{G^\eta(t)} \text{ for } t \geq t_1. \quad (7)$$

Then we have $w(t) > 0$ for $t \geq t_1$. Also, we have

$$\begin{aligned} \Delta w(t) &= \Delta \rho(t) \frac{w(t+1)}{\rho(t+1)} \\ &+ \frac{\rho(t) \Delta(p(t) \Delta(r(t) \Delta^\alpha x(t))^\eta) G^\eta(t+1) - \rho(t) p(t+1) \Delta(r(t+1) \Delta^\alpha x(t+1))^\eta \Delta G^\eta(t)}{G^\eta(t+1) G^\eta(t)} \\ &\leq \Delta \rho_+(t) \frac{w(t+1)}{\rho(t+1)} - \rho(t) q(t) - \frac{\rho(t) p(t+1) \Delta(r(t+1) \Delta^\alpha x(t+1))^\eta \Delta G^\eta(t)}{G^{2\eta}(t+1)} \end{aligned}$$

Now using the inequality (see [1])

$$x^\beta - y^\beta \geq 2^{1-\beta} (x - y)^\beta, \text{ for all } x \geq y > 0 \text{ and } \beta \geq 1 \quad (8)$$

we have

$$p(t) \Delta(r(t) \Delta^\alpha x(t))^\eta \geq p(t+1) \Delta(r(t+1) \Delta^\alpha x(t+1))^\eta \quad (9)$$

Using the above inequality, we obtain

$$\begin{aligned} \Delta w(t) &\leq \Delta \rho_+(t) \frac{w(t+1)}{\rho(t+1)} - \rho(t) q(t) - \frac{2^{1-\eta} \rho(t) p(t+1) \Delta(r(t+1) \Delta^\alpha x(t+1))^\eta (\Delta G(t))^\eta}{G^{2\eta}(t+1)} \\ &\leq \Delta \rho_+(t) \frac{w(t+1)}{\rho(t+1)} - \rho(t) q(t) - \frac{2^{1-\eta} \rho(t) \Gamma(1-\alpha)^\eta \Delta(r(t) \Delta^\alpha x(t))^\eta \frac{1}{r(t)^\eta} \sum_{s=t_1}^{t-1} \frac{1}{p(s)}}{\frac{\rho^2(t+1) p(t+1) \Delta(r(t+1) \Delta^\alpha x(t+1))^\eta}{w(t+1)^2}} \\ \Delta w(t) &\leq \Delta \rho_+(t) \frac{w(t+1)}{\rho(t+1)} - \rho(t) q(t) - \frac{2^{1-\eta} \Gamma(1-\alpha)^\eta \rho(t) \frac{p(t+1)}{p(t)} \Delta(r(t+1) \Delta^\alpha x(t+1))^\eta \frac{1}{r(t)^\eta} \sum_{s=t_1}^{t-1} \frac{1}{p(s)}}{\frac{\rho^2(t+1) p(t+1) \Delta(r(t+1) \Delta^\alpha x(t+1))^\eta}{w(t+1)^2}} \\ &\leq \Delta \rho_+(t) \frac{w(t+1)}{\rho(t+1)} - \rho(t) q(t) - \frac{2^{1-\eta} \rho(t) \Gamma(1-\alpha)^\eta \frac{1}{r(t)^\eta} \sum_{s=t_1}^{t-1} \frac{1}{p(s)}}{\rho^2(t+1) p(t)} w(t+1)^2 \\ &= \Delta \rho_+(t) \frac{w(t+1)}{\rho(t+1)} - \rho(t) q(t) - R(t) w(t+1)^2 \end{aligned} \quad (10)$$

where

$$R(t) = \frac{2^{1-\eta} \rho(t) \Gamma(1-\alpha)^\eta \frac{1}{r(t)^\eta} \sum_{s=t_1}^{t-1} \frac{1}{p(s)}}{\rho^2(t+1) p(t)}$$

Taking

$$A = R(t), \quad B = \frac{\Delta \rho_+(t)}{\rho(t+1)}, \quad \text{and } u = w(t+1).$$

and using Lemma (2.3) with $\gamma = 1$, we obtain

$$\begin{aligned} Bu - AU^2 &\leq \frac{1}{2^2} \frac{B^2}{A} \\ &= \frac{1}{4} \left(\frac{\Delta\rho_+(t)}{\rho(t+1)} \right)^2 \\ &= \frac{(\Delta\rho_+(t))^2}{4\rho^2(t+1)R(t)} \end{aligned}$$

From (5), we conclude that

$$\Delta w(t) \leq -\rho(t)q(t) + \frac{(\Delta\rho_+(t))^2}{4\rho^2(t+1)R(t)}.$$

Summing the above inequality from t_1 to $t-1$ we have

$$\sum_{s=t_1}^{t-1} \left(\rho(s)q(s) - \frac{(\Delta\rho_+(s))^2}{4\rho^2(s+1)R(s)} \right) \leq w(t_1) - w(t) \leq w(t_1) < \infty, \quad \text{for } t \geq t_1$$

Letting $t \rightarrow \infty$, we get

$$\limsup_{t \rightarrow \infty} \sum_{s=t_1}^{t-1} \left(\rho(s)q(s) - \frac{(\Delta\rho_+(s))^2}{4\rho^2(s+1)R(s)} \right) \leq w(t_1) < \infty,$$

which contradicts (3). The proof is complete.

Theorem 3.5. Suppose that conditions (H_1) and (H_2) hold. Furthermore, assume that there exists a positive sequence $\rho(t)$ such that

$$\begin{aligned} H(t,t) &= 0 \quad \text{for } t \geq 0 \quad H(t,s) > 0 \quad t > s \geq 0 \\ \Delta_2 H(t,s) &= H(t,s+1) - H(t,s) \leq 0 \quad \text{for } t \geq s \geq 0. \end{aligned}$$

If

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t,t_0)} \sum_{s=t_0}^{t-1} \left(\rho(s)q(s)H(t,s) - \frac{h_+^2(t,s)}{4H(t,s)R(s)} \right) = \infty, \tag{11}$$

$$h_+(t,s) = \Delta_2 H(t,s) + \frac{H(t,s)\Delta\rho_+(s)}{\rho(s+1)} \quad \text{and} \quad \Delta\rho_+(s) = \max[\Delta\rho(s), 0].$$

Where

every solution of (1) is oscillatory.

Then every

Proof. Suppose the contrary that $x(t)$ is a nonoscillatory solution of (1). Without loss of generality, we may assume that $x(t)$ is an eventually positive solution of (1). We proceed as in the proof of Theorem (3.4) to get (10). Multiplying (10) by $H(t,s)$ and summing from t_1 to $t-1$, we obtain

$$\begin{aligned} \sum_{s=t_1}^{t-1} \rho(s)q(s)H(t,s) &\leq -\sum_{s=t_1}^{t-1} H(t,s)\Delta w(s) + \sum_{s=t_1}^{t-1} H(t,s)\Delta\rho_+(s) \frac{w(s+1)}{b(s+1)} \\ &\quad - \sum_{s=t_1}^{t-1} H(t,s)R(s)w^2(s+1) \end{aligned} \tag{12}$$

Using summation by parts formula, we get

$$\begin{aligned}
-\sum_{s=t_1}^{t-1} H(t,s)\Delta w(s) &= -\left[H(t,s)w(s)\right]_{s=t_1}^t + \sum_{s=t_1}^{t-1} w(s+1)\Delta_2 H(t,s) \\
&= H(t,t_1)w(t_1) + \sum_{s=t_1}^{t-1} w(s+1)\Delta_2 H(t,s)
\end{aligned} \tag{13}$$

where $\Delta_2 H(t,s) = H(t,s+1) - H(t,s)$. For $t \geq t_1$ we have

$$\begin{aligned}
\sum_{s=t_1}^{t-1} \rho(s)q(s)H(t,s) &\leq H(t,t_1)w(t_1) + \sum_{s=t_1}^{t-1} w(s+1)\Delta_2 H(t,s) + \sum_{s=t_1}^{t-1} H(t,s)\Delta\rho_+(s)\frac{w(s+1)}{\rho(s+1)} - \sum_{s=t_1}^{t-1} H(t,s)R(s)w^2(s+1) \\
&\leq H(t,t_1)w(t_1) + \sum_{s=t_1}^{t-1} \left(\Delta_2 H(t,s) + \frac{H(t,s)\Delta\rho_+(s)}{b(s+1)} \right) w(s+1) - \sum_{s=t_1}^{t-1} H(t,s)R(s)w^2(s+1) \\
&\leq H(t,t_1)w(t_1) + \sum_{s=t_1}^{t-1} \left(h_+(t,s)w(s+1) - H(t,s)R(s)w^2(s+1) \right)
\end{aligned} \tag{14}$$

where

$$h_+(t,s) = \Delta_2 H(t,s) + \frac{H(t,s)\Delta\rho_+(s)}{\rho(s+1)}$$

Taking

$A = R(t)H(t,s)$, $B = h_+(t,s)$, and $u = w(t+1)$ and using Lemma 2.3 with $\gamma = 1$, we get

$$\begin{aligned}
Bu - AU^2 &\leq \frac{1}{2^2} \frac{B^2}{A} \\
&= \frac{1}{4} \frac{h_+^2(t,s)}{H(t,s)R(t)}
\end{aligned}$$

We have $\Delta_2 H(t,s) \leq 0$ for $t > s \geq t_0$, $0 < H(t,t_1) \leq H(t,t_0)$ for $t > t_1 \geq t_0$.

$$\begin{aligned}
\sum_{s=t_1}^{t-1} \rho(s)q(s)H(t,s) &\leq H(t,t_1)w(t_1) + \sum_{s=t_1}^{t-1} \frac{h_+^2(t,s)}{4H(t,s)R(s)} \\
\sum_{s=t_1}^{t-1} \left(\rho(s)q(s)H(t,s) - \frac{h_+^2(t,s)}{4H(t,s)R(s)} \right) &\leq H(t,t_1)w(t_1) \\
&\leq H(t,t_0)w(t_1).
\end{aligned}$$

Since, $0 < H(t,s) \leq H(t,t_0)$ for $t > s \geq t_0$, we have $0 < \frac{H(t,s)}{H(t,t_0)} \leq 1$ for $t > s \geq t_0$.

Hence it follows that

$$\begin{aligned}
\frac{1}{H(t,t_0)} \sum_{s=t_0}^{t-1} \left(\rho(s)q(s)H(t,s) - \frac{h_+^2(t,s)}{4H(t,s)R(s)} \right) &= \frac{1}{H(t,t_0)} \sum_{s=t_0}^{t_1-1} \left(\rho(s)q(s)H(t,s) - \frac{h_+^2(t,s)}{4H(t,s)R(s)} \right) \\
&\quad + \frac{1}{H(t,t_0)} \sum_{s=t_1}^{t-1} \left(\rho(s)q(s)H(t,s) - \frac{h_+^2(t,s)}{4H(t,s)R(s)} \right) \\
&\leq \frac{1}{H(t,t_0)} \sum_{s=t_0}^{t_1-1} \rho(s)q(s)H(t,s) + w(t_1) \\
&\leq \sum_{s=t_0}^{t_1-1} \rho(s)q(s) + w(t_1).
\end{aligned}$$

Letting $t \rightarrow \infty$, we have

$$\limsup_{t \rightarrow \infty} \frac{1}{H(t, t_0)} \sum_{s=t_0}^{t-1} \left(\rho(s)q(s)H(t, s) - \frac{h_+^2(t, s)}{4H(t, s)R(s)} \right) \leq \sum_{s=t_0}^{t-1} \rho(s)q(s) + w(t_1) < \infty,$$

which is a contradiction to (11). The proof is complete.

Example 3.6. Consider the following fractional difference equation

$$\Delta \left(t \Delta \left(t^{1/3} \Delta^\alpha (x(t)) \right)^3 \right) + t^2 \left(\sum_{s=0}^{t-1+\alpha} (t-s-1)^{(-\alpha)} x(s) \right)^3 = 0, \quad (15)$$

Here, $\alpha = \frac{1}{2}, \eta = 3, p(t) = t, q(t) = t^2$ and $r(t) = t^{1/3}$

Here,

We will apply Theorem (3.4) and it remains show that condition (6) is satisfied. Taking $\rho(s) = s$, we obtain

$$\limsup_{t \rightarrow \infty} \sum_{s=t_0}^{t-1} \left(\rho(s)q(s) - \frac{(\Delta \rho_+(s))^2}{4\rho^2(s+1)R(s)} \right) = \limsup_{t \rightarrow \infty} \sum_{s=t_0}^{t-1} s \left(s^2 - \frac{1}{\pi^{3/2} \sum_{s=t_0}^{\infty} \frac{1}{s}} \right) = \infty$$

which implies that (6) holds. Therefore, by Theorem (3.4) every solution of (15) is oscillatory.

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