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Common Fixed Point Theorem for Three Self Mappings

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Abstract. The aim of this paper is to study the concept of compatible mappings of type (R) and discussed a common fixed point theorem for three compatible mappings of type (R) satisfying contractive conditions.

1. Introduction

In 1986, Jungck [5] introduced the concept of compatible mappings by generalizing commuting mappings. Pathak, Chang and Cho [6] in 1994 introduced the concept of a new type of compatible mappings called compatible mappings of type (P). Al-Thagafi and Shahazad [3] gave the concept of occasionally weakly compatible mappings in 2008. In 2011, H. Bouhadjera [1] proved some common fixed point theorems for three and four occasionally weakly compatible mappings satisfying different types of contractive conditions.

Now the result of H. Bouhadjera [1] is extended by employing compatible mappings of type (R) instead of occasionally weakly compatible mappings.

2. Preliminaries

Definition 2.1 Self mappings A and B of a metric space (X, d) are said to be weakly commuting pair if, for all x $\in$X

$$d(ABx, BAx) \leq d(Ax, Bx)$$

Definition 2.2 Self mappings *A* and *B* of a metric space (*X*, *d*) are said to be compatible if, for all $x \in X$

$$\lim_{n\to\infty} d(ABx_n, BAx_n) = 0$$

whenever $\{x_n\}$ is a sequence in *X* and such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some $t \in X$.

Definition 2.4 Self mappings *A* and *B* of a metric space (*X*, *d*) are said to be compatible of type (P) if, for all $x \in X$

$$\lim_{n\to\infty} d(AAx_n, BBx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X and such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some $t \in X$.

Definition 2.3 Self mappings A and B of a metric space (X, d) are said to be compatible of type (R) if, for all $x \in X$

$$\lim_{n\to\infty} d(ABx_n, BAx_n) = 0 \text{ and } \lim_{n\to\infty} d(AAx_n, BBx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X and such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some $t \in X$.

Definition 2.5 Self mappings A and B of a metric space (X, d) are said to be weakly compatible if they commute at their coincidence points.

Definition 2.6 Two self mappings A and B of a set X are occasionally weakly compatible if and only if there is a point t in X which is a coincidence point of A and B at which A and B commute.

Definition 2.7 A function $\phi : [0, \infty) \to [0, \infty)$ is said to be a contractive modulus if $\phi(0) = 0$ and $\varphi(t) < t$ for $t > 0$.

Definition 2.8 A real valued function φ defined on $X \subset R$ is said to be upper semi continuous if $\lim_{n\to\infty} \varphi(t_n) \le \varphi(t)$, for every sequence $\{t_n\}$ in X with $t_n \to t$ as $n \to \infty$.

Pathak, Murthy and Cho [6] prove the following propositions.

Proposition 2.9: Let S and T be mappings from a complete metric space (X, d) into itself. If a pair{S, T} is compatible of type (R) on X and $Sz = Tz$ for $z \in X$, then STz = TSz = SSz = TTz.

Proposition 2.10: Let S and T be mappings from a complete metric space (X, d) into itself. If a pair {S, T} is compatible of type (R) on X and $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$, then we have

(i) $d(TSx_n, Sz) \to 0$ as $n \to \infty$ if S is continuous,
(ii) $d(STx_n, Tz) \to 0$ as $n \to \infty$ if T is continuous and
(iii) $STz = TSz$ and $Sz = Tz$ if S and T are continuous at z.

H. Bouhadjera [1] proved the following results:

Theorem 2.11 Let X be a set with a symmetric d. Let f, g and h be self mapping of (X, d) and φ is acontractive modulus function satisfying

$$d^2(fx, gy) \le \max\{\varphi(d(hx, hy))\varphi(d(hx, fx)),\ \varphi(d(hx, hy))\varphi(d(hy, fx)),\ \varphi(d(hx, hy))\varphi(d(hy, gy)),$$
$$\varphi(d(hx, fx))\varphi(d(hy, gy)),\ \varphi(d(hx, gy))\varphi(d(hy, fx))\}$$

for all $x, y \in X$,

the pair (f, h) and (g, h) is owc.

Then f, g and h have a unique common fixed point.

Theorem 2.11 Let X be a set endowed with a symmetric d. Suppose f, g, h and k are four self mappings of (X, d) satisfying the conditions:

$$d^2(fx, gy) \leqslant \max\{ \phi(d(hx, ky))\, \phi(d(hx, fx)),\ \phi(d(hx, ky))\, \phi(d(ky, gy)),\ \phi(d(hx, fx))\, \phi(d(ky, gy)), \varphi(d(hx, gy))\varphi(d(ky, fx))\},$$

for all x, y $\in$ X, where φ is contractive modulus,

the pair *(f, h)* and *(g,, k)* are owc.

Then *f, g, h* and k have a unique common fixed point.

3 Main Results

Theorem 3.1 Let A, B and T be self mapping of a complete metric space (X, d) and φ is a contractive modulus satisfying:

(a) $A(X) \cup B(X) \subset T(X)$.

(b) $d2(Ax,By) \leqslant \max\{ \phi(d(Tx,Ty))\, \phi(d(Tx,Ax)),\ \phi(d(Tx,Ty))\, \phi(d(Ty,Ax)),\ \phi(d(Tx,Ty))\, (d(Ty,By)), \varphi(d(Tx,Ax))\varphi(d(Ty,By)), \varphi(d(Tx,By))\varphi(d(Ty,Ax))\}$

for all x, y $\in X$.

(c) the pair(A, T) or (B, T) is compatible of type (R).

(d) If T is continuous,

then A, B and T have a unique common fixed point.

Proof: Let $x_0 \in$ X be arbitrary. Take a sequence $\{Tx_n\}$, as follows.

$$Tx_{2n+}1 = Ax_{2n},\ Tx_{2n+2} = Bx_{2n+1},\ n = 0, 1, 2,\ldots \tag{1}$$

From condition (b) we get

$$\begin{aligned}
d^2(Tx_{2n+1}, Tx_{2n+2}) &= d^2(Ax_{2n}, Bx_{2n+1}) \\
&\leq \max\{\varphi(d(Tx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n}, Ax_{2n})), \varphi(d(Tx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n+1}, Ax_{2n})), \\
&\qquad \varphi(d(Tx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n+1}, Bx_{2n+1})), \varphi(d(Tx_{2n}, Ax_{2n}))\varphi(d(Tx_{2n+1}, Bx_{2n+1})), \\
&\qquad \varphi(d(Tx_{2n}, Bx_{2n+1}))\varphi(d(Tx_{2n+1}, Ax_{2n}))\}, \\
&= \max\{\varphi(d(Tx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n}, Tx_{2n+1})), \varphi(d(Tx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n+1}, Tx_{2n+1})), \\
&\qquad \varphi(d(Tx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n+1}, Tx_{2n+2})), \varphi(d(Tx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n+1}, Tx_{2n+2})), \\
&\qquad \varphi(d(Tx_{2n}, Tx_{2n+2}))\varphi(d(Tx_{2n+1}, Tx_{2n+1}))\}, \\
&= \max\{\varphi(d(Tx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n}, Tx_{2n+1})), \varphi(d(Tx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n+1}, Tx_{2n+1}))\}.
\end{aligned}$$

This implies

$$d^2(Tx_{2n+1}, Tx_{2n+2}) \leq \varphi(d(Tx_{2n}, Tx_{2n+1})) \max\{\varphi(d(Tx_{2n}, Tx_{2n+1})), \varphi(d(Tx_{2n+1}, Tx_{2n+2}))\}$$
$$\leq d(Tx_{2n}, Tx_{2n+1}) \max\{d(Tx_{2n}, Tx_{2n+1}), d(Tx_{2n+1}, Tx_{2n+2})\}.$$

Since φ is contractive modulus. We have,

$$d(Tx_{2n+1}, Tx_{2n+2}) \leq \varphi(d(Tx_{2n}, Tx_{2n+1})) \leq d(Tx_{2n}, Tx_{2n+1}). \tag{2}$$

The sequence $\{d(Tx_{2n}, Tx_{2n+1})\}$ is decreasing. If $\{d(Tx_{2n}, Tx_{2n+1})\} \to \alpha$ then from (2) we have $\alpha \leqslant \phi(\alpha) \leqslant \alpha$ and so we must have $\alpha = 0$, hence

$$\lim_{n\to\infty} d(Tx_{2n}, Tx_{2n+1}) = 0. \tag{3}$$

We shall show that $\{Tx2n\}$ is a Cauchy sequence. If it is not so, there exist an $\in > 0$ and a sequence of integers $\{mk\}$, $\{nk\}$ with $mk > nk \geq k$, such that

$$d(Tx_{2m}, Tx_{2n}) \geq \in \tag{4}$$

$k = 1, 2, 3, \ldots\ldots$. If m_k is the smallest integer exceeding n_k for which (4) holds, then from wellordering principle, we have

$$d(Tx2m-1, Tx2n) < \in. \tag{5}$$

Since

$$d(Tx_{2m-1}, Tx_{2n+1}) \leq d(Tx_{2m-1}, Tx_{2n}) + d(Tx_{2n}, Tx2n+1),$$

and so letting $k \to \infty$, we see that

$$d(Tx_{2m-1}, Tx_{2n+1}) \leq \in. \tag{6}$$

Now

$$d(Tx_{2m}, Tx_{2n}) \leq d(Tx_{2m}, Tx_{2n+1}) + d(Tx_{2n+1}, Tx_{2n}). \tag{7}$$

But we have

$$\begin{aligned} d^2(Tx_{2m}, Tx_{2n+1}) &= d^2(Ax_{2n}, Bx_{2m-1}) \\ &\leq \max\{\varphi(d(Tx_{2n}, Tx_{2m-1}))\varphi(d(Tx_{2n}, Ax_{2n})), \\ &\qquad \varphi(d(Tx_{2n}, Tx_{2m-1}))\varphi(d(Tx_{2m-1}, Ax_{2n})), \\ &\qquad \varphi(d(Tx_{2n}, Tx_{2m-1}))\varphi(d(Tx_{2m-1}, Bx_{2m-1})), \\ &\qquad \varphi(d(Tx_{2n}, Ax_{2n}))\varphi(d(Tx_{2m-1}, Bx_{2m-1})), \\ &\qquad \varphi(d(Tx_{2n}, Bx_{2m-1}))\varphi(d(Tx_{2m-1}, Ax_{2n}))\}, \\ &= \max\{\varphi(d(Tx_{2n}, Tx_{2m-1}))\varphi(d(Tx_{2n}, Tx_{2n+1})), \\ &\qquad \varphi(d(Tx_{2n}, Tx_{2m-1}))\varphi(d(Tx_{2m-1}, Tx_{2n+1})), \\ &\qquad \varphi(d(Tx_{2n}, Tx_{2m-1}))\varphi(d(Tx_{2m-1}, Tx_{2m})), \\ &\qquad \varphi(d(Tx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2m-1}, Tx_{2m})), \\ &\qquad \varphi(d(Tx_{2n}, Tx_{2m}))\varphi(d(Tx_{2m-1}, Tx_{2n+1}))\}. \end{aligned}$$

Using (3), (5), (6), (8) and letting $k \to \infty$, we obtain

$$\lim_k d^2(Tx_{2m}, Tx_{2n+1}) \le \max\{\varphi(\in)\varphi(\in), \varphi(\in) \lim_k d(Tx_{2m}, Tx_{2n})\}$$

$$\le \varphi(\in) \max\{\varphi(\in), \lim_k d(Tx_{2m}, Tx_{2n+1})\},$$

implies

$$\lim_k d(Tx_{2m}, Tx_{2n+1}) \le \varphi(\in) < \in.$$

Then by (7)

$$\lim_k d(Tx_{2m}, Tx_{2n}) \le \varphi(\in) + 0 < \in,$$

a contradiction. Thus $\{Tx_n\}$ is a Cauchy sequence. Since X is complete, there exist a point $z \in X$ such that $Tx_n \to z$. It follows that from (1) that the sequences $\{Ax_{2n}\}$and $\{Bx_{2n+1}\}$ also converge to z.

Since T is continuous such that

$$TTx_{2n} \to Tz,\ TAx_{2n} \to Tz \text{ as } n \to \infty,$$

Since the pair (A, T) is compatible of type (R), we have

$$AAx_{2n} \to Tz \text{ as } n \to \infty$$

Then from condition (ii), we have

$$\begin{aligned} d^2(AAx_{2n}, Bx_{2n+1}) \le \max\{&\varphi(d(TAx_{2n}, Tx_{2n+1}))\varphi(d(TAx_{2n}, AAx_{2n+1})),\\ &\varphi(d(TAx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n+1}, AAx_{2n})),\\ &\varphi(d(TAx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n+1}, Bx_{2n+1})),\\ &\varphi(d(TAx_{2n}, AAx_{2n}))\varphi(d(Tx_{2n+1}, Bx_{2n+1})),\\ &\varphi(d(TAx_{2n}, Bx_{2n+1}))\varphi(d(Tx_{2n+1}, AAx_{2n}))\}. \end{aligned}$$

Letting $n \to \infty$ and we have

$$\begin{aligned} d^2(Tz, z) \le \max\{&\varphi(d(Tz, z))\varphi(d(Tz, Tz)), \varphi(d(Tz, Tz))\varphi(d(Tz, z)), \varphi(d(Tz, z))\varphi(d(z, z)),\\ &\varphi(d(Tz,Tz))\varphi(d(z, z)), \varphi(d(Tz, z))\varphi(d(z, Tz))\}\\ = &\varphi(d(Tz, z))\varphi(d(Tz, z)), \end{aligned}$$

and it implies that

$$d(Tz, z) \le \varphi(d(Tz, z)) \le d(Tz, z)$$

i.e. $\varphi(d(Tz, z)) = d(Tz, z)$. Hence $Tz = z$.

Again from (ii), we have

$$\begin{aligned} d^2(Az, Bx_{2n+1}) \le \max\{&\varphi(d(Tz, Tx_{2n+1}))\varphi(d(Tz, Az)), \varphi(d(Tz, Tx_{2n+1}))\varphi(d(Tx_{2n+1}, Bx_{2n+1})),\\ &\varphi(d(Tz, Az))\varphi(d(Tx_{2n+1}, Bx_{2n+1})), \varphi(d(Tz, Ez))\varphi(d(Tx_{2n+1}, Bx_{2n+1})),\\ &\varphi(d(Tz, Bx_{2n+1}))\varphi(d(Tx_{2n+1}, Az))\}. \end{aligned}$$

Letting as $n \to \infty$ and using $Tz = z$,

$$d^2(Az, z) \le \max\{\varphi(d(z, z))\varphi(d(z, Az)), \varphi(d(z, z))\varphi(d(z, z)), \varphi(d(z, Az))\varphi(d(z, z)), \varphi(d(z, Az))\varphi(d(z, z)), \varphi(d(z, z))\varphi(d(z, Az))\},$$

that is $d^2(Az, z) \leqslant 0$, so $d(Az, z) \leqslant 0$. But $d(Az, z) \geqslant 0$.

Therefore $d(Az, z) = 0$ and hence $Az = z$. So $Tz = Az = z$.

Again from condition (b), we have

$$d^2(Ax_{2n+1}, Bz) \le \max\{\varphi(d(Tx_{2n+1}, Tz))\varphi(d(Tx_{2n+1}, Ax_{2n+1})), \varphi(d(Tx_{2n+1}, Tz))\varphi(d(Tz, Bz)), \varphi(d(Tx_{2n+1}, Ax_{2n+1}))\varphi(d(Tz,Bz)), \varphi(d(Tx_{2n+1}, Ax_{2n+1}))\varphi(d(Tz, Bz)), \varphi(d(Tx_{2n+1}, Bz))\varphi(d(Tz, Ax_{2n+1}))\}.$$

Letting as $n \to \infty$, we have

$$d^2(z, Bz) \le \max\{\varphi(d(z, z))\varphi(d(z, z)), \varphi(d(z, z))\varphi(d(z, Bz)), \varphi(d(z, z))\varphi(d(z, Bz)), \varphi(d(z, z))\varphi(d(z, Bz)), \varphi(d(z, Bz))\varphi(d(z, z))\},$$

implies that $d(z, Bz) = 0$. Hence $z = Bz$.

Thus $z = Fz = Ez = Tz$, showing that z is a common fixed point of A, B and T. Similarly we can prove that z is a common fixed point of A, B and T when the pair (B, T) is compatible of type (R).

Uniqueness:

Let z and w be two common fixed points of A, B and T, so $z = Az = Bz = Tz$ and $w = Aw = Bw = Tw$. From condition (ii), we have

$$\begin{aligned} d^2(z, w) &= d^2(Az, Fw) \\ &\le \max\{\varphi(d(Tz, Tw))\varphi(d(Tz, Az)), \varphi(d(Tz, Tw))\varphi(d(Tw, Az)), \\ &\qquad \varphi(d(Tz, Tw))\varphi(d(Tw, Bw)), \varphi(d(Tz, Az))\varphi(d(Tw, Bw)), \\ &\qquad \varphi(d(Tz, Bw))\varphi(d(Tw, Az))\} \\ &= \max\{\varphi(d(z, w))\varphi(d(z, z)), \varphi(d(z, w))\varphi(d(w, z)), \varphi(d(z, w))\varphi(d(w, w)), \\ &\qquad \varphi(d(z, z))\varphi(d(w, w)), \varphi(d(z, w))\varphi(d(w, z))\} \\ &= \varphi(d(z, w))\varphi(d(w, z)) < d^2(z, w), \end{aligned}$$

implies $d(z, w) < d(w, z)$, a contradiction, hence the proof.

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