The Bulletin of Society for Mathematical Services and Standards
ISSN: 2277-8020, Vol. 9, pp 18-24
doi:10.18052/www.scipress.com/BSMaSS.9.18

Online: 2014-03-03

Common Fixed Point Theorem for Three Self Mappings

Oinam Budhichandra Singh

Department of Mathematics
Thambalmarik College, Oinam,
Manipur, India.

Keywords: Compatible mapping, Contractive modulus function, Occasionally weakly compatible, Fixed point, Common fixed point theorem.

Abstract. The aim of this paper is to study the concept of compatible mappings of type (R) and discussed a common fixed point theorem for three compatible mappings of type (R) satisfying contractive conditions.

1. Introduction

In 1986, Jungck [5] introduced the concept of compatible mappings by generalizing commuting mappings. Pathak, Chang and Cho [6] in 1994 introduced the concept of a new type of compatible mappings called compatible mappings of type (P). Al-Thagafi and Shahazad [3] gave the concept of occasionally weakly compatible mappings in 2008. In 2011, H. Bouhadjera [1] proved some common fixed point theorems for three and four occasionally weakly compatible mappings satisfying different types of contractive conditions.

Now the result of H. Bouhadjera [1] is extended by employing compatible mappings of type (R) instead of occasionally weakly compatible mappings.

2. Preliminaries

Definition 2.1 Self mappings A and B of a metric space (X, d) are said to be weakly commuting pair if, for all x $\in$X

$$d(ABx, BAx) \leq d(Ax, Bx)$$

Definition 2.2 Self mappings A and B of a metric space (X, d) are said to be compatible if, for all $x \in X$

$$\lim_{n\to\infty} d(ABx_n, BAx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X and such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some $t \in X$.

Definition 2.4 Self mappings A and B of a metric space (X, d) are said to be compatible of type (P) if, for all $x \in X$

$$\lim_{n\to\infty} d(AAx_n, BBx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X and such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some $t \in X$.

Definition 2.3 Self mappings A and B of a metric space (X, d) are said to be compatible of type (R) if, for all $x \in X$

$$\lim_{n\to\infty} d(ABx_n, BAx_n) = 0 \text{ and } \lim_{n\to\infty} d(AAx_n, BBx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X and such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some $t \in X$.

Definition 2.5 Self mappings A and B of a metric space (X, d) are said to be weakly compatible if they commute at their coincidence points.

Definition 2.6 Two self mappings A and B of a set X are occasionally weakly compatible if and only if there is a point t in X which is a coincidence point of A and B at which A and B commute.

Definition 2.7 A function $\phi : [0, \infty) \to [0, \infty)$ is said to be a contractive modulus if $\phi(0) = 0$ and $\varphi(t) < t$ for $t > 0$.

Definition 2.8 A real valued function φ defined on $X \subset R$ is said to be upper semi continuous if $\lim_{n\to\infty} \varphi(t_n) \leq \varphi(t)$, for every sequence $\{t_n\}$ in X with $t_n \to t$ as $n \to \infty$.

Pathak, Murthy and Cho [6] prove the following propositions.

Proposition 2.9: Let S and T be mappings from a complete metric space (X, d) into itself. If a pair{S, T} is compatible of type (R) on X and $Sz = Tz$ for $z \in X$, then STz = TSz = SSz = TTz.

Proposition 2.10: Let S and T be mappings from a complete metric space (X, d) into itself. If a pair {S, T} is compatible of type (R) on X and $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$, then we have

(i) $d(TSx_n, Sz) \to 0$ as $n \to \infty$ if S is continuous,
(ii) $d(STx_n, Tz) \to 0$ as $n \to \infty$ if T is continuous and
(iii) $STz=TSz$ and $Sz=Tz$ if S and T are continuous at z.

H. Bouhadjera [1] proved the following results:

Theorem 2.11 Let X be a set with a symmetric d. Let f, g and h be self mapping of (X, d) and φ is acontractive modulus function satisfying

$$d^2(fx, gy) \leq \max\{\varphi(d(hx, hy))\varphi(d(hx, fx)),\ \varphi(d(hx, hy))\varphi(d(hy, fx)),\ \varphi(d(hx, hy))\varphi(d(hy, gy)),$$
$$\varphi(d(hx, fx))\varphi(d(hy, gy)),\ \varphi(d(hx, gy))\varphi(d(hy, fx))\}$$

for all $x, y \in X$,

the pair (f, h) and (g, h) is owc.

Then f, g and h have a unique common fixed point.

Theorem 2.11 Let X be a set endowed with a symmetric d. Suppose f, g, h and k are four self mappings of (X, d) satisfying the conditions:

$$d^2(fx, gy) \leqslant max\{ \phi(d(hx, ky))\, \phi(d(hx, fx)),\ \phi(d(hx, ky))\, \phi(d(ky, gy)),\ \phi(d(hx, fx))\, \phi(d(ky, gy)), \varphi(d(hx, gy))\varphi(d(ky, fx))\},$$

for all x, y $\in$ X, where φ is contractive modulus,

the pair *(f, h)* and *(g,, k)* are owc.

Then *f, g, h* and k have a unique common fixed point.

3 Main Results

Theorem 3.1 Let A, B and T be self mapping of a complete metric space (X, d) and φ is a contractive modulus satisfying:

(a) $A(X) \cup B(X) \subset T(X)$.

(b) $d2(Ax,By) \leqslant max\{ \phi(d(Tx,Ty))\, \phi(d(Tx,Ax)),\ \phi(d(Tx,Ty))\, \phi(d(Ty,Ax)),\ \phi(d(Tx,Ty))\, (d(Ty,By)), \varphi(d(Tx,Ax))\varphi(d(Ty,By)), \varphi(d(Tx,By))\varphi(d(Ty,Ax))\}$

for all x, y $\in X$.

(c) the pair(A, T) or (B, T) is compatible of type (R).

(d) If T is continuous,

then A, B and T have a unique common fixed point.

Proof: Let $x_0 \in X$ be arbitrary. Take a sequence $\{Tx_n\}$, as follows.

$$Tx_{2n+}1 = Ax_{2n},\ Tx_{2n+2} = Bx_{2n+1},\ n = 0, 1, 2,\dots \tag{1}$$

From condition (b) we get

$$\begin{aligned}
d^2(Tx_{2n+1}, Tx_{2n+2}) &= d^2(Ax_{2n}, Bx_{2n+1}) \\
&\le \max\{\varphi(d(Tx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n}, Ax_{2n})), \varphi(d(Tx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n+1}, Ax_{2n})), \\
&\qquad \varphi(d(Tx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n+1}, Bx_{2n+1})), \varphi(d(Tx_{2n}, Ax_{2n}))\varphi(d(Tx_{2n+1}, Bx_{2n+1})), \\
&\qquad \varphi(d(Tx_{2n}, Bx_{2n+1}))\varphi(d(Tx_{2n+1}, Ax_{2n}))\}, \\
&= \max\{\varphi(d(Tx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n}, Tx_{2n+1})), \varphi(d(Tx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n+1}, Tx_{2n+1})), \\
&\qquad \varphi(d(Tx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n+1}, Tx_{2n+2})), \varphi(d(Tx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n+1}, Tx_{2n+2})), \\
&\qquad \varphi(d(Tx_{2n}, Tx_{2n+2}))\varphi(d(Tx_{2n+1}, Tx_{2n+1}))\}, \\
&= \max\{\varphi(d(Tx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n}, Tx_{2n+1})), \varphi(d(Tx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n+1}, Tx_{2n+1}))\}.
\end{aligned}$$

This implies

$$d^2(Tx_{2n+1}, Tx_{2n+2}) \le \varphi(d(Tx_{2n}, Tx_{2n+1})) \max\{\varphi(d(Tx_{2n}, Tx_{2n+1})), \varphi(d(Tx_{2n+1}, Tx_{2n+2}))\}$$
$$\le d(Tx_{2n}, Tx_{2n+1}) \max\{d(Tx_{2n}, Tx_{2n+1}), d(Tx_{2n+1}, Tx_{2n+2})\}.$$

Since φ is contractive modulus. We have,

$$d(Tx_{2n+1}, Tx_{2n+2}) \le \varphi(d(Tx_{2n}, Tx_{2n+1})) \le d(Tx_{2n}, Tx_{2n+1}). \tag{2}$$

The sequence $\{d(Tx_{2n}, Tx_{2n+1})\}$ is decreasing. If $\{d(Tx_{2n}, Tx_{2n+1})\} \to \alpha$ then from (2) we have $\alpha \leqslant \phi(\alpha) \leqslant \alpha$ and so we must have $\alpha = 0$, hence

$$\lim_{n\to\infty} d(Tx_{2n}, Tx_{2n+1}) = 0. \tag{3}$$

We shall show that $\{Tx2n\}$ is a Cauchy sequence. If it is not so, there exist an $\in > 0$ and a sequence of integers $\{mk\}$, $\{nk\}$ with $mk > nk \ge k$, such that

$$d(Tx_{2m}, Tx_{2n}) \ge \in \tag{4}$$

$k = 1, 2, 3, \ldots\ldots$. If m_k is the smallest integer exceeding n_k for which (4) holds, then from wellordering principle, we have

$$d(Tx2m-1, Tx2n) < \in. \tag{5}$$

Since

$$d(Tx_{2m-1}, Tx_{2n+1}) \le d(Tx_{2m-1}, Tx_{2n}) + d(Tx_{2n}, Tx2n+1),$$

and so letting $k\to\infty$, we see that

$$d(Tx_{2m-1}, Tx_{2n+1}) \le \in. \tag{6}$$

Now

$$d(Tx_{2m}, Tx_{2n}) \le d(Tx_{2m}, Tx_{2n+1}) + d(Tx_{2n+1}, Tx_{2n}). \tag{7}$$

But we have

$$\begin{aligned} d^2(Tx_{2m}, Tx_{2n+1}) = d^2(Ax_{2n}, Bx_{2m-1}) & \\ \le \max\{&\varphi(d(Tx_{2n}, Tx_{2m-1}))\varphi(d(Tx_{2n}, Ax_{2n})),\\ &\varphi(d(Tx_{2n}, Tx_{2m-1}))\varphi(d(Tx_{2m-1}, Ax_{2n})),\\ &\varphi(d(Tx_{2n}, Tx_{2m-1}))\varphi(d(Tx_{2m-1}, Bx_{2m-1})),\\ &\varphi(d(Tx_{2n}, Ax_{2n}))\varphi(d(Tx_{2m-1}, Bx_{2m-1})),\\ &\varphi(d(Tx_{2n}, Bx_{2m-1}))\varphi(d(Tx_{2m-1}, Ax_{2n}))\},\\ = \max\{&\varphi(d(Tx_{2n}, Tx_{2m-1}))\varphi(d(Tx_{2n}, Tx_{2n+1})),\\ &\varphi(d(Tx_{2n}, Tx_{2m-1}))\varphi(d(Tx_{2m-1}, Tx_{2n+1})),\\ &\varphi(d(Tx_{2n}, Tx_{2m-1}))\varphi(d(Tx_{2m-1}, Tx_{2m})),\\ &\varphi(d(Tx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2m-1}, Tx_{2m})),\\ &\varphi(d(Tx_{2n}, Tx_{2m}))\varphi(d(Tx_{2m-1}, Tx_{2n+1}))\}. \end{aligned}$$

Using (3), (5), (6), (8) and letting $k\to\infty$, we obtain

$$\lim_k d^2(Tx_{2m}, Tx_{2n+1}) \le \max\{\varphi(\in)\varphi(\in), \varphi(\in) \lim_k d(Tx_{2m}, Tx_{2n})\}$$

$$\le \varphi(\in) \max\{\varphi(\in), \lim_k d(Tx_{2m}, Tx_{2n+1})\},$$

implies

$$\lim_k d(Tx_{2m}, Tx_{2n+1}) \le \varphi(\in) < \in.$$

Then by (7)

$$\lim_k d(Tx_{2m}, Tx_{2n}) \le \varphi(\in) + 0 < \in,$$

a contradiction. Thus $\{Tx_n\}$ is a Cauchy sequence. Since X is complete, there exist a point $z \in X$ such that $Tx_n \to z$. It follows that from (1) that the sequences $\{Ax_{2n}\}$and $\{Bx_{2n+1}\}$ also converge to z.

Since *T* is continuous such that

$$TTx_{2n} \to Tz,\ TAx_{2n} \to Tz \text{ as } n\to\infty,$$

Since the pair (A, T) is compatible of type (R), we have

$$AAx_{2n} \to Tz \text{ as } n\to\infty$$

Then from condition (ii), we have

$$\begin{aligned} d^2(AAx_{2n}, Bx_{2n+1}) \le \max\{&\varphi(d(TAx_{2n}, Tx_{2n+1}))\varphi(d(TAx_{2n}, AAx_{2n+1})),\\ &\varphi(d(TAx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n+1}, AAx_{2n})),\\ &\varphi(d(TAx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n+1}, Bx_{2n+1})),\\ &\varphi(d(TAx_{2n}, AAx_{2n}))\varphi(d(Tx_{2n+1}, Bx_{2n+1})),\\ &\varphi(d(TAx_{2n}, Bx_{2n+1}))\varphi(d(Tx_{2n+1}, AAx_{2n}))\}.\end{aligned}$$

Letting $n\to\infty$ and we have

$$\begin{aligned} d^2(Tz, z) \le \max\{&\varphi(d(Tz, z))\varphi(d(Tz, Tz)), \varphi(d(Tz, Tz))\varphi(d(Tz, z)), \varphi(d(Tz, z))\varphi(d(z, z)),\\ &\varphi(d(Tz,Tz))\varphi(d(z, z)), \varphi(d(Tz, z))\varphi(d(z, Tz))\}\\ = \varphi(&d(Tz, z))\varphi(d(Tz, z)),\end{aligned}$$

and it implies that

$$d(Tz, z) \le \varphi(d(Tz, z)) \le d(Tz, z)$$

i.e. $\varphi(d(Tz, z)) = d(Tz, z)$. Hence $Tz = z$.

Again from (ii), we have

$$\begin{aligned} d^2(Az, Bx_{2n+1}) \le \max\{&\varphi(d(Tz, Tx_{2n+1}))\varphi(d(Tz, Az)), \varphi(d(Tz, Tx_{2n+1}))\varphi(d(Tx_{2n+1}, Bx_{2n+1})),\\ &\varphi(d(Tz, Az))\varphi(d(Tx_{2n+1}, Bx_{2n+1})), \varphi(d(Tz, Ez))\varphi(d(Tx_{2n+1}, Bx_{2n+1})),\\ &\varphi(d(Tz, Bx_{2n+1}))\varphi(d(Tx_{2n+1}, Az))\}.\end{aligned}$$

Letting as n→∞ and using Tz = z,

$$d^2(Az, z) \leq \max\{\varphi(d(z, z))\varphi(d(z, Az)), \varphi(d(z, z))\varphi(d(z, z)), \varphi(d(z, Az))\varphi(d(z, z)),$$
$$\varphi(d(z, Az))\varphi(d(z, z)), \varphi(d(z, z))\varphi(d(z, Az))\},$$

that is $d^2(Az, z) \leqslant 0$, so $d(Az, z) \leqslant 0$. But $d(Az, z) \geqslant 0$.

Therefore d(Az, z) = 0 and hence Az = z. So Tz = Az = z.

Again from condition (b), we have

$$d^2(Ax_{2n+1}, Bz) \leq \max\{\varphi(d(Tx_{2n+1}, Tz))\varphi(d(Tx_{2n+1}, Ax_{2n+1})), \varphi(d(Tx_{2n+1}, Tz))\varphi(d(Tz, Bz)),$$
$$\varphi(d(Tx_{2n+1}, Ax_{2n+1}))\varphi(d(Tz,Bz)), \varphi(d(Tx_{2n+1}, Ax_{2n+1}))\varphi(d(Tz, Bz)),$$
$$\varphi(d(Tx_{2n+1}, Bz))\varphi(d(Tz, Ax_{2n+1}))\}.$$

Letting as n→∞, we have

$$d^2(z, Bz) \leq \max\{\varphi(d(z, z))\varphi(d(z, z)), \varphi(d(z, z))\varphi(d(z, Bz)), \varphi(d(z, z))\varphi(d(z, Bz)),$$
$$\varphi(d(z, z))\varphi(d(z, Bz)), \varphi(d(z, Bz))\varphi(d(z, z))\},$$

implies that $d(z, Bz) = 0$. Hence $z = Bz$.

Thus $z = Fz = Ez = Tz$, showing that z is a common fixed point of A, B and T. Similarly we can prove that z is a common fixed point of A, B and T when the pair (B, T) is compatible of type (R).

Uniqueness:

Let z and w be two common fixed points of A, B and T, so $z = Az = Bz = Tz$ and $w = Aw = Bw = Tw$. From condition (ii), we have

$$\begin{aligned}
d^2(z, w) &= d^2(Az, Fw) \\
&\leq \max\{\varphi(d(Tz, Tw))\varphi(d(Tz, Az)), \varphi(d(Tz, Tw))\varphi(d(Tw, Az)), \\
&\qquad \varphi(d(Tz, Tw))\varphi(d(Tw, Bw)), \varphi(d(Tz, Az))\varphi(d(Tw, Bw)), \\
&\qquad \varphi(d(Tz, Bw))\varphi(d(Tw, Az))\} \\
&= \max\{\varphi(d(z, w))\varphi(d(z, z)), \varphi(d(z, w))\varphi(d(w, z)), \varphi(d(z, w))\varphi(d(w, w)), \\
&\qquad \varphi(d(z, z))\varphi(d(w, w)), \varphi(d(z, w))\varphi(d(w, z))\} \\
&= \varphi(d(z, w))\varphi(d(w, z)) < d^2(z, w),
\end{aligned}$$

implies $d(z, w) < d(w, z)$, a contradiction, hence the proof.

Acknowledgement: Author is thankful to Yumnam Rohen Singh, Assistant Professor, NIT Manipur, Imphal for his valuable suggestion towards the improvement of this paper.

REFERENCES

[1] H. Bouhadjera, On Common Fixed Point Theorems for Three and Four Self Mappings Satisfying Contractive Conditions, Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica 49, 1 (2010), 25 – 31.

[2] Aage, C.T., Salunke, J.N., A note on common fixed point theorems, Int. J. Math. Anal. 2, 28 (2008), 1369 – 1380.

[3] Al – Thagafi, M.A., Shahzad, N., Generalised I – nonexpansive selfmaps and invariant approximations, Acta Math. Sin. (Engl. Ser.) 24, 5 (2008), 867 – 876.

[4] Sessa S., On weakly commutativity condition in a fixed point consideration, Publ. Int. Math. 32(46)(1986), pp. 149-153.

[5] G. Jungck, Compatible mappings and common fixed points, Internat. J. Math. Math. Sci. 9, (1986), 771-779.

[6] G. Jungck, P. P. Murthy and Y. J. Cho, Compatible mappings of type (A) and common fixed points, Math. Japonica, 38, (1993), 381-386.

[7] H.K. Pathak, S.S. Chang and Y.J. Cho, Fixed point theorems for compatibles of type (P), Indian J. Math., 36 (2) (1994), 151 – 156.

[8] M. Rangamma, V. Sriniwas and R. Uma Maheswar Rao, A Common Fixed Point Theorem for Three Selfmaps, The Mathematics Education, Vol. XL, No. 3, Sept. (2006), pp 180-185.