Common Fixed Point Theorem for Three Self Mappings

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Abstract. The aim of this paper is to study the concept of compatible mappings of type (R) and discussed a common fixed point theorem for three compatible mappings of type (R) satisfying contractive conditions.

1. Introduction

In 1986, Jungck [5] introduced the concept of compatible mappings by generalizing commuting mappings. Pathak, Chang and Cho [6] in 1994 introduced the concept of a new type of compatible mappings called compatible mappings of type (P). Al-Thagafi and Shahzad [3] gave the concept of occasionally weakly compatible mappings in 2008. In 2011, H. Bouhadjera [1] proved some common fixed point theorems for three and four occasionally weakly compatible mappings satisfying different types of contractive conditions.

Now the result of H. Bouhadjera [1] is extended by employing compatible mappings of type (R) instead of occasionally weakly compatible mappings.

2. Preliminaries

Definition 2.1 Self mappings A and B of a metric space (X, d) are said to be weakly commuting pair if, for all $x \in X$

$$d(ABx, BAx) \leq d(Ax, Bx)$$

Definition 2.2 Self mappings A and B of a metric space (X, d) are said to be compatible if, for all $x \in X$

$$\lim_{n \rightarrow \infty} d(ABx_n, BAx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X and such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Bx_n = t$ for some $t \in X$.

Definition 2.4 Self mappings A and B of a metric space (X, d) are said to be compatible of type (P) if, for all $x \in X$

$$\lim_{n \rightarrow \infty} d(AAx_n, BBx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X and such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Bx_n = t$ for some $t \in X$.

Definition 2.3 Self mappings A and B of a metric space (X, d) are said to be compatible of type (R) if, for all $x \in X$

$$\lim_{n \rightarrow \infty} d(ABx_n, BAx_n) = 0 \text{ and } \lim_{n \rightarrow \infty} d(AAx_n, BBx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X and such that $\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Bx_n = t$ for some $t \in X$.

Definition 2.5 Self mappings A and B of a metric space (X, d) are said to be weakly compatible if they commute at their coincidence points.

Definition 2.6 Two self mappings A and B of a set X are occasionally weakly compatible if and only if there is a point t in X which is a coincidence point of A and B at which A and B commute.

Definition 2.7 A function $\phi : [0, \infty) \rightarrow [0, \infty)$ is said to be a contractive modulus if $\phi(0) = 0$ and $\phi(t) < t$ for $t > 0$.

Definition 2.8 A real valued function ϕ defined on $X \subset \mathbb{R}$ is said to be upper semi continuous if $\lim_{n \rightarrow \infty} \phi(t_n) \leq \phi(t)$, for every sequence $\{t_n\}$ in X with $t_n \rightarrow t$ as $n \rightarrow \infty$.

Pathak, Murthy and Cho [6] prove the following propositions.

Proposition 2.9: Let S and T be mappings from a complete metric space (X, d) into itself. If a pair $\{S, T\}$ is compatible of type (R) on X and $Sz = Tz$ for $z \in X$, then $STz = TSz = SSz = TTz$.

Proposition 2.10: Let S and T be mappings from a complete metric space (X, d) into itself. If a pair $\{S, T\}$ is compatible of type (R) on X and $\lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Tx_n = z$ for some $z \in X$, then we have

- (i) $d(TSx_n, Sz) \rightarrow 0$ as $n \rightarrow \infty$ if S is continuous,
- (ii) $d(STx_n, Tz) \rightarrow 0$ as $n \rightarrow \infty$ if T is continuous and
- (iii) $STz = TSz$ and $Sz = Tz$ if S and T are continuous at z .

H. Bouhadjera [1] proved the following results:

Theorem 2.11 Let X be a set with a symmetric d . Let f, g and h be self mapping of (X, d) and ϕ is a contractive modulus function satisfying

$$d^2(fx, gy) \leq \max \{ \phi(d(hx, hy))\phi(d(hx, fx)), \phi(d(hx, hy))\phi(d(hy, fx)), \phi(d(hx, hy))\phi(d(hy, gy)), \\ \phi(d(hx, fx))\phi(d(hy, gy)), \phi(d(hx, gy))\phi(d(hy, fx)) \}$$

for all $x, y \in X$,

the pair (f, h) and (g, h) is owc.

Then f, g and h have a unique common fixed point.

Theorem 2.11 Let X be a set endowed with a symmetric d . Suppose f, g, h and k are four self mappings of (X, d) satisfying the conditions:

$$d^2(fx, gy) \leq \max\{ \phi(d(hx, ky)) \phi(d(hx, fx)), \phi(d(hx, ky)) \phi(d(ky, gy)), \phi(d(hx, fx)) \phi(d(ky, gy)), \phi(d(hx, gy))\phi(d(ky, fx)) \},$$

for all $x, y \in X$, where ϕ is contractive modulus,

the pair (f, h) and (g, k) are owc.

Then f, g, h and k have a unique common fixed point.

3 Main Results

Theorem 3.1 Let A, B and T be self mapping of a complete metric space (X, d) and ϕ is a contractive modulus satisfying:

$$(a) A(X) \cup B(X) \subset T(X).$$

$$(b) d^2(Ax, By) \leq \max\{ \phi(d(Tx, Ty)) \phi(d(Tx, Ax)), \phi(d(Tx, Ty)) \phi(d(Ty, Ax)), \phi(d(Tx, Ty)) \phi(d(Ty, By)), \phi(d(Tx, Ax))\phi(d(Ty, By)), \phi(d(Tx, By))\phi(d(Ty, Ax)) \}$$

for all $x, y \in X$.

$$(c) \text{ the pair } (A, T) \text{ or } (B, T) \text{ is compatible of type } (R).$$

$$(d) \text{ If } T \text{ is continuous,}$$

then A, B and T have a unique common fixed point.

Proof: Let $x_0 \in X$ be arbitrary. Take a sequence $\{Tx_n\}$, as follows.

$$Tx_{2n+1} = Ax_{2n}, Tx_{2n+2} = Bx_{2n+1}, n = 0, 1, 2, \dots \quad (1)$$

From condition (b) we get

$$\begin{aligned} d^2(Tx_{2n+1}, Tx_{2n+2}) &= d^2(Ax_{2n}, Bx_{2n+1}) \\ &\leq \max\{ \phi(d(Tx_{2n}, Tx_{2n+1}))\phi(d(Tx_{2n}, Ax_{2n})), \phi(d(Tx_{2n}, Tx_{2n+1}))\phi(d(Tx_{2n+1}, Ax_{2n})), \\ &\quad \phi(d(Tx_{2n}, Tx_{2n+1}))\phi(d(Tx_{2n+1}, Bx_{2n+1})), \phi(d(Tx_{2n}, Ax_{2n}))\phi(d(Tx_{2n+1}, Bx_{2n+1})), \\ &\quad \phi(d(Tx_{2n}, Bx_{2n+1}))\phi(d(Tx_{2n+1}, Ax_{2n})) \}, \\ &= \max\{ \phi(d(Tx_{2n}, Tx_{2n+1}))\phi(d(Tx_{2n}, Tx_{2n+1})), \phi(d(Tx_{2n}, Tx_{2n+1}))\phi(d(Tx_{2n+1}, Tx_{2n+1})), \\ &\quad \phi(d(Tx_{2n}, Tx_{2n+1}))\phi(d(Tx_{2n+1}, Tx_{2n+2})), \phi(d(Tx_{2n}, Tx_{2n+1}))\phi(d(Tx_{2n+1}, Tx_{2n+2})), \\ &\quad \phi(d(Tx_{2n}, Tx_{2n+2}))\phi(d(Tx_{2n+1}, Tx_{2n+1})) \}, \\ &= \max\{ \phi(d(Tx_{2n}, Tx_{2n+1}))\phi(d(Tx_{2n}, Tx_{2n+1})), \phi(d(Tx_{2n}, Tx_{2n+1}))\phi(d(Tx_{2n+1}, Tx_{2n+1})) \}. \end{aligned}$$

This implies

$$\begin{aligned} d^2(Tx_{2n+1}, Tx_{2n+2}) &\leq \varphi(d(Tx_{2n}, Tx_{2n+1})) \max\{\varphi(d(Tx_{2n}, Tx_{2n+1})), \varphi(d(Tx_{2n+1}, Tx_{2n+2}))\} \\ &\leq d(Tx_{2n}, Tx_{2n+1}) \max\{d(Tx_{2n}, Tx_{2n+1}), d(Tx_{2n+1}, Tx_{2n+2})\}. \end{aligned}$$

Since φ is contractive modulus. We have,

$$d(Tx_{2n+1}, Tx_{2n+2}) \leq \varphi(d(Tx_{2n}, Tx_{2n+1})) \leq d(Tx_{2n}, Tx_{2n+1}). \quad (2)$$

The sequence $\{d(Tx_{2n}, Tx_{2n+1})\}$ is decreasing. If $\{d(Tx_{2n}, Tx_{2n+1})\} \rightarrow \alpha$ then from (2) we have $\alpha \leq \varphi(\alpha) \leq \alpha$ and so we must have $\alpha = 0$, hence

$$\lim_{n \rightarrow \infty} d(Tx_{2n}, Tx_{2n+1}) = 0. \quad (3)$$

We shall show that $\{Tx_{2n}\}$ is a Cauchy sequence. If it is not so, there exist an $\epsilon > 0$ and a sequence of integers $\{mk\}$, $\{nk\}$ with $mk > nk \geq k$, such that

$$d(Tx_{2m}, Tx_{2n}) \geq \epsilon \quad (4)$$

$k = 1, 2, 3, \dots$. If m_k is the smallest integer exceeding n_k for which (4) holds, then from wellordering principle, we have

$$d(Tx_{2m-1}, Tx_{2n}) < \epsilon. \quad (5)$$

Since

$$d(Tx_{2m-1}, Tx_{2n+1}) \leq d(Tx_{2m-1}, Tx_{2n}) + d(Tx_{2n}, Tx_{2n+1}),$$

and so letting $k \rightarrow \infty$, we see that

$$d(Tx_{2m-1}, Tx_{2n+1}) \leq \epsilon. \quad (6)$$

Now

$$d(Tx_{2m}, Tx_{2n}) \leq d(Tx_{2m}, Tx_{2n+1}) + d(Tx_{2n+1}, Tx_{2n}). \quad (7)$$

But we have

$$\begin{aligned} d^2(Tx_{2m}, Tx_{2n+1}) &= d^2(Ax_{2n}, Bx_{2m-1}) \\ &\leq \max\{\varphi(d(Tx_{2n}, Tx_{2m-1}))\varphi(d(Tx_{2n}, Ax_{2n})), \\ &\quad \varphi(d(Tx_{2n}, Tx_{2m-1}))\varphi(d(Tx_{2m-1}, Ax_{2n})), \\ &\quad \varphi(d(Tx_{2n}, Tx_{2m-1}))\varphi(d(Tx_{2m-1}, Bx_{2m-1})), \\ &\quad \varphi(d(Tx_{2n}, Ax_{2n}))\varphi(d(Tx_{2m-1}, Bx_{2m-1})), \\ &\quad \varphi(d(Tx_{2n}, Bx_{2m-1}))\varphi(d(Tx_{2m-1}, Ax_{2n}))\}, \\ &= \max\{\varphi(d(Tx_{2n}, Tx_{2m-1}))\varphi(d(Tx_{2n}, Tx_{2n+1})), \\ &\quad \varphi(d(Tx_{2n}, Tx_{2m-1}))\varphi(d(Tx_{2m-1}, Tx_{2n+1})), \\ &\quad \varphi(d(Tx_{2n}, Tx_{2m-1}))\varphi(d(Tx_{2m-1}, Tx_{2m})), \\ &\quad \varphi(d(Tx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2m-1}, Tx_{2m})), \\ &\quad \varphi(d(Tx_{2n}, Tx_{2m}))\varphi(d(Tx_{2m-1}, Tx_{2n+1}))\}. \end{aligned}$$

Using (3), (5), (6), (8) and letting $k \rightarrow \infty$, we obtain

$$\begin{aligned} \lim_k d^2(Tx_{2m}, Tx_{2n+1}) &\leq \max\{\varphi(\epsilon)\varphi(\epsilon), \varphi(\epsilon) \lim_k d(Tx_{2m}, Tx_{2n})\} \\ &\leq \varphi(\epsilon) \max\{\varphi(\epsilon), \lim_k d(Tx_{2m}, Tx_{2n+1})\}, \end{aligned}$$

implies

$$\lim_k d(Tx_{2m}, Tx_{2n+1}) \leq \varphi(\epsilon) < \epsilon.$$

Then by (7)

$$\lim_k d(Tx_{2m}, Tx_{2n}) \leq \varphi(\epsilon) + 0 < \epsilon,$$

a contradiction. Thus $\{Tx_n\}$ is a Cauchy sequence. Since X is complete, there exist a point $z \in X$ such that $Tx_n \rightarrow z$. It follows that from (1) that the sequences $\{Ax_{2n}\}$ and $\{Bx_{2n+1}\}$ also converge to z .

Since T is continuous such that

$$TTx_{2n} \rightarrow Tz, TAx_{2n} \rightarrow Tz \text{ as } n \rightarrow \infty,$$

Since the pair (A, T) is compatible of type (R), we have

$$AAx_{2n} \rightarrow Tz \text{ as } n \rightarrow \infty$$

Then from condition (ii), we have

$$\begin{aligned} d^2(AAx_{2n}, Bx_{2n+1}) &\leq \max\{\varphi(d(TAx_{2n}, Tx_{2n+1}))\varphi(d(TAx_{2n}, AAx_{2n+1})), \\ &\quad \varphi(d(TAx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n+1}, AAx_{2n})), \\ &\quad \varphi(d(TAx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n+1}, Bx_{2n+1})), \\ &\quad \varphi(d(TAx_{2n}, AAx_{2n}))\varphi(d(Tx_{2n+1}, Bx_{2n+1})), \\ &\quad \varphi(d(TAx_{2n}, Bx_{2n+1}))\varphi(d(Tx_{2n+1}, AAx_{2n}))\}. \end{aligned}$$

Letting $n \rightarrow \infty$ and we have

$$\begin{aligned} d^2(Tz, z) &\leq \max\{\varphi(d(Tz, z))\varphi(d(Tz, Tz)), \varphi(d(Tz, Tz))\varphi(d(Tz, z)), \varphi(d(Tz, z))\varphi(d(z, z)), \\ &\quad \varphi(d(Tz, Tz))\varphi(d(z, z)), \varphi(d(Tz, z))\varphi(d(z, Tz))\} \\ &= \varphi(d(Tz, z))\varphi(d(Tz, z)), \end{aligned}$$

and it implies that

$$d(Tz, z) \leq \varphi(d(Tz, z)) \leq d(Tz, z)$$

i.e. $\varphi(d(Tz, z)) = d(Tz, z)$. Hence $Tz = z$.

Again from (ii), we have

$$\begin{aligned} d^2(Az, Bx_{2n+1}) &\leq \max\{\varphi(d(Tz, Tx_{2n+1}))\varphi(d(Tz, Az)), \varphi(d(Tz, Tx_{2n+1}))\varphi(d(Tx_{2n+1}, Bx_{2n+1})), \\ &\quad \varphi(d(Tz, Az))\varphi(d(Tx_{2n+1}, Bx_{2n+1})), \varphi(d(Tz, Ez))\varphi(d(Tx_{2n+1}, Bx_{2n+1})), \\ &\quad \varphi(d(Tz, Bx_{2n+1}))\varphi(d(Tx_{2n+1}, Az))\}. \end{aligned}$$

Letting as $n \rightarrow \infty$ and using $Tz = z$,

$$d^2(Az, z) \leq \max \{ \varphi(d(z, z))\varphi(d(z, Az)), \varphi(d(z, z))\varphi(d(z, z)), \varphi(d(z, Az))\varphi(d(z, z)), \\ \varphi(d(z, Az))\varphi(d(z, z)), \varphi(d(z, z))\varphi(d(z, Az)) \},$$

that is $d^2(Az, z) \leq 0$, so $d(Az, z) \leq 0$. But $d(Az, z) \geq 0$.

Therefore $d(Az, z) = 0$ and hence $Az = z$. So $Tz = Az = z$.

Again from condition (b), we have

$$d^2(Ax_{2n+1}, Bz) \leq \max \{ \varphi(d(Tx_{2n+1}, Tz))\varphi(d(Tx_{2n+1}, Ax_{2n+1})), \varphi(d(Tx_{2n+1}, Tz))\varphi(d(Tz, Bz)), \\ \varphi(d(Tx_{2n+1}, Ax_{2n+1}))\varphi(d(Tz, Bz)), \varphi(d(Tx_{2n+1}, Ax_{2n+1}))\varphi(d(Tz, Bz)), \\ \varphi(d(Tx_{2n+1}, Bz))\varphi(d(Tz, Ax_{2n+1})) \}.$$

Letting as $n \rightarrow \infty$, we have

$$d^2(z, Bz) \leq \max \{ \varphi(d(z, z))\varphi(d(z, z)), \varphi(d(z, z))\varphi(d(z, Bz)), \varphi(d(z, z))\varphi(d(z, Bz)), \\ \varphi(d(z, z))\varphi(d(z, Bz)), \varphi(d(z, Bz))\varphi(d(z, z)) \},$$

implies that $d(z, Bz) = 0$. Hence $z = Bz$.

Thus $z = Fz = Ez = Tz$, showing that z is a common fixed point of A , B and T . Similarly we can prove that z is a common fixed point of A , B and T when the pair (B, T) is compatible of type (R) .

Uniqueness:

Let z and w be two common fixed points of A , B and T , so $z = Az = Bz = Tz$ and $w = Aw = Bw = Tw$. From condition (ii), we have

$$d^2(z, w) = d^2(Az, Fw) \\ \leq \max \{ \varphi(d(Tz, Tw))\varphi(d(Tz, Az)), \varphi(d(Tz, Tw))\varphi(d(Tw, Az)), \\ \varphi(d(Tz, Tw))\varphi(d(Tw, Bw)), \varphi(d(Tz, Az))\varphi(d(Tw, Bw)), \\ \varphi(d(Tz, Bw))\varphi(d(Tw, Az)) \} \\ = \max \{ \varphi(d(z, w))\varphi(d(z, z)), \varphi(d(z, w))\varphi(d(w, z)), \varphi(d(z, w))\varphi(d(w, w)), \\ \varphi(d(z, z))\varphi(d(w, w)), \varphi(d(z, w))\varphi(d(w, z)) \} \\ = \varphi(d(z, w))\varphi(d(w, z)) < d^2(z, w),$$

implies $d(z, w) < d(w, z)$, a contradiction, hence the proof.

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