

# EXISTENCE AND UNIQUENESS OF FIXED POINT THEOREMS IN PARTIALLY ORDERED METRIC SPACES

A.MURALIRAJ R.JAHIR HUSSAIN

P.G. and Research Department of Mathematics, Urumu Dhanalakshmi College, Kattur, Tiruchirappalli-620 019, TamilNadu, India.

P.G. and Research Department of Mathematics, Jamal Mohamed College(Autonomous), Tiruchirappalli-620 020, Tamil Nadu, India.

## Abstract

The purpose of this paper is to present a fixed point theorem using a contractive condition of rational type and involves combining the ideas of an iterative technique in the contraction mapping principle with those in the monotone technique in the context of partially ordered metric spaces.

## 1 Introduction

In [ 3, 6, 16 ], it is proved that some fixed theorems for a mixed monotone mapping in a metric space endowed with a partial order and the authors apply their results to problems of existence and uniqueness of solutions for some boundary value problems.

In the context of partially ordered metric spaces, the usual contractive condition is weakened but at the expense that the operator is monotone. The main idea involves combining the ideas of an iterative technique in the contraction mapping principle with those in the monotone iterative technique .

## 2. Main Result

**Definition 2.1:** Let  $( X, \leq )$  be a partially ordered set and  $T : X \rightarrow X$ . We say that  $T$  is a nondecreasing mapping if for  $x, y \in X, x \leq y \Rightarrow Tx \leq Ty$ .

**Theorem 2.2:** Let  $( X, \leq )$  be a partially ordered set and suppose that there exists a metric  $d$  in  $X$  such that  $X, d$  is a complete metric space. Let  $T : X \rightarrow X$  be a continuous and nondecreasing mapping such that

$$d(Tx, Ty) \leq \alpha \left( \frac{d(x, Tx) + d(y, Ty)}{d(x, y) + d(x, Tx)} \right) d(x, Tx) + \beta d(x, y), \text{ for } x, y \in X, x \geq y, x \neq y, \tag{1}$$

with  $\alpha, \beta < 1$  if there exists  $x_0 \in X$  with  $x_0 \leq Tx_0$ , then  $T$  has a fixed point.

**Proof:** If  $Tx_0 = x_0$ , then the proof is finished. Suppose that  $x_0 < Tx_0$ . Since  $T$  is a nondecreasing mapping, we obtain by induction that

$$x_0 < Tx_0 \leq T^2x_0 \leq \dots \leq T^nx_0 \leq T^{n+1}x_0 \leq \dots \tag{2}$$

Put  $x_{n+1} = Tx_n$ . If there exists  $n \geq 1$  such that  $x_{n+1} = x_n$ , then from  $x_{n+1} = Tx_n = x_n$ ,  $x_n$  is a fixed point and the proof is finished.

Suppose that  $x_{n+1} \neq x_n$  for  $n \geq 1$

Then, from (1) and as the elements  $x_n$  and  $x_{n-1}$  are comparable, we get, for  $n \geq 1$ ,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq \alpha \left( \frac{d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1})}{d(x_n, x_{n-1}) + d(x_n, Tx_n)} \right) d(x_n, Tx_n) + \beta d(x_n, x_{n-1}) \\ &= \alpha \left( \frac{d(x_n, x_{n+1}) + d(x_{n-1}, x_n)}{d(x_n, x_{n-1}) + d(x_n, x_{n+1})} \right) d(x_n, x_{n+1}) + \beta d(x_n, x_{n-1}) \\ &= \alpha d(x_n, x_{n+1}) + \beta d(x_n, x_{n-1}) \end{aligned} \tag{3}$$

The last inequality gives us

$$d(x_{n+1}, x_n) \leq \left( \frac{\beta}{1-\alpha} \right) d(x_n, x_{n-1}) \tag{4}$$

Again, using induction

$$d(x_{n+1}, x_n) \leq \left( \frac{\beta}{1-\alpha} \right)^n d(x_1, x_0) \tag{5}$$

$$k = \frac{\beta}{1-\alpha} < 1.$$

Put

Moreover, by the triangular inequality, we have, for  $m \geq n$ ,

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \dots + d(x_{n+1}, x_n) \\ &\leq (k^{m-1} + k^{m-2} + \dots + k^n) d(x_1, x_0) \leq \left( \frac{k^n}{1-k} \right) d(x_1, x_0), \end{aligned} \tag{6}$$

and this proves that  $d(x_m, x_n) \rightarrow 0$  as  $m, n \rightarrow \infty$ .

So,  $\{x_n\}$  is a Cauchy sequence and since  $X$  is a complete metric space, there exists  $z \in X$  such that  $\lim_{n \rightarrow \infty} x_n = z$ .

Further, the continuity of  $T$  implies

$$z = T \left( \lim_{n \rightarrow \infty} x_n \right) = \lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} x_{n+1} = z, \tag{7}$$

And this proves that  $z$  is a fixed point.

Theorem 2.3: Let  $(X, \leq)$  be a partially ordered set and suppose that there exists a metric  $d$  in  $X$  such that  $(X, d)$  is a complete metric space. Assume that the following hypothesis in  $X$  satisfies, if  $\{x_n\}$  is a nondecreasing sequence in  $X$  such that

$x_n \rightarrow x$ , then  $x = \sup \{x_n\}$ . Let  $T : X \rightarrow X$  be a nondecreasing mapping such that

$$d(Tx, Ty) \leq \alpha \left( \frac{d(x, Tx) + d(y, Ty)}{d(x, y) + d(x, Tx)} \right) d(x, Tx) + \beta d(x, y), \text{ for } x, y \in X, x \geq y, x \neq y, \tag{8}$$

with  $\alpha + \beta < 1$ . If there exists  $x_0 \in X$  with  $x_0 \leq Tx_0$ , then  $T$  has a fixed point.

**Proof:** Following the proof of Theorem 2.2, We only have to check that  $Tz = z$ .

As  $\{x_n\}$  nondecreasing sequence in  $X$  and  $x_n \rightarrow z$ , then  $z = \sup \{x_n\}$ .

Particularly,  $x_n \leq z$  for all  $n \in \mathbb{N}$ .

Since  $T$  is a nondecreasing mapping, then  $Tx_n \leq Tz$ , for all  $n \in \mathbb{N}$ , or equivalently,

$x_{n+1} \leq Tz$  for all  $n \in \mathbb{N}$ .

Moreover, as  $x_0 < x_1 \leq Tz$  and  $z = \sup \{x_n\}$ , we get  $z \leq Tz$ .

Suppose that  $z < Tz$ . Using a similar argument that in the proof of theorem 2.2, for  $x_0 \leq Tx_0$ , we obtain that  $\{T^n z\}$  is a nondecreasing sequence and

$$\lim_{n \rightarrow \infty} T^n z = y \text{ for certain } y \in X.$$

Again using hypothesis, we have that  $y = \sup \{T^n z\}$ .

Moreover from  $x_0 \leq z$ , we get  $x_n = T^n x_0 \leq T^n z$  for  $n \geq 1$  and  $x_n \leq T^n z$  for  $n \geq 1$  because  $x_n \leq z < Tz \leq T^n z$  for  $n \geq 1$ .

As  $x_n$  and  $T^n z$  are comparable and distinct for  $n \geq 1$ , applying the contractive condition we get,

$$\begin{aligned} d(x_{n+1}, T^n z) &= d(Tx_n, T(T^n z)) \\ &\leq \alpha \left( \frac{d(x_n, Tx_n) + d(T^n z, T^{n+1} z)}{d(x_n, T^n z) + d(x_n, Tx_n)} \right) d(x_n, Tx_n) + \beta d(x_n, T^n z) \\ &= \alpha \left( \frac{d(x_n, x_{n+1}) + d(T^n z, T^{n+1} z)}{d(x_n, T^n z) + d(x_n, x_{n+1})} \right) d(x_n, x_{n+1}) + \beta d(x_n, T^n z) \end{aligned} \tag{9}$$

$$\text{Making } n \rightarrow \infty \text{ in the last inequality, we obtain } d(z, y) \leq \beta d(z, y) \tag{10}$$

As  $\beta < 1$ ,  $d(z, y) = 0$ , thus  $z = y$ .

Particularly,  $z = y = \sup \{T^n z\}$  and consequently,  $Tz \leq z$  and this is a contradiction.

Hence, we conclude that  $Tz = z$ .

**Theorem 2.3:** Let  $(X, \leq)$  be a partially ordered set and suppose that there exists a metric  $d$  in  $X$  such that  $(X, d)$  is a complete metric space. Assume that for  $x, y \in X$ , there exists  $z \in X$  which is comparable to  $x$  and  $y$ . Let  $T : X \rightarrow X$  be a nondecreasing mapping such that

$$d(Tx, Ty) \leq \alpha \left( \frac{d(x, Tx) + d(y, Ty)}{d(x, y) + d(x, Tx)} \right) d(x, Tx) + \beta d(x, y), \text{ for } x, y \in X, x \geq y, x \neq y, \tag{11}$$

with  $\alpha + \beta < 1$ . If there exists  $x_0 \in X$  with  $x_0 \leq Tx_0$ , then  $T$  has a fixed point.

**Proof :** Suppose that there exists  $z, y \in X$  which are fixed point. We distinguish two cases.

**Case (1) :** If  $y$  and  $z$  are comparable and  $y \neq z$ , then using the contractive condition we have,

$$\begin{aligned} d(y,z) = d(Ty,Tz) &\leq \alpha \left( \frac{d(y,Ty) + d(z,Tz)}{d(y,z) + d(y,Ty)} \right) d(y,Ty) + \beta d(y,z) \\ &= \alpha \left( \frac{d(y,y) + d(z,z)}{d(y,z) + d(y,y)} \right) d(y,y) + \beta d(y,z) \\ &= \beta d(y,z). \end{aligned} \tag{12}$$

As  $\beta < 1$  is the last inequality, it is a contradiction. Thus  $y = z$ .

**Case (2) :** If  $y$  is not comparable to  $z$ , then there exists  $x \in X$  comparable to  $y$  and  $z$ . Monotonicity implies that  $T^n x$  is comparable to  $T^n y = y$  and  $T^n z = z$  for  $n = 0, 1, 2, \dots$ .

If there exists  $n_0 \geq 1$  such that  $T^{n_0} x = y$ , then as  $y$  is a fixed point, the sequence

$$\{ T^n x : n \geq n_0 \} \text{ is constant and consequently } \lim_{n \rightarrow \infty} T^n x = y.$$

Hence, we conclude that  $\lim_{n \rightarrow \infty} T^n x = y$ .

Using a similar argument, we can prove that

$$\lim_{n \rightarrow \infty} T^n x = z.$$

Now, the uniqueness of the limit gives us  $y = z$ .

Hence proved the theorem.

**Remark 2.5 :** It is easily proved that the space  $C[0,1] = \{ x : [0,1] \rightarrow \mathbb{R}, \text{ continuous} \}$  with the partial order given by  $x \leq y \Leftrightarrow x(t) \leq y(t), t \in [0,1]$ ,

and metric is given by  $d(x,y) = \int_0^1 |x(t) - y(t)| dt \in [0,1]$

satisfies condition : if  $\{x_n\}$  is a nondecreasing sequence in  $X$  such that

$x_n \rightarrow x$ , then  $x = \sup \{x_n\}$ .

Moreover, as for  $x, y \in C[0,1]$ , the function  $\max(x, y)(t) = \max \{ x(t), y(t) \}$  is continuous,  $(C[0,1], \leq)$  satisfies also condition :

For  $x, y \in X$  there exists  $z \in X$  which is comparable to  $x$  and  $y$ .

### 3. Some Remarks

**Remark 3.1 :** Using the theorems of Section 2,  $\beta=0$ , we obtain the following fixed point theorem in partially ordered complete metric space.

**Theorem 3.2 :** Let  $(X, \leq)$  be a partially ordered set and suppose that there exists a metric  $d$  in  $X$  such that  $(X, d)$  is a complete metric space. Let  $T : X \rightarrow X$  be a nondecreasing mapping such that there exists  $a \in [0, 1)$  satisfying

$$d(Tx,Ty) \leq \alpha \left( \frac{d(x,Tx) + d(y,Ty)}{d(x,y) + d(x,Tx)} \right) d(x,Tx), \text{ for } x,y \in X, x \geq y, x \neq y \tag{13}$$

Suppose also that either  $T$  is continuous or  $X$  satisfies condition : If  $\{x_n\}$  is a nondecreasing

sequence in  $X$  such that  $x_n \rightarrow x$ , then  $x = \sup \{x_n\}$ .

If there exists  $x_0 \in X$  with  $x_0 \leq Tx_0$ , then  $T$  has a fixed point.

Besides, if  $(X, \leq)$  satisfies the condition: For  $x, y \in X$  there exists  $z \in X$  which is comparable to  $x$  and  $y$ , then one obtains uniqueness of the fixed point.

**Example 3.3:** Let  $X = \{(0, 1), (1, 0), (1, 1)\}$  and consider in  $X$  the partial order given by  $R = \{(x, x) : x \in X\}$ . Notice that elements in  $X$  are only comparable to themselves. Besides  $(X, d_2)$  is a complete metric space considering  $d_2$  the Euclidean distance.

Let  $T : X \rightarrow X$  be defined by  $T(0, 1) = (1, 0)$ ,  $T(1, 0) = (0, 1)$ ,  $T(1, 1) = (1, 1)$ .

$T$  is trivially continuous and nondecreasing and

$$d(Tx, Ty) \leq \alpha \left( \frac{d(x, Tx) + d(y, Ty)}{d(x, y) + d(x, Tx)} \right) d(x, Tx) + \beta d(x, y)$$

is satisfied. Since elements in  $X$  are only comparable to themselves. Moreover  $(1, 1) \leq T(1, 1) = (1, 1)$  and by theorem 2.2  $T$  has a fixed point. (Obviously this fixed point is  $(1, 1)$ ).

On the other hand, for  $x = (0, 1)$ ,  $y = (1, 0) \in X$ , we have  $d(Tx, Ty) = \sqrt{2}$ ,  $d(x, Tx) = \sqrt{2}$ ,  $d(y, Ty) = \sqrt{2}$ ,  $d(x, y) = \sqrt{2}$  and the contractive condition is not satisfied because

$$\begin{aligned} d(Tx, Ty) = \sqrt{2} &\leq \alpha \left( \frac{d(x, Tx) + d(y, Ty)}{d(x, y) + d(x, Tx)} \right) d(x, Tx) + \beta d(x, y) \\ &= \alpha \left( \frac{\sqrt{2} + \sqrt{2}}{\sqrt{2} + \sqrt{2}} \right) \sqrt{2} + \beta \sqrt{2} = (\alpha + \beta) \sqrt{2} \end{aligned}$$

and thus  $\alpha + \beta \geq 1$ .

Moreover, notice that in this example we have uniqueness of fixed point and  $(X, \leq)$  does not satisfy Condition: for  $x, y \in X$ , there exists  $z \in X$  which is comparable to  $x$  and  $y$ . This condition is not necessary condition for the uniqueness of the fixed point.

## References

- [1] D. S. Jaggi, "Some unique fixed point theorems," *Indian Journal of Pure and Applied Mathematics*, vol. 8, no. 2, pp. 223–230, 1977.
- [2] P. P. Agarwal, M. A. El-Gebeily, and D. O'Regan, "Generalized contractions in partially ordered metric spaces," *Applicable Analysis*, vol. 87, no. 1, pp. 109–116, 2008.
- [3] T. Choudhury-Bhaskar and V. Lakshmikantham, "Fixed point theorems in partially ordered metric spaces and applications," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 65, no. 7, pp. 1379–1393, 2006.
- [4] J. Harjani and K. Sadarangani, "Fixed point theorems for weakly contractive mappings in partially ordered sets," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 7-8, pp. 3403–3410, 2009.
- [5] J. Harjani and K. Sadarangani, "Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 3-4, pp. 1188–1197, 2010.
- [6] V. Lakshmikantham and L. Ćirić, "Coupled fixed point theorems for nonlinear contractions

- in partially ordered metric spaces,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 70, no. 12, pp. 4341–4349, 2009.
- [7] J. J. Nieto and R. Rodríguez-López, “Existence of extremal solutions for quadratic fuzzy equations,” *Fixed Point Theory and Applications*, vol. 2005, no. 3, pp. 321–342, 2005.
- [8] J. J. Nieto and R. Rodríguez-López, “Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations,” *Order*, vol. 22, no. 3, pp. 223–239, 2005.
- [9] J. J. Nieto and R. Rodríguez-López, “Applications of contractive-like mapping principles to fuzzy equations,” *Revista Matemática Complutense*, vol. 19, no. 2, pp. 361–383, 2006.
- [10] J. J. Nieto, R. L. Pouso, and R. Rodríguez-López, “Fixed point theorems in ordered contract spaces,” *Proceedings of the American Mathematical Society*, vol. 135, no. 8, pp. 2505–2517, 2007.
- [11] J. J. Nieto and R. Rodríguez-López, “Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations,” *Acta Mathematica Sinica*, vol. 23, no. 12, pp.2205–2212, 2007.
- [12] D. O’Regan and A. Petruşel, “Fixed point theorems for generalized contractions in ordered metric spaces,” *Journal of Mathematical Analysis and Applications*, vol. 341, no. 2, pp. 1241–1252, 2008.
- [13] A. Petruşel and I. A. Rus, “Fixed point theorem in ordered L-spaces,” *Proceedings of the American Mathematical Society*, vol. 134, no. 2, pp. 411–418, 2006.8
- [14] A. C. M. Ran and M. C. B. Reurings, “Fixed point theorem in partially ordered sets and some applications to matrix equations,” *Proceedings of the American Mathematical Society*, vol. 132, no. 5, pp.1435–1443, 2004.
- [15] Y. Wu, “New fixed point theorem and applications of mixed monotone operator,” *Journal of Mathematical Analysis and Applications*, vol. 341, no. 2, pp. 883–893, 2008.
- [16] A. Cabada and J. J. Nieto, “Fixed points and approximate solutions for nonlinear operator equations,” *Journal of Computational and Applied Mathematics*, vol. 113, no. 1-2, pp. 17–25, 2000.