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On the Complete Elliptic Integrals and Babylonian Identity XII: The Complete Elliptic Integral of first kind and Power Series Representation

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Abstract. We developed a new power series representation of the complete elliptic integral of first kind.

1. INTRODUCTION

We developed the following power series representation:

$$K(k)=\frac{1}{2}\sum_{n=0}^{\infty}\frac{((-1)^n+1)\Gamma^2\left(\frac{n+1}{2}\right)}{n\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+2}{2}\right)}k^n,$$

for $0<k<1$.

2. THEOREM

Theorem 1. *If* $0<k<1$, *then*

$$K(k)=\frac{1}{2}\sum_{n=0}^{\infty}\frac{((-1)^n+1)\Gamma^2\left(\frac{n+1}{2}\right)}{n\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+2}{2}\right)}k^n,$$

where $K(k)$ is the complete elliptic integral of first kind.

Proof. Expand the expression

$$\frac{1}{\sqrt{1-A^2B^2}}=\frac{1}{\sqrt{1-AB}\times\sqrt{1+AB}}=\frac{1}{\sqrt{1-AB}\times\sqrt{1+AB}}$$

$$=\sum_{r=0}^{\infty}(-1)^r\binom{-\frac{1}{2}}{r}A^rB^r\sum_{s=0}^{\infty}\binom{-\frac{1}{2}}{s}A^sB^s$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}(-1)^r\binom{-\frac{1}{2}}{r}\binom{-\frac{1}{2}}{s}(AB)^{r+s}. \tag{8}$$

The coefficient of $(AB)^n$ in this double series is

$$\sum_{r=0}^{n}(-1)^r\binom{-\frac{1}{2}}{r}\binom{-\frac{1}{2}}{n-r}=\frac{((-1)^n+1)\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}n\Gamma\left(\frac{n}{2}\right)}. \tag{9}$$

From (8) and (9), I conclude that

$$\frac{1}{\sqrt{1-A^2B^2}}=\frac{1}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{((-1)^n+1)\Gamma\left(\frac{n+1}{2}\right)}{n\Gamma\left(\frac{n}{2}\right)}(AB)^n \tag{10}$$

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Put AB by kt into (10)

$$\frac{1}{\sqrt{1-k^2t^2}} = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{((-1)^n+1)\Gamma\left(\frac{n+1}{2}\right)}{n\Gamma\left(\frac{n}{2}\right)} k^n t^n. \tag{11}$$

Multiplying both sides of (11) by $\frac{1}{\sqrt{1-t^2}}$ and integrating from 0 at ∞ with respect to t, we have

$$\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{((-1)^n+1)\Gamma\left(\frac{n+1}{2}\right)}{n\Gamma\left(\frac{n}{2}\right)} k^n \int_0^1 \frac{t^n}{\sqrt{1-t^2}} dt, \tag{12}$$

hence, it follows forthwith

$$K(k) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{((-1)^n + 1)\Gamma^2\left(\frac{n+1}{2}\right)}{n\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+2}{2}\right)} k^n. \square$$