

On the Complete Elliptic Integrals and Babylonian Identity XII: The Complete Elliptic Integral of first kind and Power Series Representation

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Abstract. We developed a new power series representation of the complete elliptic integral of first kind.

1. INTRODUCTION

We developed the following power series representation:

$$K(k) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{((-1)^n + 1) \Gamma^2\left(\frac{n+1}{2}\right)}{n \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{n+2}{2}\right)} k^n,$$

for $0 < k < 1$.

2. THEOREM

Theorem 1. *If $0 < k < 1$, then*

$$K(k) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{((-1)^n + 1) \Gamma^2\left(\frac{n+1}{2}\right)}{n \Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{n+2}{2}\right)} k^n,$$

where $K(k)$ is the complete elliptic integral of first kind.

Proof. Expand the expression

$$\begin{aligned} \frac{1}{\sqrt{1-A^2B^2}} &= \frac{1}{\sqrt{1-AB} \times \sqrt{1+AB}} = \frac{1}{\sqrt{1-AB} \times \sqrt{1+AB}} \\ &= \sum_{r=0}^{\infty} (-1)^r \binom{-\frac{1}{2}}{r} A^r B^r \sum_{s=0}^{\infty} \binom{-\frac{1}{2}}{s} A^s B^s \\ &= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (-1)^r \binom{-\frac{1}{2}}{r} \binom{-\frac{1}{2}}{s} (AB)^{r+s}. \end{aligned} \tag{8}$$

The coefficient of $(AB)^n$ in this double series is

$$\sum_{r=0}^n (-1)^r \binom{-\frac{1}{2}}{r} \binom{-\frac{1}{2}}{n-r} = \frac{((-1)^{n+1} \Gamma\left(\frac{n+1}{2}\right))}{\sqrt{\pi} n \Gamma\left(\frac{n}{2}\right)}. \tag{9}$$

From (8) and (9), I conclude that

$$\frac{1}{\sqrt{1-A^2B^2}} = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{((-1)^{n+1} \Gamma\left(\frac{n+1}{2}\right))}{n \Gamma\left(\frac{n}{2}\right)} (AB)^n \tag{10}$$

Put AB by kt into (10)

$$\frac{1}{\sqrt{1-k^2t^2}} = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{((-1)^{n+1})\Gamma\left(\frac{n+1}{2}\right)}{n\Gamma\left(\frac{n}{2}\right)} k^n t^n. \quad (11)$$

Multiplying both sides of (11) by $\frac{1}{\sqrt{1-t^2}}$ and integrating from 0 at ∞ with respect to t , we have

$$\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{((-1)^{n+1})\Gamma\left(\frac{n+1}{2}\right)}{n\Gamma\left(\frac{n}{2}\right)} k^n \int_0^1 \frac{t^n}{\sqrt{1-t^2}} dt, \quad (12)$$

hence, it follows forthwith

$$K(k) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{((-1)^n + 1)\Gamma^2\left(\frac{n+1}{2}\right)}{n\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n+2}{2}\right)} k^n. \square$$