

An Investigation on the Beta Function III: Explicit Evaluation of the Harmonic Number $H_{\frac{3}{2}}$

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Abstract. In previous paper, I developed new versions of the Euler beta function, which given a closed form for the harmonic number $H_{\frac{3}{2}}$.

1. INTRODUCTION

In this paper, I prove that

$$\sum_{k=1}^{\infty} \frac{(2k)!}{4^k k!} \sum_{n=1}^j \frac{\Gamma\left(n + \frac{1}{2}\right)}{\Gamma\left(n + k + \frac{3}{2}\right)} = H_j - H_{j+\frac{1}{2}} + 2 - 2\ln 2,$$

which provided the special value

$$H_{\frac{3}{2}} = \frac{8}{3} - 2\ln 2.$$

2. THEOREM

THEOREM 1. *I have*

$$H_{\frac{3}{2}} = \frac{8}{3} - 2\ln 2,$$

where H_j is the harmonic number.

Proof. In [1], I prove that

$$\frac{1}{n\Gamma\left(n+\frac{1}{2}\right)} = \sum_{k=0}^{\infty} \frac{(2k)!}{4^k k! \Gamma\left(n+k+\frac{3}{2}\right)} \Rightarrow \frac{1}{n} = \sum_{k=0}^{\infty} \frac{(2k)! \Gamma\left(n+\frac{1}{2}\right)}{4^k k! \Gamma\left(n+k+\frac{3}{2}\right)} = \frac{2}{2n+1} + \sum_{k=1}^{\infty} \frac{(2k)! \Gamma\left(n+\frac{1}{2}\right)}{4^k k! \Gamma\left(n+k+\frac{3}{2}\right)}, \quad (1)$$

I sum in both sides of equation above from 1 at j with respect to n

$$\sum_{n=1}^j \frac{1}{n} = 2 \sum_{n=1}^j \frac{1}{2n+1} + \sum_{k=1}^{\infty} \frac{(2k)!}{4^k k!} \sum_{n=1}^j \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(n+k+\frac{3}{2}\right)}, \quad (2)$$

ergo,

$$\sum_{k=1}^{\infty} \frac{(2k)!}{4^k k!} \sum_{n=1}^j \frac{\Gamma\left(n + \frac{1}{2}\right)}{\Gamma\left(n + k + \frac{3}{2}\right)} = H_j - H_{j+\frac{1}{2}} + 2 - 2\ln 2$$

Let $j = 1$ in Theorem 1, then

$$H_{\frac{3}{2}} = \frac{8}{3} - 2\ln 2.$$

I not can calculate other special values, since, for $j > 1$, the summation

$$\sum_{k=1}^{\infty} \frac{(2k)!}{4^k k!} \sum_{n=1}^j \frac{\Gamma\left(n + \frac{1}{2}\right)}{\Gamma\left(n + k + \frac{3}{2}\right)}$$

is a divergent series. \square

REFERENCES

- [1] Guedes, Edigles, *An Investigation on the Beta Function II: The Summations of $1/\sqrt{\pi}$* , 2013.