The Bulletin of Society for Mathematical Services and Standards Online: 2013-12-02
ISSN: 2277-8020, Vol. 8, pp 26-34
doi:10.18052/www.scipress.com/BSMaSS.8.26

Some anti-solutions of the Pillai's conjecture and proof of Fermat's Last Theorem.

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Abstract. Let us show several; including the part already well known variants of find uncountable set solutions of equation

$$\mathrm{A}x^m - By^n = C\ [\mathbf{1}]$$

for natural numbers $\mathrm{A, B, C, x, y, m, n}$ of specified values $\mathrm{A, B, C}$,in contrast to the Pillai's conjecture, in which it is assumed that the set of solutions of equation $[\mathbf{1}]$ is finite, and proof of Fermat's Last Theorem.

Solution

§ 1

$$m = n = 1$$

If the numbers A and B are coprime $[(A, B) = 1]$, then the equation

$$Ax + By = C$$

always has results in integers (positive) integer numbers and its solutions will always be all the pairs (x_t, y_t), where

$$x_t = C \times A^{\varphi(B)-1} + B \times t,$$

$$y_t = C \times \frac{1 - A^{\varphi(B)}}{B} - A \times t,$$

t - any integer,

$\varphi(B)$ - Euler function.

N.N. Vorobiev, "Divisibility" (3) Science Glavfizmatgiz, Moscow, 1974, p. 47.

§ 2

2.1.

$$m = 2; n = 2; A = 1; C = 1.$$

The equation

$$x^2 - By^2 = 1,$$

where $"\mathrm{B}" = D$ is not exact square, has an infinite set of solutions and all of its solutions in the smallest natural numbers are the formulas:

$$x_{k+1} = x_1 x_k + D y_1 y_k$$
$$y_{k+1} = y_1 x_k + x_1 y_k \quad (k = 1,2,3,4, \dots .).$$

For $D = 2,3,5,6,7,8,10,11,12,13,29$ ... and etc.

$$(x, y) = (3,2), (2,1), (9,4), (5,2), (8,3),$$
$$(3,1), (19,6), (10,3), (7,2), (649,180), (4901,1820)$$ and etc.

2.1.1. Then it is possible another variant

$$Ax^2 - y^2 = 1; \left(x_0\sqrt{A} + y_0\right)^k = x\sqrt{A} + y,$$

where k is odd number.

$$A = 2; x_0 = 1; y_0 = 1, k = 7,$$
$$x = 13^2 = 169; y = 239; 2 \times 13^4 - 239^2 = 1.$$

2.2. The complete solution issue see. V.Sierpinski, "The solution of equations in integers" (1) Fizmatgiz, Moscow, 1961, pp. 29-33..

A.O. Gelfond, "Solving equations in integers" (2) "Science", Glavfizmatgiz, Moscow, 1978.

§ 3

3.1.

$$m = n = 2; c = 1$$

The equation

$$Ax^2 - By^2 = 1,$$

where A and B is not exact square, has an infinite set of solutions are natural (whole) numbers, which can be found from the equation

$$x\sqrt{A} + y\sqrt{B} = (x_0\sqrt{A} + y_0\sqrt{B})^k \ [\mathbf{2}],$$

where k – is odd number.

3.2. The equations

$$3x^2 - 2y^2 = 1; \quad (A - B = 1)$$
$$x\sqrt{3} + y\sqrt{2} = (1 \times \sqrt{3} + 1 \times \sqrt{2})^k,$$

since they have an obvious solution $(1,1)$.

Put "k" values $3,5,7 \ldots$, we obtain $(9,11), (89,109)$ and etc.

3.3. U.S. Davydov, "Exercises in theoretical arithmetic of integers", the Ministry of Education Uchpedgiz BSSR, Minsk, 1963, N266, p.61 (4).

§ 4

Statement

4.1. **"**For each an arbitrary set of four natural numbers

$$x^m, y^n, p^k, q^t,$$

where all 8 parameters not related to each other (autonomous), there are at least two equations, such that

$$\begin{cases} Ax^m - By^n = C \\ Ap^k - Bq^t = C \end{cases} [\mathbf{3}],$$

where A, B, C –are natural numbers (for each equation are the same), the values of which depend only on the valuesx^m, y^n, p^k, q^t".

Proof.

4.2. If

$$A\vartheta_1 - Bu_1 = C\ [\mathbf{4}],$$

Then

$$\begin{cases} \vartheta_2 = \vartheta_1 + B \\ u_2 = u_1 + A \end{cases} [\mathbf{5}]$$

$$A\vartheta_2 - Bu_2 = C\ [\mathbf{6}].$$

Can be verified directly by substituting $[\mathbf{5}]$ in $[\mathbf{6}]$.

4.3. From $[\mathbf{5}]$

$$\begin{cases} B = \vartheta_2 - \vartheta_1 \\ A = u_2 - u_1 \end{cases} [\mathbf{7}].$$

Substituting in $[\mathbf{4}]$ and $[\mathbf{6}]$, we obtain

$$\begin{cases} (u_2 - u_1)\vartheta_1 - (\vartheta_2 - \vartheta_1)u_1 = C \\ (u_2 - u_1)\vartheta_2 - (\vartheta_2 - \vartheta_1)u_2 = C \end{cases} [\mathbf{8}]$$

and

$$C = u_2\vartheta_1 - \vartheta_2 u_1.$$

4.4. Let

$$u_2 = 5^3; u_1 = 2^3; \vartheta_2 = 3^4; \vartheta_1 = 4^3.$$

Therefore,

$$\begin{cases}(5^3 - 2^3)4^3 - (3^4 - 4^3)2^3 = 7352 \\ (5^3 - 2^3)3^4 - (3^4 - 4^3)5^3 = 7352\end{cases} \; [\mathbf{9}]$$

and

$$117 \times 3^4 - 17 \times 5^3 = 7352$$

$$117 \times 4^3 - 17 \times 2^3 = 7352.$$

This completes the proof of statement.

P.S. If $(\vartheta_1, \vartheta_2, u_1, u_2)$ - arbitrary natural numbers in even powers such that $(\vartheta_2 - \vartheta_1)$; $(u_2 - u_1)$ – are inexact squares, then using the method **§ 5** , can be get infinite set of solutions of equations $[\mathbf{8}]$ we say that rational numbers for each fixed triple A, B, C.

§ 5

5.1. We obtain identities:

$$A(A + 3B)^2 - B(3A + B)^2 \equiv (A - B)^3 \; [\mathbf{10}]$$

and

$$\frac{A}{(A - B)^3}(A + 3B)^2 - \frac{B}{(A - B)^3}(3A + B)^2 \equiv 1 \; [\mathbf{11}]$$

5.2. Since $[\mathbf{11}]$ we have

$$x\sqrt{\frac{A}{(A - B)^3}} + y\sqrt{\frac{B}{(A - B)^3}} =$$

$$= (x_0\sqrt{\frac{A}{(A - B)^3}} + y_0\sqrt{\frac{B}{(A - B)^3}})^k =$$

$$= [(A + 3B)\sqrt{\frac{A}{(A - B)^3}} + (3A + B)\sqrt{\frac{B}{(A - B)^3}}]^k \; [\mathbf{12}],$$

where "k" -is arbitrary odd natural number.

5.3. For every specific value A and B, where A, B –are inexact squares of arbitrary natural numbers, such as,

1) A and , B such that both are odd

$$A - B = 2.$$

Similarly, the equation $[12]$ provides infinite set of solutions"x" and "y" of natural numbers for each pair A and B for constant

$$A, B, C = (A - B)^3.$$

2) In the case, for example, where A and B are numbers with different parity (inexact squares), the equation $[12]$ provides infinite set of solutions "x" and "y" of rational numbers for each pair A and B for constant

$$A, B, C = (A - B)^3.$$

Examples.

If $A = 5; B = 3; k = 3$;from $[13]$

$$x_0 = A + 3B = 5 + 3 \times 3 = 14;$$

$$y_0 = 3A + B = 3 \times 5 + 3 = 18.$$

$$5 \times 14^2 - 3 \times 18^2 = 8$$

5.4. Follows from $[15]$

$$x\sqrt{\frac{5}{8}} + y\sqrt{\frac{3}{8}} = \left(14\sqrt{\frac{5}{8}} + 18\sqrt{\frac{3}{8}}\right)^3 =$$

$$14^3 \times \frac{5}{8} \times \sqrt{\frac{5}{8}} + 3 \times 14^2 \times \frac{5}{8} \times 18 \times \sqrt{\frac{3}{8}} +$$

$$+3 \times 14\sqrt{\frac{5}{8}} \times 18^2 \times \frac{3}{8} + 18^3 \times \frac{3}{8}\sqrt{\frac{3}{8}};$$

$$x = 14^3 \times \frac{5}{8} + 3 \times 14 \times 18^2 \times \frac{3}{8} =$$

$$= 5 \times 7^3 + 3^6 \times 7 = 6818$$

$$y = 3 \times 14^2 \times \frac{5}{8} \times 18 + 18^3 \frac{3}{8} =$$

$$= 3^3 \times 5 \times 7^2 + 3^7 = 8802$$

and

$$5 \times 6818^2 - 3 \times 8802^2 = (5-3)^3 = 2^3 =$$
$$= 232425620 - 232425612 = 8 \text{ and etc.}$$

5.5. Thus,

$$Ax^2 - By^2 = C; 5x^2 - 3y^2 = 8 \; [\mathbf{13}]$$

where x, y - are infinite numbers of the equation $[\mathbf{13}]$ of natural numbers for constant $A = 5; B = 3; C = 8$ for the case where $C \neq 1$.

§ 6

6.1. We obviously have the identity:

$$(y^t + k)^{z-q}(y^t + k)^q - y^t \times \frac{(y^t + k)^z - k^z}{(y^t + k) - k} \equiv k^z \; [\mathbf{14}]$$

6.2. If

z -is inexact square arbitrary odd natural number;

q – is even natural number;

t - is even natural number;

$(z - q)$ – is inexact square;

"k" – arbitrary natural number,

then using the method of the previous section, we can get infinite set of solutions, we shall say that, of rational numbers in the equation $[\mathbf{14}]$ for constant A, B, C.

§ 7

The proof of the Fermat's Last Theorem (another version)

7.1. Let us prove that in the equation

$$x^n + y^n = z^n \;\; [\mathbf{15}]$$

x_0, y_0, z_0 –are solution of $[\mathbf{15}]$ in natural numbers for $n > 2$.

7.2. Then, follows from $[\mathbf{15}]$

$$z^n - x^n = y^n;$$
$$z^m z^{n-m} - x^z x^{n-r} = y^n;$$

$$\frac{z^m}{y^n} z^{n-m} - \frac{x^r}{y^n} x^{n-r} = 1 \; [\mathbf{16}],$$

where $n > 2$ – is arbitrary odd natural number;

$m < n, r < n$ – are inexact squares arbitrary odd natural numbers.

$n - m = 2p, n - r = 2q;$

$$\frac{z_0^m}{y_0^n} (z_0^p)^2 - \frac{x_0^r}{y_0^n} \left(x_0^q\right)^2 = 1 \; [\mathbf{17}].$$

7.3. Using the method of $\S\, \mathbf{5}$,we have

$$z \times \sqrt{\frac{z_0^m}{y_0^n}} - x \times \sqrt{\frac{x_0^r}{y_0^n}} = \left(z_0^p \times \sqrt{\frac{z_0^m}{y_0^n}} - x_0^q \times \sqrt{\frac{x_0^r}{y_0^n}} \right)^k \quad [\mathbf{18}],$$

where k - is arbitrary odd natural number, and hence infinite set of solutions z, x, y of equations $[\mathbf{18}], [\mathbf{17}], [\mathbf{16}], [\mathbf{15}]$ for each odd $n > 2$ in natural (rational) numbers; it is clear that using research of G. Faltings **(5)** the equation $[\mathbf{15}]$ has no solutions in the natural (rational) numbers for arbitrary natural $n > 2$, because equations $[\mathbf{15}]$ has no solution for $n = 4$,as we know, has proven by Fermat himself.

7.4. Indeed, if 3^x equation having at least one solution (**§ 2,3,4,5,6**) such that allow obtain from it type equation $[\mathbf{2}], [\mathbf{12}], [\mathbf{16}], [\mathbf{17}], [\mathbf{18}]$ then the solutions of are countless.

7.5. To conclude the proof, it remains to note that with not coprime solutions of $[\mathbf{15}]$ in **(6)**, **1.2., 2.1.,2.2.,2.3.** and **(7), 2.6.9**. We obtain a complete solution to the Fermat's Last Theorem.

7.6. Indeed, as in the general case, in **(6)** and **(7)**, if

$$x^n + y^n = z^n,$$

then

$$x^n (x^{35} y^{84} z^{20})^n + y^n (x^{35} y^{84} z^{20})^n =$$

$$= z^n (x^{35} y^{84} z^{20})^n \; [\boldsymbol{I}]$$

$$[(x^9 y^{21} z^5)^4]^n + [(x^7 y^{17} z^4)^5]^n =$$

$$= [(x^5 y^{12} z^3)^7]^n \; [\boldsymbol{II}]$$

$$[\boldsymbol{I}] = [\boldsymbol{II}].$$

$$x^n (x^{315} y^{224} z^{160})^n + y^n (x^{315} y^{224} z^{160})^n =$$

$$z^n (x^{315} y^{224} z^{160})^n \; [\boldsymbol{III}]$$

$$[(x^{79}y^{56}z^{40})^4]^n + [(x^{63}y^{45}z^{32})^5]^n =$$

$$= [(x^{45}y^{32}z^{23})^7]^n \; [\boldsymbol{IV}]$$

$$[\boldsymbol{III}] = [\boldsymbol{IV}] \quad \text{and etc.}$$

These different solutions in parentheses for each natural "n" and each arbitrary fixed triple coprime exponents in the square brackets (4, 5, 7) - are countless.

7.7. From the initial same, but slightly modified considerations that used in **7.6.** , we obtain follows identity:

$$[x^{3p+1}y^{3q}z^{2(3t-2)}]^{3t-2} + [x^{3p}y^{3q+1}z^{2(3t-2)}]^{3t-2} \equiv$$

$$\equiv \left[x^{3p}y^{3q}z^{3(2t-1)}\right]^{3t-2} \; [\boldsymbol{V}],$$

if

$$x^{3t-2} + y^{3t-2} = z^{3t-2}.$$

Here, there are (independent from each other),

$$p = 1,2,3,4, \ldots.$$

$$q = 1,2,3,4 \ldots$$

$$t = 1,2,3,4 \ldots.$$

$$3t - 2 = n = 1,4,7,10, \ldots.$$

7.8. The identity of type $[\boldsymbol{V}]$ the same arguments may be obtained for an arbitrary natural"n".

7.9. As a consequence of the preceding paragraphs of the section in **(7)**, **1.2.** all members of the sequence

$$\frac{\sqrt[n-1]{(x^{n-1}+y^{n-1})^n}}{\sqrt[n]{(x^n+y^n)^{n-1}}} = \frac{\sqrt[n-1]{z_{n-1}^n}}{\sqrt[n]{z_n^{n-1}}} \; [\mathbf{19}]$$

for $n > 2$ are only irrational numbers. Therefore, the proof of insolubility in natural numbers of the equation

$$x^n + y^n = z^n$$

for $n > 2$ in the **(7)** becomes a complete and fully correct.

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