

Some anti-solutions of the Pillai's conjecture and proof of Fermat's Last Theorem.

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Abstract. Let us show several; including the part already well known variants of find uncountable set solutions of equation

$$Ax^m - By^n = C \quad [1]$$

for natural numbers A, B, C, x, y, m, n of specified values A, B, C , in contrast to the Pillai's conjecture, in which it is assumed that the set of solutions of equation [1] is finite, and proof of Fermat's Last Theorem.

Solution

§ 1

$$m = n = 1$$

If the numbers A and B are coprime $[(A, B) = 1]$, then the equation

$$Ax + By = C$$

always has results in integers (positive) integer numbers and its solutions will always be all the pairs (x_t, y_t) , where

$$x_t = C \times A^{\varphi(B)-1} + B \times t,$$
$$y_t = C \times \frac{1 - A^{\varphi(B)}}{B} - A \times t,$$

t - any integer,

$\varphi(B)$ - Euler function.

N.N. Vorobiev, "Divisibility" (3) Science Glavfizmatgiz, Moscow, 1974, p. 47.

§ 2

2.1.

$$m = 2; n = 2; A = 1; C = 1.$$

The equation

$$x^2 - By^2 = 1,$$

where " B " = D is not exact square, has an infinite set of solutions and all of its solutions in the smallest natural numbers are the formulas:

$$\begin{aligned}x_{k+1} &= x_1x_k + Dy_1y_k \\y_{k+1} &= y_1x_k + x_1y_k \quad (k = 1, 2, 3, 4, \dots).\end{aligned}$$

For $D = 2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 29 \dots$ and etc.

$$\begin{aligned}(x, y) &= (3, 2), (2, 1), (9, 4), (5, 2), (8, 3), \\(3, 1), (19, 6), (10, 3), (7, 2), (649, 180), (4901, 1820) &\text{ and etc.}\end{aligned}$$

2.1.1. Then it is possible another variant

$$Ax^2 - y^2 = 1; (x_0\sqrt{A} + y_0)^k = x\sqrt{A} + y,$$

where k is odd number.

$$\begin{aligned}A &= 2; x_0 = 1; y_0 = 1, k = 7, \\x &= 13^2 = 169; y = 239; 2 \times 13^4 - 239^2 = 1.\end{aligned}$$

2.2. The complete solution issue see. V.Sierpinski, "The solution of equations in integers" (1) Fizmatgiz, Moscow, 1961, pp. 29-33..

A.O. Gelfond, "Solving equations in integers" (2) "Science", Glavfizmatgiz, Moscow, 1978.

§ 3

3.1.

$$m = n = 2; c = 1$$

The equation

$$Ax^2 - By^2 = 1,$$

where A and B is not exact square, has an infinite set of solutions are natural (whole) numbers, which can be found from the equation

$$x\sqrt{A} + y\sqrt{B} = (x_0\sqrt{A} + y_0\sqrt{B})^k \quad [2],$$

where k – is odd number.

3.2. The equations

$$\begin{aligned}3x^2 - 2y^2 &= 1; \quad (A - B = 1) \\x\sqrt{3} + y\sqrt{2} &= (1 \times \sqrt{3} + 1 \times \sqrt{2})^k,\end{aligned}$$

since they have an obvious solution (1,1).

Put "k" values 3,5,7 ..., we obtain (9,11), (89,109) and etc.

3.3. U.S. Davydov, "Exercises in theoretical arithmetic of integers", the Ministry of Education Uchpedgiz BSSR, Minsk, 1963, N266, p.61 (4).

§ 4

Statement

4.1. "For each an arbitrary set of four natural numbers

$$x^m, y^n, p^k, q^t,$$

where all 8 parameters not related to each other (autonomous), there are at least two equations, such that

$$\begin{cases} Ax^m - By^n = C \\ Ap^k - Bq^t = C \end{cases} \quad [3],$$

where A, B, C –are natural numbers (for each equation are the same), the values of which depend only on the values x^m, y^n, p^k, q^t .

Proof.

4.2. If

$$A\vartheta_1 - Bu_1 = C \quad [4],$$

Then

$$\begin{cases} \vartheta_2 = \vartheta_1 + B \\ u_2 = u_1 + A \end{cases} \quad [5]$$

$$A\vartheta_2 - Bu_2 = C \quad [6].$$

Can be verified directly by substituting [5] in [6].

4.3. From [5]

$$\begin{cases} B = \vartheta_2 - \vartheta_1 \\ A = u_2 - u_1 \end{cases} \quad [7].$$

Substituting in [4] and [6], we obtain

$$\begin{cases} (u_2 - u_1)\vartheta_1 - (\vartheta_2 - \vartheta_1)u_1 = C \\ (u_2 - u_1)\vartheta_2 - (\vartheta_2 - \vartheta_1)u_2 = C \end{cases} \quad [8]$$

and

$$C = u_2\vartheta_1 - \vartheta_2u_1.$$

4.4. Let

$$u_2 = 5^3; u_1 = 2^3; v_2 = 3^4; v_1 = 4^3.$$

Therefore,

$$\begin{cases} (5^3 - 2^3)4^3 - (3^4 - 4^3)2^3 = 7352 \\ (5^3 - 2^3)3^4 - (3^4 - 4^3)5^3 = 7352 \end{cases} \quad [9]$$

and

$$117 \times 3^4 - 17 \times 5^3 = 7352$$

$$117 \times 4^3 - 17 \times 2^3 = 7352.$$

This completes the proof of statement.

P.S. If (v_1, v_2, u_1, u_2) - arbitrary natural numbers in even powers such that $(v_2 - v_1)$; $(u_2 - u_1)$ - are inexact squares, then using the method § 5, can be get infinite set of solutions of equations [8] we say that rational numbers for each fixed triple A, B, C .

§ 5

5.1. We obtain identities:

$$A(A + 3B)^2 - B(3A + B)^2 \equiv (A - B)^3 \quad [10]$$

and

$$\frac{A}{(A - B)^3} (A + 3B)^2 - \frac{B}{(A - B)^3} (3A + B)^2 \equiv 1 \quad [11]$$

5.2. Since [11] we have

$$\begin{aligned} & x \sqrt{\frac{A}{(A - B)^3}} + y \sqrt{\frac{B}{(A - B)^3}} = \\ & = (x_0 \sqrt{\frac{A}{(A - B)^3}} + y_0 \sqrt{\frac{B}{(A - B)^3}})^k = \\ & = [(A + 3B) \sqrt{\frac{A}{(A - B)^3}} + (3A + B) \sqrt{\frac{B}{(A - B)^3}}]^k \quad [12], \end{aligned}$$

where " k " - is arbitrary odd natural number.

5.3. For every specific value A and B , where A, B are inexact squares of arbitrary natural numbers, such as,

1) A and B such that both are odd

$$A - B = 2.$$

Similarly, the equation [12] provides infinite set of solutions " x " and " y " of natural numbers for each pair A and B for constant

$$A, B, C = (A - B)^3.$$

2) In the case, for example, where A and B are numbers with different parity (inexact squares), the equation [12] provides infinite set of solutions " x " and " y " of rational numbers for each pair A and B for constant

$$A, B, C = (A - B)^3.$$

Examples.

If $A = 5; B = 3; k = 3$; from [13]

$$x_0 = A + 3B = 5 + 3 \times 3 = 14;$$

$$y_0 = 3A + B = 3 \times 5 + 3 = 18.$$

$$5 \times 14^2 - 3 \times 18^2 = 8$$

5.4. Follows from [15]

$$\begin{aligned} x \sqrt{\frac{5}{8}} + y \sqrt{\frac{3}{8}} &= \left(14 \sqrt{\frac{5}{8}} + 18 \sqrt{\frac{3}{8}} \right)^3 = \\ &= 14^3 \times \frac{5}{8} \times \sqrt{\frac{5}{8}} + 3 \times 14^2 \times \frac{5}{8} \times 18 \times \sqrt{\frac{3}{8}} + \\ &+ 3 \times 14 \sqrt{\frac{5}{8}} \times 18^2 \times \frac{3}{8} + 18^3 \times \frac{3}{8} \sqrt{\frac{3}{8}}; \\ x &= 14^3 \times \frac{5}{8} + 3 \times 14 \times 18^2 \times \frac{3}{8} = \\ &= 5 \times 7^3 + 3^6 \times 7 = 6818 \\ y &= 3 \times 14^2 \times \frac{5}{8} \times 18 + 18^3 \frac{3}{8} = \end{aligned}$$

$$= 3^3 \times 5 \times 7^2 + 3^7 = 8802$$

and

$$\begin{aligned} 5 \times 6818^2 - 3 \times 8802^2 &= (5 - 3)^3 = 2^3 = \\ &= 232425620 - 232425612 = 8 \text{ and etc.} \end{aligned}$$

5.5. Thus,

$$Ax^2 - By^2 = C; 5x^2 - 3y^2 = 8 \text{ [13]}$$

where x, y - are infinite numbers of the equation [13] of natural numbers for constant $A = 5; B = 3; C = 8$ for the case where $C \neq 1$.

§ 6

6.1. We obviously have the identity:

$$(y^t + k)^{z-q}(y^t + k)^q - y^t \times \frac{(y^t + k)^z - k^z}{(y^t + k) - k} \equiv k^z \text{ [14]}$$

6.2. If

z - is inexact square arbitrary odd natural number;

q - is even natural number;

t - is even natural number;

$(z - q)$ - is inexact square;

" k " - arbitrary natural number,

then using the method of the previous section, we can get infinite set of solutions, we shall say that, of rational numbers in the equation [14] for constant A, B, C .

§ 7

The proof of the Fermat's Last Theorem (another version)

7.1. Let us prove that in the equation

$$x^n + y^n = z^n \text{ [15]}$$

x_0, y_0, z_0 - are solution of [15] in natural numbers for $n > 2$.

7.2. Then, follows from [15]

$$\begin{aligned} z^n - x^n &= y^n; \\ z^m z^{n-m} - x^z x^{n-r} &= y^n; \end{aligned}$$

$$\frac{z^m}{y^n} z^{n-m} - \frac{x^r}{y^n} x^{n-r} = 1 \quad [16],$$

where $n > 2$ – is arbitrary odd natural number;

$m < n, r < n$ – are inexact squares arbitrary odd natural numbers.

$n - m = 2p, n - r = 2q$;

$$\frac{z_0^m}{y_0^n} (z_0^p)^2 - \frac{x_0^r}{y_0^n} (x_0^q)^2 = 1 \quad [17].$$

7.3. Using the method of § 5, we have

$$z \times \sqrt{\frac{z_0^m}{y_0^n}} - x \times \sqrt{\frac{x_0^r}{y_0^n}} = \left(z_0^p \times \sqrt{\frac{z_0^m}{y_0^n}} - x_0^q \times \sqrt{\frac{x_0^r}{y_0^n}} \right)^k \quad [18],$$

where k - is arbitrary odd natural number, and hence infinite set of solutions z, x, y of equations [18], [17], [16], [15] for each odd $n > 2$ in natural (rational) numbers; it is clear that using research of G. Faltings (5) the equation [15] has no solutions in the natural (rational) numbers for arbitrary natural $n > 2$, because equations [15] has no solution for $n = 4$, as we know, has proven by Fermat himself.

7.4. Indeed, if 3^x equation having at least one solution (§ 2,3,4,5,6) such that allow obtain from it type equation [2], [12], [16], [17], [18] then the solutions of are countless.

7.5. To conclude the proof, it remains to note that with not coprime solutions of [15] in (6), 1.2., 2.1.,2.2.,2.3. and (7), 2.6.9. We obtain a complete solution to the Fermat's Last Theorem.

7.6. Indeed, as in the general case, in (6) and (7), if

$$x^n + y^n = z^n,$$

then

$$\begin{aligned} x^n(x^{35}y^{84}z^{20})^n + y^n(x^{35}y^{84}z^{20})^n &= \\ &= z^n(x^{35}y^{84}z^{20})^n \quad [I] \\ [(x^9y^{21}z^5)^4]^n + [(x^7y^{17}z^4)^5]^n &= \\ &= [(x^5y^{12}z^3)^7]^n \quad [II] \\ [I] &= [II]. \end{aligned}$$

$$\begin{aligned} x^n(x^{315}y^{224}z^{160})^n + y^n(x^{315}y^{224}z^{160})^n &= \\ z^n(x^{315}y^{224}z^{160})^n & \quad [III] \end{aligned}$$

$$\begin{aligned}
 & [(x^{79}y^{56}z^{40})^4]^n + [(x^{63}y^{45}z^{32})^5]^n = \\
 & = [(x^{45}y^{32}z^{23})^7]^n \text{ [IV]} \\
 & \text{[III]} = \text{[IV]} \text{ and etc.}
 \end{aligned}$$

These different solutions in parentheses for each natural "n" and each arbitrary fixed triple coprime exponents in the square brackets (4, 5, 7) - are countless.

7.7. From the initial same, but slightly modified considerations that used in 7.6. , we obtain follows identity:

$$\begin{aligned}
 & [x^{3p+1}y^{3q}z^{2(3t-2)}]^{3t-2} + [x^{3p}y^{3q+1}z^{2(3t-2)}]^{3t-2} \equiv \\
 & \equiv [x^{3p}y^{3q}z^{3(2t-1)}]^{3t-2} \text{ [V]},
 \end{aligned}$$

if

$$x^{3t-2} + y^{3t-2} = z^{3t-2}.$$

Here, there are (independent from each other),

$$p = 1,2,3,4, \dots$$

$$q = 1,2,3,4 \dots$$

$$t = 1,2,3,4 \dots$$

$$3t - 2 = n = 1,4,7,10, \dots$$

7.8. The identity of type [V] the same arguments may be obtained for an arbitrary natural "n".

7.9. As a consequence of the preceding paragraphs of the section in (7), 1.2. all members of the sequence

$$\frac{\sqrt[n-1]{(x^{n-1} + y^{n-1})^n}}{\sqrt[n]{(x^n + y^n)^{n-1}}} = \frac{\sqrt[n-1]{z_{n-1}^n}}{\sqrt[n]{z_n^{n-1}}} \text{ [19]}$$

for $n > 2$ are only irrational numbers. Therefore, the proof of insolubility in natural numbers of the equation

$$x^n + y^n = z^n$$

for $n > 2$ in the (7) becomes a complete and fully correct.

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