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The methods of solving equations $A^x + B^y = C^z$ with coprime $\mathrm{A, B, C}$, where $\mathrm{x} \geq 2, \mathrm{y} \geq 2, \mathrm{z} \geq 2$ are natural numbers, equal the two only in one of the three possible cases. The proof of Catalan's Conjecture.

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Abstract. One of the principal problems of the Beal's conjecture, as we see that, is methods for finding a pairwise coprime solution which is defined below. First found methods and identities, allowing receiving infinite number solutions of equations as

$$\mathrm{A}^x + B^y = C^z$$

for coprime integers arranged in a pair $(\mathrm{A, B, C}) = 1$ are natural (whole) numbers, where a fixed permutation $(\mathrm{x, y, z})$ corresponds to each of the permutations $(2,3,4), (2,4,3), (4,3,2)$. ere we obtain also our method and identities of all not recurrent and not coprime solutions of the above type, part of which has already been published, in contrast to the method of obtaining the recurrence not coprime solutions of this type from [(1), W. Sierpiński, p. 21-25, 63]. As the solution of the main problem appeared additional problems that solved by obtained appropriate identities. Given as two equal proofs of Catalan's Conjecture.

Solution

§ 1

Preliminary in this section we obtain some identities that are required in below to achieve the main results.

1.1. Multiplying both sides of Pythagorean equation

$$\mathrm{x}^2 + y^2 = z^2$$

by

$$9(x^4 + y^4 + z^4) \div x^2[9(x^4 + y^4 + z^4)] = z^2[9(x^4 + y^4 + z^4)] \ [\mathbf{1}]$$

Adding term by term, the following equation:

$$(x \times x^2)^2 + [y \times (3z^2 + y^2]^2 = [z \times (3y^2 + z^2]^2 \ [\mathbf{2}],$$

$$[x \times (3z^2 + x^2)]^2 + (y \times y^2)^2 = [z \times (3x^2 + z^2)]^2 \ [\mathbf{3}],$$

$$[x \times (3y^2 - x^2)]^2 + [y \times (3x^2 - y^2)]^2 = (z \times z)^2 \ [\mathbf{4}],$$

we obtain

$$[\mathbf{2}]+[\mathbf{3}]+[\mathbf{4}]=[\mathbf{1}].$$

1.2. We get three identities needed for the task.

1.2.1.

$$(mx \pm ny)^2+(my \mp nx)^2=(m^2+n^2)(x^2+y^2)=(m^2+n^2)^3.$$

Here,

$$x=m^2-n^2; y=2mn; z=m^2+n^2, (m,n)=1,$$

m, n -are arbitrary natural numbers.

1.2.2.

$$(m^2+n^2)^3 \equiv [m(m^2-n^2) \pm n(2mn)]^2+[m(2mn \mp n(m^2-n^2)]^2\ [\mathbf{5}]$$

$$(m^2+n^2)^3 \equiv [m(m^2+n^2)]^2+[n(m^2+n^2)]^2\ \ [\mathbf{5_1}]$$

$$(m^2+n^2)^3 \equiv [m(m^2-3n^2)]^2+[n(3m^2-n^2)]^2\ \ [\mathbf{6}].$$

1.2.3. Follows from $[\mathbf{6}]$ for $n=ni$, where $i=\sqrt{-1}$,

$$(m^2-n^2)^3 \equiv [m(m^2+3n^2)^2-[n(3m^2+n^2)]^2\ \ [\mathbf{7}].$$

§ 2

2.1. From $[\mathbf{7}]$ we need some new notations $m=y$ and $n=1$ then we obtain the identity:

$$[y(y^2+3)]^2-(3y^2+1)^2 \equiv (y^2-1)^3 \quad [\mathbf{8}].$$

2.2. Let

$$3y^2+1=x^2\ \ [\mathbf{9}].$$

Therefore,

$$[y(y^2+3)^2-x^4=(y^2-1)^3\ [\mathbf{10}]$$

and

$$x^2-3y^2=1\ [\mathbf{11}].$$

2.3. Using W. Sierpiński [(1),p. 29-30] we get solutions in natural (whole) number of equations types $[\mathbf{11}]$ and, accordingly $[\mathbf{10}]$.

2.3.1.

$$\begin{cases} x_{n+1}=x_1x_n+3y_1y_n \\ y_{n+1}=y_1x_n+x_1y_n \end{cases}\ [\mathbf{12}],$$

where $n = 1,2,3,4,\ldots.$

1)

$$x_1 = 2;\ y_1 = 1;\ 2^2 - 3 \times 1^2 = 1$$

2)

$$x_{1+1} = x_2 = x_1 x_1 + 3y_1 y_1 = 2 \times 2 + 3 \times 1 \times 1 = 7$$

$$y_{1+1} = 1 \times 1 + 3 \times 1 \times 1 = 4;\ 7^2 - 3 \times 4^2 = 1.$$

3)

$$x_{2+1} = x_3 = 2 \times 7 + 3 \times 1 \times 4 = 26;\ y_3 = 1 \times 7 + 2 \times 4 = 15$$

$$26^2 - 3 \times 15^2 = 1$$

4)

$$x_{3+1} = x_4 = 2 \times 26 + 3 \times 1 \times 15 = 97;\ y_4 = 1 \times 26 + 2 \times 15 = 56$$

$$97^2 - 3 \times 56^2 = 1$$

2.3.2. From $[\mathbf{10}]$:

1)

$$[4 \times (4^2 + 3)]^2 - 7^4 = (4^2 - 1)^3$$

$$76^2 - 7^4 = 15^3$$

2)

$$[15 \times (15^2 + 3)]^2 - 26^4 = (15^2 - 1)^3$$

$$3420^2 - 26^4 = 224^3$$

3)

$$[56(56^2 + 3)]^2 - 97^4 = (56^2 - 1)^3$$

$$175784^2 - 97^4 = 3135^3$$

and etc.

2.4. Infinite set of all pairwise coprime solutions of equations $[\mathbf{10}]$ and $[\mathbf{11}]$ we obtain for odd "$\boldsymbol{x}$".

We have

$$x_2^2 - 3y_2^2 = 1$$

and

$$(x_2^2 + 3y_2^2)^2 - 3(2x_2y_2)^2 = 1,$$

where

$$x_4 = x_2^2 + 3y_2^2;\ y_4 = 2x_2y_2,$$

In other words, if

$$x_2 = 7; y_2 = 4,$$

then

$$x_4 = 7^2 + 3 \times 4^2 = 97;\ y_4 = 2 \times 7 \times 4 = 56;$$

and etc.

2.4.1. In general, if

$$x_{2n}^2 - 3y_{2n}^2 = 1,$$

then

$$(x_{2n}^2 + 3y_{2n}^2)^2 - 3(2x_{2n}y_{2n})^2 = 1$$

and

$$x_{2n+2} = x_{2n}^2 + 3y_{2n}^2;\ y_{2n+2} = 2 \times x_{2n} \times y_{2n},$$

where

$$n = 1,2,3,4 \ldots$$

$$x_{2\times2+2} = x_6 = 97^2 + 3 \times 56^2 = 18817$$

$$y_{2\times2+2} = y_6 = 2 \times 97 \times 56 = 10864$$

$$18817^2 - 3 \times 10864^2 = 1.$$

2.5. Accordingly, thus recurrently, we obtain infinite set of all pairwise coprime solutions of $[\mathbf{10}]$**.**

§ 3

3.1. Let us give another version of getting all the solutions of equation

$$x^2 - 3y^2 = 1$$

3.2. There is an identity:

$$(k^2 \mp 1)^2 - (k^2 \mp 2)k^2 \equiv 1\ [\mathbf{13}].$$

With the upper signs in$[\mathbf{13}]$ and

1) $k = 1 \times \sqrt{3}$ $\quad 2^2 - (3-2) \times 3 \times 1^2 = 1,$

$$2^2 - 3 \times 1^2 = 1$$

2)

$$(2^2 + 3 \times 1^2)^2 - 3 \times (2 \times 2 \times 1)^2 = 1, \quad 7^2 - 3 \times 4^2 = 1$$

and etc , see **2.4.** and **2.5.**

3.3. Comment.

For $k = 2$ $\ 3^2 - 2^3 = 1\ [\mathbf{14}].$

The identity $[\mathbf{13}]$ using to prove the Catalan's Conjecture:

“The equation

$$x^z - y^z = 1$$

has a unique solution

$$3^2 - 2^3 = 1".$$

Proof.

3.3.1. We first prove that the equation

$$(k^2 - 1)^2 - k^3 = 1$$

has a unique solution [**14**].

Let from [**13**] it follows that

$$k^2 - 2 = k$$

then,

$$k = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 2} = \frac{1}{2} \pm \frac{3}{2}; k_1 = 2; k_2 = -1;$$

$$(2^2 - 1)^2 - (2^2 - 2)2^2 = 1;$$

and

$$3^2 - 2^3 = 1$$

3.3.2. Using our proof in Bulletin of Mathematical Sciences & Applications ISSN: 2278-9634 Vol. 2 No. 3 (2013), pp. 53-60(2) exponent of one of the elements

$$A^x + B^y = C^z$$

can not be more than two in[**14**].

3.3.3. Let

$$k' = k^{2n};$$

Then,

$$k^{2n} - 2 = k^m,$$

where

$$m < 2n, k^{2n} - k^m = 2,\ k^m(k^{2n-m} - 1) = 2$$

and

$$k = 2; n = 1, m = 1,$$

This completes the proof.

3.4. The second variant of the proof of Catalan's Conjecture

Proof.

3.4.1. Suppose

$$m - n = 1 \; [\mathbf{I}].$$

then,

$$m + n = 2p + 1 \; [\mathbf{II}]$$

where

$$p = 1,2,3,4, \dots \dots$$

Multiplying [**I**] by [**II**]**.** we obviously have,

$$m^2 - n^2 = 2p + 1; \; m^2 - (n^2 + 2p) = 1 \; [\mathbf{III}]$$

from [**III**]

$$n^2 + 2p = a^t, p = \frac{a^t - n^2}{2} \; [\mathbf{IV}].$$

Then, follows that [**II**] using [**IV**]

$$m = a^t - n^2 - n + 1 \; [\mathbf{V}].$$

Subtracting from the left and right sides of equation [**V**] "n".

We get,

$$m - n = 1 = a^t - n^2 - 2n + 1$$

and

$$n^2 + 2n - a^t = 0, n = -1 \pm \sqrt{1 + a^t} \; .$$

3.4.2. To prove

$$1 + 2^3 = 3^2$$

is a unique solution of equation

$$1 + a^t = b^2$$

for natural $t > 1$.

Let

$$t = c + k,$$

then,

$$a^t = (b-1)(b+1) = a^c a^k$$

and

$$a^c = b-1; a^k = b+1;\ a^k + a^c = 2b,$$

$$a^k - a^c = 2, b = a^c + 1{,}1 = (a^c+1)^2 - a^{c+k}$$

$$a^{2c} + 2a^c - a^{c+k} = 0, a^c(a^c + 2 - a^k) = 0$$

and

$$a^k - a^c = 2\ [\boldsymbol{VII}].$$

3.4.2.1.

$$a = 2q + 1.$$

Let us prove that, from $[\boldsymbol{VII}]$ for $k > c$

$$(2q+1)^c[(2q+1)^{k-c} - 1] = 2$$

and

$$a = 2q + 1 = 3; q = 1; 3 - 1 = 2$$

3.4.2.2.

$$a = 2q; a = 2; k = 2; c = 1$$

Unique solution $[\boldsymbol{VII}]$

$2^2 - 2^1 = 2,$ here $a = 2; k = 2; c = 1.$

Similarly,

$$t = 3; a = 2; b = 3$$

It is clear that $1 + 2^3 = 3^2$- is a unique solution of equation $1 + a^t = b^2$ for natural $t > 1$.

3.4.3. Without loss of generality it can be assumed that from $[\boldsymbol{VI}]$

$$n = -1 \pm 3;\ n_1 = 2;\ n_2 = -4.$$

This means that, from $[\boldsymbol{V}]$ $m = 3.$

3.4.4. Using W. Sierpiński [(1), p. 76,77] proved that the equation

$$x^z - 2^t = 1$$

In natural numbers $x, z, t,$ >1, has only one solution

$$x = 3; z = 2; t = 3.$$

3.5. The application to the proof of Theorem Catalana. W. Sierpiński, "On solution of equations in integers" (Russian), Fizmatgiz, Moscow, 1961, **p.76-77**

INTERNET LIBRARY: http://ilib.mccme.ru/djvu/serp-int_eq.htm

§ **4**

4.1. From $[\mathbf{7}]$ for $n = n^2$ we have:

$$(m^2 - n^4)^3 \equiv [m(m^2 + 3n^4)]^2 - [n^2(3m^2 + n^4)]^2 \ [\mathbf{15}].$$

Let

$$m = 2ab; 3m^2 = 3 \times 4a^2b^2 = (6a)(2ab^2) = (2a)(6ab^2) =$$

$$= (6a^2)(2b^2) = (2a^2)(6b^2) \ [\mathbf{16}].$$

$$3m^2 + n^4 = q^2;\ 3m^2 = q^2 - n^4 = (q - n^2)(q + n^2) \ [\mathbf{17}].$$

4.1.1. It follows from $[\mathbf{16}]$ and $[\mathbf{17}]$

$$\begin{cases} 6a = q - n^2 \\ 2ab^2 = q + n^2 \end{cases}$$

$$n^2 = a(b^2 - 3); n = b^2 - 3; a = b^2 - 3$$

$$q = a(b^2 + 3) = b^4 - 9$$

$$m = 2b(b^2 - 3);$$

4.1.2. From $[\mathbf{15}]$ and with respect to **4.1.1.**

$$\{2b(b^2 - 3)^3[3(b^2 - 3)^2 + 4b^2]\}^2 - [(b^2 - 3)^2(b^2 + 3)]^4 \equiv$$

$$\{(b^2 - 3)^2[4b^2 - (b^2 - 3)^2]\}^3 \quad [\mathbf{18}].$$

For $b = 2$

$$76^2 = 15^3 + 7^4$$

is unique pairwise coprime solution to this version. Other solutions to this version are not pairwise coprime $(b \neq 1; b \neq 3)$, "b" - are arbitrary natural (integer) numbers.

For $b = 4$

$$10035896^2 + 17745^3 = 3211^4;$$

$$(2^3 \times 13^3 \times 571)^2 + (3 \times 5 \times 7 \times 13^2)^3 = (13^2 \times 19)^4$$

and etc.

4.2. Let $[\mathbf{18}]$ be

$$b^2 + 3 = c^2.$$

We obtain:

$$(\alpha \pm 2\beta)^2 + 3(\alpha^2 - \beta^2) \equiv (2\alpha \pm \beta)^2. [\mathbf{19}]$$

and

$$\frac{(\alpha \pm 2\beta)^2}{\alpha^2 - \beta^2} + 3 \equiv \frac{(2\alpha \pm \beta)^2}{\alpha^2 - \beta^2}$$

Since $[\mathbf{2}]$ we have

$$\alpha = z(3y^2 + z^2);\ \beta = y(3z^2 + y^2).$$

Therefore,

$$\alpha^2 - \beta^2 = x^6$$

where

$$x = m^2 - n^2; y = 2mn; z = m^2 + n^2.$$

It follows that,

$$b^2 + 3 = (\frac{\alpha \pm 2\beta}{x^3})^2 + 3 \equiv \left(\frac{2\alpha \pm \beta}{x^3}\right)^2 = c^2,$$

$$c = \frac{2\alpha \pm \beta}{x^3}; b = \frac{\alpha \pm \beta}{x^3}.$$

Remove in $[\mathbf{18}]$ the denominator, after appropriate substitutions; we obtain the identity of the type

$$A^2 + B^8 \equiv C^3.$$

§ 5

Using the methods outlined and possible variants (with$[\mathbf{2}]$,$[\mathbf{3}]$,$[\mathbf{4}]$,$[\mathbf{5}]$,$[\mathbf{6}]$, $[\mathbf{7}]$,$[\mathbf{16}]$**)**, we get infinite set of solutions of equations of type

$$A^2 + B^4 = C^3, A^2 + B^8 = C^3$$

in natural numbers as pairwise coprime, and not coprime.

References:

[1] W. Sierpiński, "On solution of equations in integers" (Russian), Fizmatgiz, Moscow, 1961. INTERNET LIBRARY: http://ilib.mccme.ru/djvu/serp-int_eq.htm

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