NOTES ON BIPOLAR-VALUED FUZZY SUBGROUPS OF A GROUP
M.S.ANITHA¹, K.L.MURUGANANTHA PRASAD² & K.ARJUNAN²

¹Research Scholar, Department of Mathematics, H.H. The Rajahs College, Pudukkottai – 622001 Tamilnadu, India.
²Department of Mathematics, H.H. The Rajahs College, Pudukkottai – 622001 Tamilnadu, India.

Keywords: Bipolar-valued fuzzy set, bipolar-valued fuzzy subgroup, product, bipolar-valued fuzzy normal subgroup, bipolar-valued fuzzy coset.

Abstract: In this paper, we study some of the properties of bipolar-valued fuzzy subgroup and prove some results on these.

Introduction
In 1965, Zadeh [12] introduced the notion of a fuzzy subset of a set, fuzzy sets are a kind of useful mathematical structure to represent a collection of objects whose boundary is vague. Since then it has become a vigorous area of research in different domains, there have been a number of generalizations of this fundamental concept such as intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets, soft sets etc [5]. Lee [7] introduced the notion of bipolar-valued fuzzy sets. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval [0, 1] to [−1, 1]. In a bipolar-valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degree (0, 1] indicates that elements somewhat satisfy the property and the membership degree [−1, 0) indicates that elements somewhat satisfy the implicit counter property. Bipolar-valued fuzzy sets and intuitionistic fuzzy sets look similar each other. However, they are different each other [7, 8]. We introduce the concept of bipolar-valued fuzzy subgroup and established some results.

1. Preliminaries
1.1 Definition: A bipolar-valued fuzzy set (BVFS) A in X is defined as an object of the form A = (A+ (x), A− (x) ▷ x ∈ X), where A+ : X→ [0, 1] and A− : X→ [−1, 0]. The positive membership degree A+ (x) denotes the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set A and the negative membership degree A− (x) denotes the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar-valued fuzzy set A. If A+ (x) ≠ 0 and A− (x) = 0, it is the situation that x is regarded as having only positive satisfaction for A and if A+ (x) = 0 and A− (x) ≠ 0, it is the situation that x does not satisfy the property of A, but somewhat satisfies the counter-property of A. It is possible for an element x to be such that A+ (x) ≠ 0 and A− (x) ≠ 0 when the membership function of the property overlaps that of its counter property over some portion of X.

1.1 Example: A = {< a, 0.5, −0.3 >, < b, 0.1, −0.7 >, < c, 0.5, −0.4 >} is a bipolar-valued fuzzy subset of X = {a, b, c}.

1.2 Definition: Let G be a group. A bipolar-valued fuzzy subset A of G is said to be a bipolar-valued fuzzy subgroup of G (BVFSG) if the following conditions are satisfied,

(i) A+ (xy) ≥ min{ A+ (x), A+ (y) },
(ii) A+ (x−1) ≥ A+ (x),
(iii) A− (xy) ≤ max{ A− (x), A− (y) },
(iv) A− (x−1) ≥ A− (x).
(iv) $A^-(x^{-1}) \leq A^-(x)$, for all $x$ and $y$ in $G$.

1.2 Example: Let $G = \{1, -1, i, -i\}$ be a group with respect to the ordinary multiplication. Then $A = \{< 1, 0.5, -0.6 >, < -1, 0.4, -0.5 >, < i, 0.2, -0.4 >, < -i, 0.2, -0.4 >\}$ is a bipolar-valued fuzzy subgroup of $G$.

1.3 Definition: Let $A = \langle A^+, A^- \rangle$ and $B = \langle B^+, B^- \rangle$ be any two bipolar-valued fuzzy subsets of sets $G$ and $H$, respectively. The product of $A$ and $B$, denoted by $A \times B$, is defined as $A \times B = \{ (x, y), (A \times B)^+(x, y), (A \times B)^-(x, y) \}$ for all $x$ in $G$ and $y$ in $H$, where $(A \times B)^+(x, y) = \min \{ A^+(x), B^+(y) \}$ and $(A \times B)^-(x, y) = \max \{ A^-(x), B^-(y) \}$, for all $x$ in $G$ and $y$ in $H$.

1.4 Definition: Let $G$ be a group. A bipolar-valued fuzzy subgroup $A$ of $G$ is said to be a bipolar-valued fuzzy normal subgroup of $G$ if

(i) $A^+(xy) = A^+(yx)$, for all $x$ and $y$ in $G$.

(ii) $A^-(xy) = A^-(yx)$, for all $x$ and $y$ in $G$.

1.5 Definition: Let $A$ be a bipolar-valued fuzzy subgroup of a group $G$. For any $a \in G$, $aA$ defined by $(aA)(x) = A(a^{-1}x)$ and $(Aa)(x) = A(a^{-1}x)$, for every $x \in G$ is called the bipolar-valued fuzzy coset of the group $G$.

1.6 Definition: Let $A$ be a bipolar-valued fuzzy subgroup of a group $G$ and $H = \{ x \in G / A^+(x) = A^+(e) \text{ and } A^-(x) = A^-(e) \}$, then $o(A)$, order of $A$ is defined as $o(A) = o(H)$.

1.7 Definition: Let $A$ and $B$ be two bipolar-valued fuzzy subgroups of a group $G$. Then $A$ and $B$ are said to be conjugate bipolar-valued fuzzy subgroup of $G$ if for some $g \in G$, $A^+(x) = B^+(g^{-1}xg)$ and $A^-(x) = B^-(g^{-1}xg)$, for every $x \in G$.

1.8 Definition: Let $A$ be a bipolar-valued fuzzy subgroup of a group $G$. Then for any $a$ and $b$ in $G$, a bipolar-valued fuzzy middle coset $aAb$ of $G$ is defined by $(aA \ast b)(x) = A^+(a^{-1}xb^{-1})$ and $(aA \ast b)(x) = A^-(a^{-1}xb^{-1})$, for every $x \in G$.

2. PROPERTIES:

2.1 Theorem: Let $A = \langle A^+, A^- \rangle$ be a bipolar-valued fuzzy subgroup of a group $G$. If $A^+(x) < A^+(y)$ and $A^-(x) > A^-(y)$, for some $x$ and $y$ in $G$, then

(i) $A^+(xy) = A^+(x) = A^+(yx)$ and (ii) $A^-(xy) = A^-(x) = A^-(yx)$.

proof: Let $A = \langle A^+, A^- \rangle$ be a bipolar-valued fuzzy subgroup of a group $G$. Let $A^+(x) < A^+(y)$ and $A^-(x) > A^-(y)$, for some $x$ and $y$ in $G$.

Now, $A^+(xy) \geq \min \{ A^+(x), A^+(y) \} = A^+(x)$; and

$A^+(x) = A^+(xyy^{-1}) \geq \min \{ A^+(xy), A^+(y) \} = A^+(xy)$.

Also, $A^+(xy) \geq \min \{ A^+(y), A^+(x) \} = A^+(x)$; and

$A^+(x) = A^+(yx^{-1}y) \geq \min \{ A^+(y), A^+(yx) \} = A^+(yx)$.

Therefore $A^+(xy) = A^+(x) = A^+(yx)$. Hence (i) is proved.

Now, $A^-(xy) \leq \max \{ A^-(x), A^-(y) \} = A^-(x)$; and

$A^-(x) = A^-(xyy^{-1}) \leq \max \{ A^-(xy), A^-(y) \} = A^-(xy)$.

Also, $A^-(xy) \leq \max \{ A^-(y), A^-(x) \} = A^-(x)$; and

$A^-(x) = A^-(yx^{-1}y) \leq \max \{ A^-(y), A^-(yx) \} = A^-(yx)$.

Therefore $A^-(xy) = A^-(x) = A^-(yx)$. Hence (ii) is proved.
2.2 Theorem: Let $A=\langle A^+, A^- \rangle$ be a bipolar-valued fuzzy subgroup of a group $G$. If $A^+(x) < A^+(y)$ and $A^-(x) < A^-(y)$, for some $x$ and $y$ in $G$, then

(i) $A^+(xy) = A^+(x) = A^+(yx)$ and (ii) $A^-(xy) = A^-(y) = A^-(yx)$.

**proof:** Let $A=\langle A^+, A^- \rangle$ be a bipolar-valued fuzzy subgroup of a group $G$. Let $A^+(x) < A^+(y)$ and $A^-(x) < A^-(y)$, for some $x$ and $y$ in $G$.

Now, $A^+(xy) \geq \min \{ A^+(x), A^+(y) \} = A^+(x)$; and

$A^+(x) = A^+(xyy^{-1}) \geq \min \{ A^+(xy), A^-(y) \} = A^+(xy)$.

And, $A^-(xy) \geq \min \{ A^-(y), A^-(x) \} = A^+(x)$; and

$A^-(x) = A^-(y^{-1}yx) \geq \min \{ A^-(y), A^-(yx) \} = A^-(yx)$.

Therefore $A^+(xy) = A^+(x) = A^+(yx)$. Hence (i) is proved.

Now, $A^-(xy) \leq \max \{ A^-(x), A^-(y) \} = A^-(y)$; and

$A^-(y) = A^-(x^{-1}xy) \leq \max \{ A^-(x), A^-(xy) \} = A^-(xy)$.

And, $A^-(yx) \leq \max \{ A^-(y), A^-(x) \} = A^-(y)$; and

$A^-(y) = A^-(yx^{-1}) \leq \max \{ A^-(yx), A^-(x) \} = A^-(yx)$.

Therefore $A^-(xy) = A^-(y) = A^-(yx)$. Hence (ii) is proved.

2.3 Theorem: Let $A=\langle A^+, A^- \rangle$ be a bipolar-valued fuzzy subgroup of a group $G$. If $A^+(x) > A^+(y)$ and $A^-(x) > A^-(y)$, for some $x$ and $y$ in $G$, then

(i) $A^+(xy) = A^+(y) = A^+(yx)$ and (ii) $A^-(xy) = A^-(x) = A^-(yx)$.

**Proof:** It is trivial.

2.4 Theorem: Let $A=\langle A^+, A^- \rangle$ be a bipolar-valued fuzzy subgroup of a group $G$. If $A^+(x) > A^+(y)$ and $A^-(x) < A^-(y)$, for some $x$ and $y$ in $G$, then

(i) $A^+(xy) = A^+(y) = A^+(yx)$ and (ii) $A^-(xy) = A^-(y) = A^-(yx)$.

**Proof:** It is trivial.

2.5 Theorem: Let $A=\langle A^+, A^- \rangle$ be a bipolar-valued fuzzy subgroup of a finite group $G$, then $o(A)$ divides $o(G)$.

**Proof:** Let $A$ be a bipolar-valued fuzzy subgroup of a finite group $G$ with $e$ as its identity element. Clearly $H = \{ x \in G / A^+(x) = A^+(e) \}$ is a subgroup of $G$. By Lagrange's theorem $o(H) / o(G)$. Hence by the definition of the order of the bipolar-valued fuzzy subgroup of the group $G$, we have $o(A) / o(G)$.

2.6 Theorem: Let $A=\langle A^+, A^- \rangle$ and $B=\langle B^+, B^- \rangle$ be two bipolar-valued fuzzy subsets of an abelian group $G$. Then $A$ and $B$ are conjugate bipolar-valued fuzzy subsets of the group $G$ if and only if $A = B$.

**Proof:** Let $A$ and $B$ be conjugate bipolar-valued fuzzy subsets of group $G$, then for some $y \in G$, we have $A^+(x) = B^+(y^{-1}xy) = B^-(y^{-1}yx) = B^+(ex) = B^+(x)$.

Therefore $A^+(x) = B^+(x)$.

And, $A^-(x) = B^-(y^{-1}xy) = B^-(y^{-1}yx) = B^-(ex) = B^-(x)$.

Therefore $A^-(x) = B^-(x)$. Hence $A = B$.

Conversely if $A = B$ then for the identity element $e$ of group $G$, we have $A^+(x) = B^+(e^{-1}xe)$ and $A^-(x) = B^-(e^{-1}xe)$ for every $x \in G$.

Hence $A$ and $B$ are conjugate bipolar-valued fuzzy subsets of the group $G$. 


2.7 Theorem: If \( A = \langle A^+, A^- \rangle \) and \( B = \langle B^+, B^- \rangle \) are conjugate bipolar-valued fuzzy subgroups of the group \( G \), then \( o(A) = o(B) \).

Proof: Let \( A \) and \( B \) are conjugate bipolar-valued fuzzy subgroups of the group \( G \). 
Now, \( o(A) = \text{order of} \{x \in G / A^+(x) = A^+(e) \text{ and } A^-(x) = A^-(e) \} \) 
\[ = \text{order of} \{x \in G / B^+(y^{-1}xy) = B^+(y^{-1}ey) \text{ and } B^-(y^{-1}xy) = B^-(y^{-1}ey) \} \] 
\[ = \text{order of} \{x \in G / B^+(x) = B^+(e) \text{ and } B^-(x) = B^-(e) \} \] 
\[ = o(B). \text{ Hence } o(A) = o(B). \]

2.8 Theorem: Let \( A = \langle A^+, A^- \rangle \) be a bipolar-valued fuzzy normal subgroup of a group \( G \). Then for any \( y \in G \) we have \( A^+(xy^{-1}) = A^+(y^{-1}xy) \) and \( A^-(xy^{-1}) = A^-(y^{-1}xy) \), for every \( x \in G \).

Proof: Let \( A \) be a bipolar-valued fuzzy normal subgroup of a group \( G \).

For any \( y \) in \( G \). Then we have, \( A^+(xy^{-1}) = A^+(x) = A^+(x) = A^+(y^{-1}xy) \).

Therefore \( A^+(xy^{-1}) = A^+(y^{-1}xy) \).

And, \( A^-((xy^{-1}) = A^-(x) = A^-(x) = A^-(y^{-1}xy) \).

Therefore \( A^-(xy^{-1}) = A^-((y^{-1}xy) \).

2.9 Theorem: A bipolar-valued fuzzy subgroup \( A = \langle A^+, A^- \rangle \) of a group \( G \) is normalized if and only if \( A^+(e) = 1 \) and \( A^-(e) = 0 \), where \( e \) is the identity element of the group \( G \).

Proof: If \( A \) is normalized then there exists \( x \in G \) such that \( A^+(x) = 1 \) and \( A^-(x) = 0 \), but by properties of a bipolar-valued fuzzy subgroup \( A \) of the group \( G \), \( A^+(x) \leq A^+(e) \) and \( A^-(x) \geq A^-(e) \) for every \( x \in G \).

since \( A^+(x) = 1 \) and \( A^-(x) = 0 \) and \( A^+(x) \leq A^+(e) \) and \( A^-(x) \geq A^-(e) \).

Therefore \( 1 \leq A^+(e) \) and \( 0 \geq A^-(e) \). But \( 1 \geq A^+(e) \) and \( 0 \leq A^-(e) \).

Hence \( A^+(e) = 1 \) and \( A^-(e) = 0 \).

Conversely if \( A^+(e) = 1 \) and \( A^-(e) = 0 \), then by the definition of normalized bipolar-valued fuzzy subset \( A \) is normalized.

2.10 Theorem: If \( A = \langle A^+, A^- \rangle \) is a bipolar-valued fuzzy subgroup of a group \( G \), then for any \( a \) in \( G \) the bipolar-valued fuzzy middle coset \( aAa^{-1} \) of \( A \) is also a bipolar-valued fuzzy subgroup of a group \( G \).

Proof: Let \( A \) is a bipolar-valued fuzzy subgroup of a group \( G \) and \( a \) in \( G \). To prove \( aAa^{-1} = (x, aAa^{-1}, aAa^{-1}) \) is a bipolar-valued fuzzy subgroup of \( G \). Let \( x \) and \( y \) in \( G \).

Then \( (aAa^{-1})(xy^{-1}) = A^+(a^{-1}xy^{-1}a) \)
\[ = A^+(a^{-1}xaa^{-1}y^{-1}a) \]
\[ = A^+(a^{-1}xa(a^{-1}ya)) \]
\[ \geq \min \{ A^+(a^{-1}xa), A^+(a^{-1}ya) \} \]
\[ = \min \{ (aAa^{-1})(x), (aAa^{-1})(y) \}. \]

Therefore \( (aAa^{-1})(xy^{-1}) \geq \min \{ (aAa^{-1})(x), (aAa^{-1})(y) \}. \)
And \( (aAa^{-1})(xy^{-1}) = A^+(a^{-1}xy^{-1}a) \)
\[ = A^-(a^{-1}xa^{-1}y^{-1}a) \]
\[ A = A^{-1}(a^{-1}xa^{-1}) \]
\[ \leq \max \{ A^{-1}(a^{-1}xa), A^{-1}(a^{-1}ya) \} \]
\[ = \max \{ (a A^{-1})(x), (a A^{-1})(y) \}. \]
Therefore \((a A^{-1})(xy^{-1}) \leq \max \{ (a A^{-1})(x), (a A^{-1})(y) \}. \)
Hence \(aAa^{-1}\) is a bipolar-valued fuzzy subgroup of a group \(G\).

2.11 Theorem: Let \(A=\langle A^+, A^-\rangle\) be a bipolar-valued fuzzy subgroup of a group \(G\) and \(aAa^{-1}\) be a bipolar-valued fuzzy middle coset of the group \(G\), then \(o(aAa^{-1}) = o(A)\), for any \(a \in G\).

Proof: Let \(A\) be a bipolar-valued fuzzy subgroup of a group \(G\) and \(a \in G\). By Theorem 2.10, the bipolar-valued fuzzy middle coset \(aAa^{-1}\) is a bipolar-valued fuzzy subgroup of a group \(G\).

Further by the definition of a bipolar-valued fuzzy middle coset of the group \(G\) we have \((A^+)(x) = A^+(a^{-1}xa)\) and \((a A^{-1})(x) = A^{-1}(a^{-1}xa)\), for every \(x \in G\).

Hence for any \(a \in G\), \(A\) and \(aAa^{-1}\) are conjugate bipolar-valued fuzzy subgroup of the group \(G\) as there exists \(a \in G\) such that \((A^+)(x) = A^+(a^{-1}xa)\) and \((a A^{-1})(x) = A^{-1}(a^{-1}xa)\) for every \(x \in G\). By Theorem 2.6, \(o(aAa^{-1}) = o(A)\) for any \(a \in G\).

2.12 Theorem: Let \(A=\langle A^+, A^-\rangle\) be a bipolar-valued fuzzy subgroup of a group \(G\) and \(B=\langle B^+, B^-\rangle\) be a bipolar-valued fuzzy subset of a group \(G\). If \(A\) and \(B\) are conjugate bipolar-valued fuzzy subsets of the group \(G\) then \(B\) is a bipolar-valued fuzzy subgroup of a group \(G\).

Proof: Let \(A\) be a bipolar-valued fuzzy subgroup of a group \(G\) and \(B\) be a bipolar-valued fuzzy subset of a group \(G\). And let \(A\) and \(B\) be conjugate bipolar-valued fuzzy subsets of the group \(G\). To prove \(B\) is a bipolar-valued fuzzy subgroup of the group \(G\).

Let \(x\) and \(y\) in \(G\). Then \(xy^{-1}\) in \(G\).

Now, \(B^+(xy^{-1}) = A^+(g^{-1}xy^{-1}g) = A^+(g^{-1}xg^{-1}y^{-1}g) = A^+(g^{-1}xg^{-1}yg^{-1}g) \geq \min \{ A^+(g^{-1}xg), A^-(g^{-1}y) \}. \)

And \(B^-(xy^{-1}) = A^-(g^{-1}xy^{-1}g) = A^-(g^{-1}xg^{-1}y^{-1}g) = A^-(g^{-1}xg^{-1}yg^{-1}g) \leq \max \{ A^-(g^{-1}xg), A^-(g^{-1}y) \}. \)

Therefore \(B^-(xy^{-1}) \leq \max \{ B^-(x), B^-(y) \}. \) Hence \(B\) is a bipolar-valued fuzzy subgroup of the group \(G\).

2.13 Theorem: Let a bipolar-valued fuzzy subgroup \(A = \langle A^+, A^-\rangle\) of a group \(G\) be conjugate to a bipolar-valued fuzzy subgroup \(M = \langle M^+, M^-\rangle\) of \(G\) and a bipolar-valued fuzzy subgroup \(B = \langle B^+, B^-\rangle\) of a group \(H\) be conjugate to a bipolar-valued fuzzy subgroup \(N = \langle N^+, N^-\rangle\) of \(H\). Then a bipolar-valued fuzzy subgroup \(A \times B = \langle (A \times B)^+, (A \times B)^-\rangle\) of a group \(G \times H\) is conjugate to a bipolar-valued fuzzy subgroup \(M \times N = \langle (M \times N)^+, (M \times N)^-\rangle\) of \(G \times H\).

Proof: Let \(A\) and \(B\) be bipolar-valued fuzzy subgroups of the groups \(G\) and \(H\). Let \(x, x^{-1}\) and \(f\) be in \(G\) and \(y, y^{-1}\) and \(g\) be in \(H\). Then \((x, y), (x^{-1}, y^{-1})\) and \((f, g)\) are in \(G \times H\).

Now, \((A \times B)^+(f, g) = \min \{ A^+(f), B^+(g) \} = \min \{ M^+(xfx^{-1}), N^+(yg) \} \}
\[ = (M \times N)^+[x(yf)(g)(x^{-1}, y^{-1})] \]
\[ = (M \times N)^+[x(yf)(g)(x^{-1}, y^{-1})] \]

Therefore, \((A \times B)^+(f, g) = (M \times N)^+[x(yf)(g)(x^{-1}, y^{-1})] \)

And, \((A \times B)^-(f, g) = \max \{ A^-(f), B^-(g) \} = \max \{ M^-(xfx^{-1}), N^-(yg) \} \}
\[ = (M \times N)^-[x(yf)(g)(x^{-1}, y^{-1})] \]
\[ = (M \times N)^-[x(yf)(g)(x^{-1}, y^{-1})] \]
Therefore, 
\[(A\times B)^{-} (f, g) = (M\times N)^{-} \left[ (x, y)(f, g)(x, y)^{-1} \right].\]

Hence a bipolar-valued fuzzy subgroup \(A\times B\) of a group \(G\times H\) is conjugate to a bipolar-valued fuzzy subgroup \(M\times N\) of \(G\times H\).

References


