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NOTES ON BIPOLAR-VALUED FUZZY SUBGROUPS OF A GROUP

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Abstract: In this paper, we study some of the properties of bipolar-valued fuzzy subgroup and prove some results on these.

Introduction

In 1965, Zadeh [12] introduced the notion of a fuzzy subset of a set, fuzzy sets are a kind of useful mathematical structure to represent a collection of objects whose boundary is vague. Since then it has become a vigorous area of research in different domains, there have been a number of generalizations of this fundamental concept such as intuitionistic fuzzy sets, interval- valued fuzzy sets, vague sets, soft sets etc [5]. Lee [7] introduced the notion of bipolar-valued fuzzy sets. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval [0, 1] to [−1, 1]. In a bipolar-valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degree (0, 1] indicates that elements somewhat satisfy the property and the membership degree [−1, 0) indicates that elements somewhat satisfy the implicit counter property. Bipolar-valued fuzzy sets and intuitionistic fuzzy sets look similar each other. However, they are different each other [7, 8]. We introduce the concept of bipolar-valued fuzzy subgroup and established some results.

1. Preliminaries

1.1 Definition: A bipolar-valued fuzzy set (BVFS) A in X is defined as an object of the form $A = \{< x, A^+(x), A^-(x) >/ x\in X\}$, where $A^+ : X\rightarrow [0, 1]$ and $A^- : X\rightarrow [-1, 0]$. The positive membership degree $A^+(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set A and the negative membership degree $A^-(x)$ denotes the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar-valued fuzzy set A. If $A^+(x) \neq 0$ and $A^-(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for A and if $A^+(x) = 0$ and $A^-(x) \neq 0$, it is the situation that x does not satisfy the property of A, but somewhat satisfies the counter property of A. It is possible for an element x to be such that $A^+(x) \neq 0$ and $A^-(x) \neq 0$ when the membership function of the property overlaps that of its counter property over some portion of X.

1.1 Example: $A = \{ < a, 0.5, -0.3 >, < b, 0.1, -0.7 >, < c, 0.5, -0.4 >\}$ is a bipolar-valued fuzzy subset of $X= \{a, b, c \}$.

1.2 Definition: Let G be a group. A bipolar-valued fuzzy subset A of G is said to be a bipolar-valued fuzzy subgroup of G (BVFSG) if the following conditions are satisfied,

(i) $A^+(xy) \geq \min\{ A^+(x), A^+(y) \}$,

(ii) $A^+(x^{-1}) \geq A^+(x)$,

(iii) $A^-(xy) \leq \max\{ A^-(x), A^-(y) \}$,

(iv) $A^{-}(x^{-1}) \leq A^{-}(x)$, for all x and y in G.

1.2 Example: Let G ={1, –1, i, –i } be a group with respect to the ordinary multiplication. Then A = { < 1, 0.5, – 0.6 >, < –1, 0.4, – 0.5 > , < i, 0.2, –0.4 >, < –i, 0.2, –0.4 > } is a bipolar-valued fuzzy subgroup of G.

1.3 Definition: Let A = $\langle A^{+}, A^{-} \rangle$ and B = $\langle B^{+}, B^{-} \rangle$ be any two bipolar-valued fuzzy subsets of sets G and H, respectively. The product of A and B, denoted by A×B, is defined as $A\times B = \{ \langle (x, y), (A\times B)^{+}(x, y), (A\times B)^{-}(x, y) \rangle$ / for all x in G and y in H }, where $(A\times B)^{+}(x, y) = \min \{ A^{+}(x), B^{+}(y) \}$ and $(A\times B)^{-}(x, y) = \max \{ A^{-}(x), B^{-}(y) \}$, for all x in G and y in H.

1.4 Definition: Let G be a group. A bipolar-valued fuzzy subgroup A of G is said to be a bipolar-valued fuzzy normal subgroup of G if

(i) $A^{+}(xy) = A^{+}(yx)$,

(ii) $A^{-}(xy) = A^{-}(yx)$, for all x and y in G.

1.5 Definition: Let A be a bipolar-valued fuzzy subgroup of a group G. For any a∈G, aA defined by $(aA^{+})(x) = A^{+}(a^{-1}x)$ and $(aA^{-})(x) = A^{-}(a^{-1}x)$, for every x∈G is called the bipolar-valued fuzzy coset of the group G.

1.6 Definition: Let A be a bipolar-valued fuzzy subgroup of a group G and $H = \{x\in G \,/\, A^{+}(x) = A^{+}(e)$ and $A^{-}(x) = A^{-}(e) \}$, then o(A), order of A is defined as o(A) = o(H).

1.7 Definition: Let A and B be two bipolar-valued fuzzy subgroups of a group G. Then A and B are said to be conjugate bipolar-valued fuzzy subgroup of G if for some g∈G, $A^{+}(x) = B^{+}(g^{-1}xg)$ and $A^{-}(x) = B^{-}(g^{-1}xg)$, for every x∈G.

1.8 Definition: Let A be a bipolar-valued fuzzy subgroup of a group G. Then for any a and b in G, a bipolar-valued fuzzy middle coset aAb of G is defined by $(aA^{+}b)(x) = A^{+}(a^{-1}xb^{-1})$ and $(aA^{-}b)(x) = A^{-}(a^{-1}x\,b^{-1})$, for every x∈G.

2. PROPERTIES:

2.1 Theorem: Let A = $\langle A^{+}, A^{-} \rangle$ be a bipolar-valued fuzzy subgroup of a group G. If $A^{+}(x) < A^{+}(y)$ and $A^{-}(x) > A^{-}(y)$, for some x and y in G, then

(i) $A^{+}(xy) = A^{+}(x) = A^{+}(yx)$ and (ii) $A^{-}(xy) = A^{-}(x) = A^{-}(yx)$.

proof: Let A = $\langle A^{+}, A^{-} \rangle$ be a bipolar-valued fuzzy subgroup of a group G.
Let $A^{+}(x) < A^{+}(y)$ and $A^{-}(x) > A^{-}(y)$, for some x and y in G.
Now, $A^{+}(xy) \geq \min\{ A^{+}(x), A^{+}(y) \}= A^{+}(x)$; and
$A^{+}(x) = A^{+}(xyy^{-1}) \geq \min\{ A^{+}(xy), A^{+}(y)\} = A^{+}(xy)$.
Also, $A^{+}(yx) \geq \min\{ A^{+}(y), A^{+}(x)\} = A^{+}(x)$; and
$A^{+}(x) = A^{+}(y^{-1}yx) \geq \min\{ A^{+}(y), A^{+}(yx)\} = A^{+}(yx)$.
Therefore $A^{+}(xy) = A^{+}(x) = A^{+}(yx)$. Hence (i) is proved.
Now, $A^{-}(xy) \leq \max \{ A^{-}(x), A^{-}(y) \}= A^{-}(x)$; and
$A^{-}(x) = A^{-}(xyy^{-1}) \leq \max \{ A^{-}(xy), A^{-}(y) \}= A^{-}(xy)$.
Also, $A^{-}(yx) \leq \max \{ A^{-}(y), A^{-}(x) \}= A^{-}(x)$; and
$A^{-}(x) = A^{-}(y^{-1}yx) \leq \max \{ A^{-}(y), A^{-}(yx) \}= A^{-}(yx)$.
Therefore $A^{-}(xy) = A^{-}(x) = A^{-}(yx)$. Hence (ii) is proved.

2.2 Theorem: Let $A=\langle A^+, A^- \rangle$ be a bipolar-valued fuzzy subgroup of a group G. If $A^+(x) < A^+(y)$ and $A^-(x) < A^-(y)$, for some x and y in G, then

(i) $A^+(xy) = A^+(x) = A^+(yx)$ and (ii) $A^-(xy) = A^-(y) = A^-(yx)$.

proof: Let $A=\langle A^+, A^- \rangle$ be a bipolar-valued fuzzy subgroup of a group G.
Let $A^+(x) < A^+(y)$ and $A^-(x) < A^-(y)$, for some x and y in G.
Now, $A^+(xy) \geq \min\{ A^+(x), A^+(y)\} = A^+(x)$; and
$A^+(x) = A^+(xyy^{-1}) \geq \min\{ A^+(xy), A^+(y)\} = A^+(xy)$.
And, $A^+(yx) \geq \min\{ A^+(y), A^+(x) \}= A^+(x)$; and
$A^+(x) = A^+(y^{-1}yx) \geq \min\{ A^+(y), A^+(yx) \}= A^+(yx)$.
Therefore $A^+(xy) = A^+(x) = A^+(yx)$. Hence (i) is proved .
Now, $A^-(xy) \leq \max \{ A^-(x), A^-(y) \}= A^-(y)$; and
$A^-(y) = A^-(x^{-1}xy) \leq \max \{ A^-(x), A^-(xy) \}= A^-(xy)$.
And, $A^-(yx) \leq \max \{ A^-(y), A^-(x) \}= A^-(y)$; and
$A^-(y) = A^-(yxx^{-1}) \leq \max \{ A^-(yx), A^-(x) \}= A^-(yx)$.
Therefore $A^-(xy) = A^-(y) = A^-(yx)$. Hence (ii) is proved.

2.3 Theorem: Let $A = \langle A^+, A^- \rangle$ be a bipolar-valued fuzzy subgroup of a group G. If

$A^+(x) > A^+(y)$ and $A^-(x) > A^-(y)$, for some x and y in G, then

(i) $A^+(xy) = A^+(y) = A^+(yx)$ and (ii) $A^-(xy) = A^-(x) = A^-(yx)$. **Proof:** It is trivial.

2.4 Theorem: Let $A = \langle A^+, A^- \rangle$ be a bipolar-valued fuzzy subgroup of a group G. If $A^+(x) > A^+(y)$ and $A^-(x) < A^-(y)$, for some x and y in G, then

(i) $A^+(xy) = A^+(y) = A^+(yx)$ and (ii) $A^-(xy) = A^-(y) = A^-(yx)$.

Proof: It is trivial.

2.5 Theorem: Let $A = \langle A^+, A^- \rangle$ be a bipolar-valued fuzzy subgroup of a finite group G, then o(A) divides o(G).

Proof: Let A be a bipolar-valued fuzzy subgroup of a finite group G with e as its identity element. Clearly $H = \{ x\in G \ / \ A^+(x) = A^+(e) \text{ and } A^-(x) = A^-(e) \}$ is a subgroup of the group G. By Lagranges theorem o(H) | o(G).
Hence by the definition of the order of the bipolar-valued fuzzy subgroup of the group G, we have o(A) | o(G).

2.6 Theorem: Let $A= \langle A^+, A^- \rangle$ and $B= \langle B^+, B^- \rangle$ be two bipolar-valued fuzzy subsets of a abelian group G. Then A and B are conjugate bipolar-valued fuzzy subsets of the group G if and only if A = B.

Proof: Let A and B be conjugate bipolar-valued fuzzy subsets of group G, then for some $y\in G$, we have $A^+(x) = B^+(y^{-1}xy) = B^+(y^{-1}yx) = B^+(ex) = B^+(x)$.

Therefore $A^+(x) = B^+(x)$.
And, $A^-(x) = B^-(y^{-1}xy) = B^-(y^{-1}yx) = B^-(ex) = B^-(x)$.
Therefore $A^-(x) = B^-(x)$. Hence A = B.
Conversely if A = B then for the identity element e of group G,
we have $A^+(x) = B^+(e^{-1}xe)$ and $A^-(x) = B^-(e^{-1}xe)$ for every $x\in G$.
Hence A and B are conjugate bipolar-valued fuzzy subsets of the group G.

2.7 Theorem: If A= $\langle A^+, A^- \rangle$ and B= $\langle B^+, B^- \rangle$ are conjugate bipolar-valued fuzzy subgroups of the group G, then o(A) = o(B).

Proof: Let A and B are conjugate bipolar-valued fuzzy subgroups of the group G.

Now, o(A) = order of $\{ x \in G \,/\, A^+(x) = A^+(e)$ and $A^-(x) = A^-(e) \}$

$= \text{order of } \{ x \in G \,/\, B^+(y^{-1}xy) = B^+(y^{-1}ey) \text{ and } B^-(y^{-1}xy) = B^-(y^{-1}ey) \}$

$= \text{order of } \{ x \in G \,/\, B^+(x) = B^+(e) \text{ and } B^-(x) = B^-(e) \}$

= o(B). Hence o(A) = o(B).

2.8 Theorem: Let A= $\langle A^+, A^- \rangle$ be a bipolar-valued fuzzy normal subgroup of a group G. Then for any y in G we have $A^+(yxy^{-1}) = A^+(y^{-1}xy)$ and $A^-(yxy^{-1}) = A^-(y^{-1}xy)$, for every $x \in G$.

Proof: Let A be a bipolar-valued fuzzy normal subgroup of a group G.

For any y in G. Then we have, $A^+(yxy^{-1}) = A^+(x) = A^+(xyy^{-1}) = A^+(y^{-1}xy)$.

Therefore $A^+(yxy^{-1}) = A^+(y^{-1}xy)$.

And, $A^-(yxy^{-1}) = A^-(x) = A^-(xyy^{-1}) = A^-(y^{-1}xy)$.

Therefore $A^-(yxy^{-1}) = A^-(y^{-1}xy)$.

2.9 Theorem: A bipolar-valued fuzzy subgroup A= $\langle A^+, A^- \rangle$ of a group G is normalized if and only if $A^+(e) = 1$ and $A^-(e) = 0$, where e is the identity element of the group G.

Proof: If A is normalized then there exists $x \in G$ such that $A^+(x) = 1$ and $A^-(x) = 0$, but by properties of a bipolar-valued fuzzy subgroup A of the group G, $A^+(x) \leq A^+(e)$ and $A^-(x) \geq A^-(e)$ for every $x \in G$.

since $A^+(x) = 1$ and $A^-(x) = 0$ and $A^+(x) \leq A^+(e)$ and $A^-(x) \geq A^-(e)$.
Therefore $1 \leq A^+(e)$ and $0 \geq A^-(e)$. But $1 \geq A^+(e)$ and $0 \leq A^-(e)$.

Hence $A^+(e) = 1$ and $A^-(e) = 0$.

Conversely if $A^+(e) = 1$ and $A^-(e) = 0$, then by the definition of normalized bipolar-valued fuzzy subset A is normalized.

2.10 Theorem: If A= $\langle A^+, A^- \rangle$ is a bipolar-valued fuzzy subgroup of a group G, then for any a in G the bipolar-valued fuzzy middle coset aAa^{-1} of G is also a bipolar-valued fuzzy subgroup of a group G.

Proof: Let A is a bipolar-valued fuzzy subgroup of a group G and a in G. To prove $aAa^{-1} = (x, aA^+a^{-1}, aA^-a^{-1})$ is a bipolar-valued fuzzy subgroup of G. Let x and y in G.

Then $(a A^+a^{-1})(xy^{-1}) = A^+(a^{-1}xy^{-1}a)$

$= A^+(a^{-1}xaa^{-1}y^{-1}a)$

$= A^+(a^{-1}xa(a^{-1}ya)^{-1})$

$\geq \min \{ A^+(a^{-1}xa), A^+(a^{-1}ya) \}$

$= \min \{ (a A^+a^{-1})(x), (a A^+a^{-1})(y) \}$.

Therefore $(a A^+a^{-1})(xy^{-1}) \geq \min \{ (a A^+a^{-1})(x), (a A^+a^{-1})(y) \}$.
And $(a A^-a^{-1})(xy^{-1}) = A^-(a^{-1}xy^{-1}a)$

$= A^-(a^{-1}xaa^{-1}y^{-1}a)$

$= A^{-}(a^{-1}xa(a^{-1}ya)^{-1})$

$\leq \max \{ A^{-}(a^{-1}xa), A^{-}(a^{-1}ya) \}$

$= \max \{ (a A^{-}a^{-1})(x), (a A^{-}a^{-1})(y) \}$.

Therefore $(a A^{-}a^{-1})(xy^{-1}) \leq \max \{ (a A^{-}a^{-1})(x), (a A^{-}a^{-1})(y) \}$. Hence aAa^{-1} is a bipolar-valued fuzzy subgroup of a group G.

2.11 Theorem: Let $A= \langle A^{+}, A^{-} \rangle$ be a bipolar-valued fuzzy subgroup of a group G and aAa^{-1} be a bipolar-valued fuzzy middle coset of the group G, then $o(aAa^{-1}) = o(A)$, for any $a \in G$.

Proof: Let A be a bipolar-valued fuzzy subgroup of a group G and $a \in G$. By Theorem 2.10, the bipolar-valued fuzzy middle coset aAa^{-1} is a bipolar-valued fuzzy subgroup of a group G. Further by the definition of a bipolar-valued fuzzy middle coset of the group G we have $(a A^{+}a^{-1})(x) = A^{+}(a^{-1}xa)$ and $(a A^{-}a^{-1})(x) = A^{-}(a^{-1}xa)$, for every x in G.
Hence for any a in G, A and aAa^{-1} are conjugate bipolar-valued fuzzy subgroup of the group G as there exists $a \in G$ such that $(a A^{+}a^{-1})(x) = A^{+}(a^{-1}xa)$ and $(a A^{-}a^{-1})(x) = A^{-}(a^{-1}xa)$ for every x in G. By Theorem 2.6, $o(aAa^{-1}) = o(A)$ for any a in G.

2.12 Theorem: Let $A=\langle A^{+}, A^{-} \rangle$ be a bipolar-valued fuzzy subgroup of a group G and $B =\langle B^{+}, B^{-} \rangle$ be a bipolar-valued fuzzy subset of a group G. If A and B are conjugate bipolar-valued fuzzy subsets of the group G then B is a bipolar-valued fuzzy subgroup of a group G.

Proof: Let A be a bipolar-valued fuzzy subgroup of a group G and B be a bipolar-valued fuzzy subset of a group G. And let A and B are conjugate bipolar-valued fuzzy subsets of the group G. To prove B is a bipolar-valued fuzzy subgroup of the group G.
Let x and y in G. Then xy^{-1} in G.
Now, $B^{+}(xy^{-1}) = A^{+}(g^{-1}xy^{-1}g) = A^{+}(g^{-1}xgg^{-1}y^{-1}g) = A^{+}(g^{-1}xg(g^{-1}yg)^{-1}) \geq \min \{ A^{+}(g^{-1}xg), A^{+}(g^{-1}yg)\}= \min\{ B^{+}(x), B^{+}(y) \}$. Therefore $B^{+}(xy^{-1}) \geq \min \{ B^{+}(x), B^{+}(y) \}$.
And $B^{-}(xy^{-1}) = A^{-}(g^{-1}xy^{-1}g) = A^{-}(g^{-1}xgg^{-1}y^{-1}g) = A^{-}(g^{-1}xg(g^{-1}yg)^{-1}) \leq \max \{ A^{-}(g^{-1}xg), A^{-}(g^{-1}yg) \}= \max\{ B^{-}(x), B^{-}(y) \}$.
Therefore $B^{-}(xy^{-1}) \leq \max \{ B^{-}(x), B^{-}(y) \}$. Hence B is a bipolar-valued fuzzy subgroup of the group G.

2.13 Theorem: Let a bipolar-valued fuzzy subgroup $A = \langle A^{+}, A^{-} \rangle$ of a group G be conjugate to a bipolar-valued fuzzy subgroup $M= \langle M^{+}, M^{-} \rangle$ of G and a bipolar-valued fuzzy subgroup $B= \langle B^{+}, B^{-}\rangle$ of a group H be conjugate to a bipolar-valued fuzzy subgroup $N= \langle N^{+}, N^{-} \rangle$ of H. Then a bipolar-valued fuzzy subgroup $A\times B = \langle (A\times B)^{+}, (A\times B)^{-} \rangle$ of a group $G\times H$ is conjugate to a bipolar-valued fuzzy subgroup $M\times N = \langle (M\times N)^{+}, (M\times N)^{-} \rangle$ of $G\times H$.

Proof: Let A and B be bipolar-valued fuzzy subgroups of the groups G and H. Let x, x^{-1} and f be in G and y, y^{-1} and g be in H. Then (x, y) ,(x^{-1}, y^{-1}) and (f, g) are in $G\times H$.

Now, $(A\times B)^{+}(f, g) = \min \{ A^{+}(f), B^{+}(g) \}= \min\{ M^{+}(xf x^{-1}), N^{+}(yg y^{-1}) \}$

$= (M\times N)^{+} (xfx^{-1}, ygy^{-1}) = (M\times N)^{+}[(x, y)(f, g)(x^{-1}, y^{-1})]$

$= (M\times N)^{+}[(x, y)(f, g)(x, y)^{-1}]$.

Therefore, $(A\times B)^{+} (f, g) = (M\times N)^{+}[(x, y)(f, g)(x, y)^{-1}]$.

And, $(A\times B)^{-} (f, g) = \max \{ A^{-}(f), B^{-}(g) \}= \max\{ M^{-}(xf x^{-1}), N^{-}(yg y^{-1})\}$

$= (M\times N)^{-} (xf x^{-1}, yg y^{-1}) = (M\times N)^{-}[(x, y)(f, g)(x^{-1}, y^{-1})]$

$= (M\times N)^{-}[(x, y)(f, g)(x, y)^{-1}]$.

Therefore, $(A\times B)^{-}(f, g) = (M\times N)^{-}[(x, y)(f, g)(x, y)^{-1}]$.

Hence a bipolar-valued fuzzy subgroup A×B of a group G×H is conjugate to a bipolar-valued fuzzy subgroup M×N of G×H.

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