NOTES ON BIPOLAR-VALUED FUZZY SUBGROUPS OF A GROUP

M.S.ANITHA¹, K.L.MURUGANANTHA PRASAD² & K.ARJUNAN²

¹Research Scholar, Department of Mathematics, H.H. The Rajahs College, Pudukkottai – 622001 Tamilnadu, India.
²Department of Mathematics, H.H. The Rajahs College, Pudukkottai – 622001 Tamilnadu, India.

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Abstract: In this paper, we study some of the properties of bipolar-valued fuzzy subgroup and prove some results on these.

Introduction

In 1965, Zadeh [12] introduced the notion of a fuzzy subset of a set, fuzzy sets are a kind of useful mathematical structure to represent a collection of objects whose boundary is vague. Since then it has become a vigorous area of research in different domains, there have been a number of generalizations of this fundamental concept such as intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets, soft sets etc [5]. Lee [7] introduced the notion of bipolar-valued fuzzy sets. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval [0, 1] to [−1, 1]. In a bipolar-valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degree (0, 1] indicates that elements somewhat satisfy the property and the membership degree [−1, 0) indicates that elements somewhat satisfy the implicit counter property. Bipolar-valued fuzzy sets and intuitionistic fuzzy sets look similar each other. However, they are different each other [7, 8]. We introduce the concept of bipolar-valued fuzzy subgroup and established some results.

1. Preliminaries

1.1 Definition: A bipolar-valued fuzzy set (BVFS) A in X is defined as an object of the form A = {< x, A⁺(x), A⁻(x) >| x ∈ X], where A⁺ : X → [0, 1] and A⁻ : X → [−1, 0]. The positive membership degree A⁺(x) denotes the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set A and the negative membership degree A⁻(x) denotes the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar-valued fuzzy set A. If A⁺(x) ≠ 0 and A⁻(x) = 0, it is the situation that x is regarded as having only positive satisfaction for A and if A⁺(x) = 0 and A⁻(x) ≠ 0, it is the situation that x does not satisfy the property of A, but somewhat satisfies the counter property of A. It is possible for an element x to be such that A⁺(x) ≠ 0 and A⁻(x) ≠ 0 when the membership function of the property overlaps that of its counter property over some portion of X.

1.1 Example: A = {< a, 0.5, −0.3 >, < b, 0.1, −0.7 >, < c, 0.5, −0.4 >} is a bipolar-valued fuzzy subset of X = {a, b, c}.

1.2 Definition: Let G be a group. A bipolar-valued fuzzy subset A of G is said to be a bipolar-valued fuzzy subgroup of G (BVFSG) if the following conditions are satisfied,

(i) A⁺(xy) ≥ min{ A⁺(x), A⁺(y) },
(ii) A⁺(x⁻¹) ≥ A⁺(x),
(iii) A⁻(xy) ≤ max{ A⁻(x), A⁻(y) },

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1.2 Example: Let $G = \{1, -1, i, -i \}$ be a group with respect to the ordinary multiplication. Then $A$ = $\{ < 1, 0.5, -0.6 >, < -1, 0.4, -0.5 >, < i, 0.2, -0.4 >, < -i, 0.2, -0.4 > \}$ is a bipolar-valued fuzzy subgroup of $G$.

1.3 Definition: Let $A = \langle A^+, A^- \rangle$ and $B = \langle B^+, B^- \rangle$ be any two bipolar-valued fuzzy subsets of sets $G$ and $H$, respectively. The product of $A$ and $B$, denoted by $A \times B$, is defined as $A \times B = \{ (x, y), (A \times B)^+(x, y), (A \times B)^-(x, y) \}$ for all $x$ in $G$ and $y$ in $H$, where $(A \times B)^+(x, y) = \min \{ A^+(x), B^+(y) \}$ and $(A \times B)^-(x, y) = \max \{ A^-(x), B^-(y) \}$, for all $x$ in $G$ and $y$ in $H$.

1.4 Definition: Let $G$ be a group. A bipolar-valued fuzzy subgroup $A$ of $G$ is said to be a bipolar-valued fuzzy normal subgroup of $G$ if

(i) $A^+(xy) = A^+(yx)$,

(ii) $A^-(xy) = A^-(yx)$, for all $x$ and $y$ in $G$.

1.5 Definition: Let $A$ be a bipolar-valued fuzzy subgroup of a group $G$. For any $a \in G$, $aA$ defined by $(aA^+)(x) = A^+(a^{-1}x)$ and $(aA^-)(x) = A^-(a^{-1}x)$, for every $x \in G$ is called the bipolar-valued fuzzy coset of the group $G$.

1.6 Definition: Let $A$ be a bipolar-valued fuzzy subgroup of a group $G$ and $H = \{ x \in G / A^+(x) = A^+(e)$ and $A^-(x) = A^-(e) \}$, then $o(A)$, order of $A$ is defined as $o(A) = o(H)$.

1.7 Definition: Let $A$ and $B$ be two bipolar-valued fuzzy subgroups of a group $G$. Then $A$ and $B$ are said to be conjugate bipolar-valued fuzzy subgroup of $G$ if for some $g \in G$, $A^+(x) = B^+(g^{-1}xg)$ and $A^-(x) = B^-(g^{-1}xg)$, for every $x \in G$.

1.8 Definition: Let $A$ be a bipolar-valued fuzzy subgroup of a group $G$. Then for any $a$ and $b$ in $G$, a bipolar-valued fuzzy middle coset $aAb$ of $G$ is defined by $(aA^b)(x) = A^+(a^{-1}xb^{-1})$ and $(aA^-b)(x) = A^-(a^{-1}xb^{-1})$, for every $x \in G$.

2. PROPERTIES:

2.1 Theorem: Let $A = \langle A^+, A^- \rangle$ be a bipolar-valued fuzzy subgroup of a group $G$. If $A^+(x) < A^+(y)$ and $A^-(x) > A^-(y)$, for some $x$ and $y$ in $G$, then

(i) $A^+(xy) = A^+(x) = A^+(yx)$ and (ii) $A^-(xy) = A^-(x) = A^-(yx)$.

Proof: Let $A = \langle A^+, A^- \rangle$ be a bipolar-valued fuzzy subgroup of a group $G$. Let $A^+(x) < A^+(y)$ and $A^-(x) > A^-(y)$, for some $x$ and $y$ in $G$.

Now, $A^+(xy) \geq \min \{ A^+(x), A^+(y) \} = A^+(x)$; and

$A^-(x) = A^-(xy^{-1}) \geq \min \{ A^-(x), A^-(y) \} = A^-(x)$.

Also, $A^+(xy) \geq \min \{ A^+(y), A^+(x) \} = A^+(x)$; and

$A^-(x) = A^-(y^{-1}yx) \geq \min \{ A^-(y), A^-(yx) \} = A^-(yx)$.

Therefore $A^+(xy) = A^+(x) = A^+(yx)$. Hence (i) is proved.

Now, $A^-(xy) \leq \max \{ A^-(x), A^-(y) \} = A^-(x)$; and

$A^-(x) = A^-(xy^{-1}) \leq \max \{ A^-(x), A^-(y) \} = A^-(x)$.

Also, $A^-(xy) \leq \max \{ A^-(y), A^-(x) \} = A^-(y)$; and

$A^-(x) = A^-(y^{-1}yx) \leq \max \{ A^-(y), A^-(yx) \} = A^-(yx)$.

Therefore $A^-(xy) = A^-(x) = A^-(yx)$. Hence (ii) is proved.
2.2 Theorem: Let $A = \langle A^+, A^- \rangle$ be a bipolar-valued fuzzy subgroup of a group $G$. If $A^+(x) < A^+(y)$ and $A^-(x) < A^-(y)$, for some $x$ and $y$ in $G$, then

(i) $A^+(xy) = A^+(x) = A^+(yx)$ and (ii) $A^-(xy) = A^-(y) = A^-(yx)$.

Proof: Let $A = \langle A^+, A^- \rangle$ be a bipolar-valued fuzzy subgroup of a group $G$. Let $A^+(x) < A^+(y)$ and $A^-(x) < A^-(y)$, for some $x$ and $y$ in $G$.

Now, $A^+(xy) \geq \min \{ A^+(x), A^+(y) \} = A^+(x)$; and

$A^+(x) = A^+(xyx^{-1}) \geq \min \{ A^+(xy), A^+(y) \} = A^+(xy)$.

And, $A^-(y) \geq \min \{ A^-(y), A^-(x) \} = A^-(x)$; and

$A^-(y) = A^-(yxx^{-1}) \geq \min \{ A^-(yx), A^-(x) \} = A^-(yx)$.

Therefore $A^+(xy) = A^-(x) = A^-(yx)$. Hence (i) is proved.

Now, $A^-(xy) \leq \max \{ A^-(x), A^-(y) \} = A^-(y)$; and

$A^- (y) = A^-(x) = A^- (x)$; and

$A^- (y) = A^-(yx) \leq \max \{ A^- (yx), A^- (x) \} = A^- (yx)$.

Therefore $A^- (xy) = A^-(y) = A^- (yx)$. Hence (ii) is proved.

2.3 Theorem: Let $A = \langle A^+, A^- \rangle$ be a bipolar-valued fuzzy subgroup of a group $G$. If

$A^+(x) > A^+(y)$ and $A^-(x) > A^-(y)$, for some $x$ and $y$ in $G$, then

(i) $A^+(xy) = A^+(y) = A^+(yx)$ and (ii) $A^-(xy) = A^-(x) = A^-(yx)$. Proof: It is trivial.

2.4 Theorem: Let $A = \langle A^+, A^- \rangle$ be a bipolar-valued fuzzy subgroup of a group $G$. If $A^+(x) > A^+(y)$ and $A^-(x) < A^-(y)$, for some $x$ and $y$ in $G$, then

(i) $A^+(xy) = A^+(y) = A^+(yx)$ and (ii) $A^- (xy) = A^- (y) = A^- (yx)$.

Proof: It is trivial.

2.5 Theorem: Let $A = \langle A^+, A^- \rangle$ be a bipolar-valued fuzzy subgroup of a finite group $G$, then $o(A)$ divides $o(G)$.

Proof: Let $A$ be a bipolar-valued fuzzy subgroup of a finite group $G$ with $e$ as its identity element. Clearly $H = \{ x \in G / A^+(x) = A^+(e) \ and A^-(x) = A^- (e) \}$ is a subgroup of the group $G$. By Lagrange's theorem $o(H) | o(G)$.

Hence by the definition of the order of the bipolar-valued fuzzy subgroup of the group $G$, we have $o(A) | o(G)$.

2.6 Theorem: Let $A = \langle A^+, A^- \rangle$ and $B = \langle B^+, B^- \rangle$ be two bipolar-valued fuzzy subsets of an abelian group $G$. Then $A$ and $B$ are conjugate bipolar-valued fuzzy subsets of the group $G$ if and only if $A = B$.

Proof: Let $A$ and $B$ be conjugate bipolar-valued fuzzy subsets of group $G$, then for some $y \in G$, we have $A^+(x) = B^+(y^x y) = B^+(y^x) = B^+(y) = B^+(x)$.

Therefore $A^+(x) = B^+(y^x)$.

And, $A^-(x) = B^-(y^x y) = B^-(y^x) = B^-(x)$.

Therefore $A^-(x) = B^-(x)$. Hence $A = B$.

Conversely if $A = B$ then for the identity element $e$ of group $G$, we have $A^+(x) = B^+(e^x x)$ and $A^-(x) = B^-(e^x x)$ for every $x \in G$.

Hence $A$ and $B$ are conjugate bipolar-valued fuzzy subsets of the group $G$. 
2.7 Theorem: If \( A= \langle A^+, A^- \rangle \) and \( B= \langle B^+, B^- \rangle \) are conjugate bipolar-valued fuzzy subgroups of the group \( G \), then \( o(A) = o(B) \).

**Proof:** Let \( A \) and \( B \) are conjugate bipolar-valued fuzzy subgroups of the group \( G \). Now, \( o(A) = \text{order of } \{ x \in G / A^+(x) = A^+(e) \text{ and } A^-(x) = A^-(e) \} \)

\[
= \text{order of } \{ x \in G / B^+(y^{-1}xy) = B^+(y^{-1}ey) \text{ and } B^-(y^{-1}xy) = B^-(y^{-1}ey) \} \\
= \text{order of } \{ x \in G / B^+(x) = B^+(e) \text{ and } B^-(x) = B^-(e) \} \\
= o(B). \text{ Hence } o(A) = o(B).
\]

2.8 Theorem: Let \( A= \langle A^+, A^- \rangle \) be a bipolar-valued fuzzy normal subgroup of a group \( G \). Then for any \( y \) in \( G \), we have \( A^+(yx^{-1}) = A^+(y^{-1}xy) \) and \( A^-(yx^{-1}) = A^-(y^{-1}xy) \), for every \( x \in G \).

**Proof:** Let \( A \) be a bipolar-valued fuzzy normal subgroup of a group \( G \).

For any \( y \) in \( G \). Then we have, \( A^+(yx^{-1}) = A^+(x) = A^+(x) = A^+(y^{-1}xy) \).
Therefore \( A^+(yx^{-1}) = A^+(y^{-1}xy) \).
And, \( A^-(yx^{-1}) = A^-(x) = A^-(x) = A^-(y^{-1}xy) \).
Therefore \( A^-(yx^{-1}) = A^-(y^{-1}xy) \).

2.9 Theorem: A bipolar-valued fuzzy subgroup \( A= \langle A^+, A^- \rangle \) of a group \( G \) is normalized if and only if \( A^+(e) = 1 \) and \( A^-(e) = 0 \), where \( e \) is the identity element of the group \( G \).

**Proof:** If \( A \) is normalized then there exists \( x \in G \) such that \( A^+(x) = 1 \) and \( A^-(x) = 0 \), but by properties of a bipolar-valued fuzzy subgroup \( A \) of the group \( G \), \( A^+(x) \leq A^+(e) \) and \( A^-(x) \geq A^-(e) \) for every \( x \in G \).

since \( A^+(x) = 1 \) and \( A^-(x) = 0 \) and \( A^+(x) \leq A^+(e) \) and \( A^-(x) \geq A^-(e) \).
Therefore \( 1 \leq A^+(e) \) and \( 0 \geq A^-(e) \). But \( 1 \geq A^+(e) \) and \( 0 \leq A^-(e) \).

Hence \( A^+(e) = 1 \) and \( A^-(e) = 0 \).
Conversely if \( A^+(e) = 1 \) and \( A^-(e) = 0 \), then by the definition of normalized bipolar-valued fuzzy subset \( A \) is normalized.

2.10 Theorem: If \( A= \langle A^+, A^- \rangle \) is a bipolar-valued fuzzy subgroup of a group \( G \), then for any \( a \) in \( G \) the bipolar-valued fuzzy middle coset \( aAa^{-1} \) of \( G \) is also a bipolar-valued fuzzy subgroup of a group \( G \).

**Proof:** Let \( A \) is a bipolar-valued fuzzy subgroup of a group \( G \) and \( a \) in \( G \). To prove \( aAa^{-1} = (x, aa^+a^{-1}, aa^-a^{-1}) \) is a bipolar-valued fuzzy subgroup of \( G \). Let \( x \) and \( y \) in \( G \).

Then \( (aAa^{-1})(xy^{-1}) = A^+(a^+1xy^{-1}a) \)
\[
= A^+(a^+1xax^{-1}y^{-1}a) \\
= A^+(a^+1xa(a^+1y)1) \\
\geq \min \{ A^+(a^+1xa), A^+(a^+1y) \} \\
= \min \{ (aAa^{-1})(x), (aAa^{-1})(y) \}.
\]
Therefore \( (aAa^{-1})(xy^{-1}) \geq \min \{ (aAa^{-1})(x), (aAa^{-1})(y) \} \).
And \( (aAa^{-1})(xy^{-1}) = A^-a^{-1}(a^+1xy^{-1}a) \)
\[
= A^-a^{-1}(a^+1xaax^{-1}y^{-1}a)
\]
= A\(^{-1}\)xa\(^{-1}\)ya\(^{-1}\)
\leq \max \{ A\(^{-1}\)xa, A\(^{-1}\)ya \}
= \max \{ (a A\(^{-1}\)a\(^{-1}\))(x), (a A\(^{-1}\)a\(^{-1}\))(y) \}.
Therefore (a A\(^{-1}\))(xy\(^{-1}\)) \leq \max \{ (a A\(^{-1}\))(x), (a A\(^{-1}\))(y) \}. Hence aAa\(^{-1}\) is a bipolar-valued fuzzy subgroup of a group G.

2.11 Theorem: Let A= \( A^+, A^- \) be a bipolar-valued fuzzy subgroup of a group G and aAa\(^{-1}\) be a bipolar-valued fuzzy middle coset of the group G, then o(aAa\(^{-1}\)) = o(A), for any a \in G.

Proof: Let A be a bipolar-valued fuzzy subgroup of a group G and a \in G. By Theorem 2.10, the bipolar-valued fuzzy middle coset aAa\(^{-1}\) is a bipolar-valued fuzzy subgroup of a group G. Further by the definition of a bipolar-valued fuzzy middle coset of the group G we have (a A\(^{-1}\))(x) = A\(^{-1}\)xa\(^{-1}\)x and (a A\(^{-1}\))(x) = A\(^{-1}\)xa\(^{-1}\)x, for every x \in G.

Hence for any a \in G, A and aAa\(^{-1}\) are conjugate bipolar-valued fuzzy subgroup of the group G as there exists a \in G such that (a A\(^{-1}\))(x) = A\(^{-1}\)xa\(^{-1}\)x and (a A\(^{-1}\))(x) = A\(^{-1}\)xa\(^{-1}\)x for every x \in G. By Theorem 2.6, o(aAa\(^{-1}\)) = o(A) for any a \in G.

2.12 Theorem: Let A=\( A^+, A^- \) be a bipolar-valued fuzzy subgroup of a group G and B=\( B^+, B^- \) be a bipolar-valued fuzzy subset of a group G. If A and B are conjugate bipolar-valued fuzzy subsets of the group G then B is a bipolar-valued fuzzy subgroup of a group G.

Proof: Let A be a bipolar-valued fuzzy subgroup of a group G and B be a bipolar-valued fuzzy subset of a group G. And let A and B are conjugate bipolar-valued fuzzy subsets of the group G. To prove B is a bipolar-valued fuzzy subgroup of the group G.

Let x and y in G. Then xy\(^{-1}\) in G.

Now, \( B^+(xy\(^{-1}\)) = A^+(g^-1xy\(^{-1}\)g) = A^+(g^-1xg^-1yg^-1) = A^+(g^-1xg) \leq \min \{ A^+(g^-1x), A^{-}(g^-1yg) \} \leq \max \{ B^+(x), B^{-}(y) \} \).

And \( B^-(xy\(^{-1}\)) = A^{-}(g^-1xy\(^{-1}\)g) = A^{-}(g^-1xg^-1yg^-1) \leq \max \{ A^{-}(g^-1x), A^{-}(g^-1yg) \} = \max \{ B^-(x), B^{-}(y) \} \).

Therefore B^-(xy\(^{-1}\)) \leq \max \{ B^-(x), B^{-}(y) \}. Hence B is a bipolar-valued fuzzy subgroup of the group G.

2.13 Theorem: Let a bipolar-valued fuzzy subgroup A = \( A^+, A^- \) of a group G be conjugate to a bipolar-valued fuzzy subgroup M= \( M^+, M^- \) of G and a bipolar-valued fuzzy subgroup B= \( B^+, B^- \) of a group H be conjugate to a bipolar-valued fuzzy subgroup N= \( N^+, N^- \) of H. Then a bipolar-valued fuzzy subgroup A \times B = \( (A \times B)^+, (A \times B)^- \) of a group G \times H is conjugate to a bipolar-valued fuzzy subgroup M \times N = \( (M \times N)^+, (M \times N)^- \) of G \times H.

Proof: Let A and B be bipolar-valued fuzzy subgroups of the groups G and H. Let x, x\(^{-1}\) and f be in G and y, y\(^{-1}\) and g be in H. Then (x, y), (x\(^{-1}\), y\(^{-1}\)) and (f, g) are in G \times H.

Now, \( (A \times B)^+(f, g) = \min \{ A^+(f), B^+(g) \} = \min \{ M^+(xf x\(^{-1}\), N^-(yg y\(^{-1}\)) \} = (M \times N)^+(xf x\(^{-1}\), yg y\(^{-1}\)) \) [ (x, y)(f, g)(x\(^{-1}\), y\(^{-1}\)) ]

Therefore, \( (A \times B)^+(f, g) = (M \times N)^+[ (x, y)(f, g)(x, y)^{-1}) ] \).

And, \( (A \times B)^-(f, g) = \max \{ A^-(f), B^-(g) \} = \max \{ M^-(xf x\(^{-1}\), N^-(yg y\(^{-1}\)) \} = (M \times N)^-(xf x\(^{-1}\), yg y\(^{-1}\)) \) [ (x, y)(f, g)(x\(^{-1}\), y\(^{-1}\)) ]

Therefore, \( (A \times B)^-(f, g) = (M \times N)^-[ (x, y)(f, g)(x, y)^{-1}) ] \)
Therefore, \((A \times B)^{-1}(f, g) = (M \times N)^{-1}[(x, y)(f, g)(x, y)^{-1}]\).

Hence a bipolar-valued fuzzy subgroup \(A \times B\) of a group \(G \times H\) is conjugate to a bipolar-valued fuzzy subgroup \(M \times N\) of \(G \times H\).

References