NOTES ON BIPOLAR-VALUED FUZZY SUBGROUPS OF A GROUP

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Abstract: In this paper, we study some of the properties of bipolar-valued fuzzy subgroup and prove some results on these.

Introduction

In 1965, Zadeh [12] introduced the notion of a fuzzy subset of a set, fuzzy sets are a kind of useful mathematical structure to represent a collection of objects whose boundary is vague. Since then it has become a vigorous area of research in different domains, there have been a number of generalizations of this fundamental concept such as intuitionistic fuzzy sets, interval- valued fuzzy sets, vague sets, soft sets etc [5]. Lee [7] introduced the notion of bipolar-valued fuzzy sets. Bipolar-valued fuzzy sets are an extension of fuzzy sets whose membership degree range is enlarged from the interval $[0, 1]$ to $[-1, 1]$. In a bipolar-valued fuzzy set, the membership degree 0 means that elements are irrelevant to the corresponding property, the membership degree $(0, 1]$ indicates that elements somewhat satisfy the property and the membership degree $[-1, 0)$ indicates that elements somewhat satisfy the implicit counter property. Bipolar-valued fuzzy sets and intuitionistic fuzzy sets look similar each other. However, they are different each other [7, 8]. We introduce the concept of bipolar-valued fuzzy subgroup and established some results.

1. Preliminaries

1.1 Definition: A bipolar-valued fuzzy set (BVFS) A in X is defined as an object of the form $A = \{ \langle x, A^+(x), A^-(x) \rangle / x \in X \}$, where $A^+ : X \rightarrow [0, 1]$ and $A^- : X \rightarrow [-1, 0]$. The positive membership degree $A^+(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set A and the negative membership degree $A^-(x)$ denotes the satisfaction degree of an element x to some implicit counter-property corresponding to a bipolar-valued fuzzy set A . If $A^+(x) \neq 0$ and $A^-(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for A and if $A^+(x) = 0$ and $A^-(x) \neq 0$, it is the situation that x does not satisfy the property of A , but somewhat satisfies the counter property of A . It is possible for an element x to be such that $A^+(x) \neq 0$ and $A^-(x) \neq 0$ when the membership function of the property overlaps that of its counter property over some portion of X .

1.1 Example: $A = \{ \langle a, 0.5, -0.3 \rangle, \langle b, 0.1, -0.7 \rangle, \langle c, 0.5, -0.4 \rangle \}$ is a bipolar-valued fuzzy subset of $X = \{a, b, c\}$.

1.2 Definition: Let G be a group. A bipolar-valued fuzzy subset A of G is said to be a bipolar-valued fuzzy subgroup of G (BVFSG) if the following conditions are satisfied,

- (i) $A^+(xy) \geq \min \{ A^+(x), A^+(y) \}$,
- (ii) $A^+(x^{-1}) \geq A^+(x)$,
- (iii) $A^-(xy) \leq \max \{ A^-(x), A^-(y) \}$,

(iv) $A^-(x^{-1}) \leq A^-(x)$, for all x and y in G .

1.2 Example: Let $G = \{1, -1, i, -i\}$ be a group with respect to the ordinary multiplication. Then $A = \{ \langle 1, 0.5, -0.6 \rangle, \langle -1, 0.4, -0.5 \rangle, \langle i, 0.2, -0.4 \rangle, \langle -i, 0.2, -0.4 \rangle \}$ is a bipolar-valued fuzzy subgroup of G .

1.3 Definition: Let $A = \langle A^+, A^- \rangle$ and $B = \langle B^+, B^- \rangle$ be any two bipolar-valued fuzzy subsets of sets G and H , respectively. The product of A and B , denoted by $A \times B$, is defined as $A \times B = \{ \langle (x, y), (A \times B)^+(x, y), (A \times B)^-(x, y) \rangle / \text{for all } x \text{ in } G \text{ and } y \text{ in } H \}$, where $(A \times B)^+(x, y) = \min \{ A^+(x), B^+(y) \}$ and $(A \times B)^-(x, y) = \max \{ A^-(x), B^-(y) \}$, for all x in G and y in H .

1.4 Definition: Let G be a group. A bipolar-valued fuzzy subgroup A of G is said to be a bipolar-valued fuzzy normal subgroup of G if

$$(i) \quad A^+(xy) = A^+(yx),$$

$$(ii) \quad A^-(xy) = A^-(yx), \text{ for all } x \text{ and } y \text{ in } G.$$

1.5 Definition: Let A be a bipolar-valued fuzzy subgroup of a group G . For any $a \in G$, aA defined by $(aA^+)(x) = A^+(a^{-1}x)$ and $(aA^-)(x) = A^-(a^{-1}x)$, for every $x \in G$ is called the bipolar-valued fuzzy coset of the group G .

1.6 Definition: Let A be a bipolar-valued fuzzy subgroup of a group G and $H = \{x \in G / A^+(x) = A^+(e) \text{ and } A^-(x) = A^-(e)\}$, then $o(A)$, order of A is defined as $o(A) = o(H)$.

1.7 Definition: Let A and B be two bipolar-valued fuzzy subgroups of a group G . Then A and B are said to be conjugate bipolar-valued fuzzy subgroup of G if for some $g \in G$, $A^+(x) = B^+(g^{-1}xg)$ and $A^-(x) = B^-(g^{-1}xg)$, for every $x \in G$.

1.8 Definition: Let A be a bipolar-valued fuzzy subgroup of a group G . Then for any a and b in G , a bipolar-valued fuzzy middle coset aAb of G is defined by $(aA^+b)(x) = A^+(a^{-1}xb^{-1})$ and $(aA^-b)(x) = A^-(a^{-1}xb^{-1})$, for every $x \in G$.

2. PROPERTIES:

2.1 Theorem: Let $A = \langle A^+, A^- \rangle$ be a bipolar-valued fuzzy subgroup of a group G . If $A^+(x) < A^+(y)$ and $A^-(x) > A^-(y)$, for some x and y in G , then

$$(i) \quad A^+(xy) = A^+(x) = A^+(yx) \text{ and } (ii) \quad A^-(xy) = A^-(x) = A^-(yx).$$

proof: Let $A = \langle A^+, A^- \rangle$ be a bipolar-valued fuzzy subgroup of a group G .

Let $A^+(x) < A^+(y)$ and $A^-(x) > A^-(y)$, for some x and y in G .

Now, $A^+(xy) \geq \min \{ A^+(x), A^+(y) \} = A^+(x)$; and

$$A^+(x) = A^+(xyy^{-1}) \geq \min \{ A^+(xy), A^+(y) \} = A^+(xy).$$

Also, $A^+(yx) \geq \min \{ A^+(y), A^+(x) \} = A^+(x)$; and

$$A^+(x) = A^+(y^{-1}yx) \geq \min \{ A^+(y), A^+(yx) \} = A^+(yx).$$

Therefore $A^+(xy) = A^+(x) = A^+(yx)$. Hence (i) is proved.

Now, $A^-(xy) \leq \max \{ A^-(x), A^-(y) \} = A^-(x)$; and

$$A^-(x) = A^-(xyy^{-1}) \leq \max \{ A^-(xy), A^-(y) \} = A^-(xy).$$

Also, $A^-(yx) \leq \max \{ A^-(y), A^-(x) \} = A^-(x)$; and

$$A^-(x) = A^-(y^{-1}yx) \leq \max \{ A^-(y), A^-(yx) \} = A^-(yx).$$

Therefore $A^-(xy) = A^-(x) = A^-(yx)$. Hence (ii) is proved.

2.2 Theorem: Let $A = \langle A^+, A^- \rangle$ be a bipolar-valued fuzzy subgroup of a group G . If $A^+(x) < A^+(y)$ and $A^-(x) < A^-(y)$, for some x and y in G , then

$$(i) \quad A^+(xy) = A^+(x) = A^+(yx) \text{ and } (ii) \quad A^-(xy) = A^-(y) = A^-(yx).$$

proof: Let $A = \langle A^+, A^- \rangle$ be a bipolar-valued fuzzy subgroup of a group G .

Let $A^+(x) < A^+(y)$ and $A^-(x) < A^-(y)$, for some x and y in G .

Now, $A^+(xy) \geq \min\{A^+(x), A^+(y)\} = A^+(x)$; and

$$A^+(x) = A^+(xyy^{-1}) \geq \min\{A^+(xy), A^+(y)\} = A^+(xy).$$

And, $A^+(yx) \geq \min\{A^+(y), A^+(x)\} = A^+(x)$; and

$$A^+(x) = A^+(y^{-1}yx) \geq \min\{A^+(y), A^+(yx)\} = A^+(yx).$$

Therefore $A^+(xy) = A^+(x) = A^+(yx)$. Hence (i) is proved.

Now, $A^-(xy) \leq \max\{A^-(x), A^-(y)\} = A^-(y)$; and

$$A^-(y) = A^-(x^{-1}xy) \leq \max\{A^-(x), A^-(xy)\} = A^-(xy).$$

And, $A^-(yx) \leq \max\{A^-(y), A^-(x)\} = A^-(y)$; and

$$A^-(y) = A^-(yxx^{-1}) \leq \max\{A^-(yx), A^-(x)\} = A^-(yx).$$

Therefore $A^-(xy) = A^-(y) = A^-(yx)$. Hence (ii) is proved.

2.3 Theorem: Let $A = \langle A^+, A^- \rangle$ be a bipolar-valued fuzzy subgroup of a group G . If

$A^+(x) > A^+(y)$ and $A^-(x) > A^-(y)$, for some x and y in G , then

$$(i) \quad A^+(xy) = A^+(y) = A^+(yx) \text{ and } (ii) \quad A^-(xy) = A^-(x) = A^-(yx). \text{ **Proof:** It is trivial.}$$

2.4 Theorem: Let $A = \langle A^+, A^- \rangle$ be a bipolar-valued fuzzy subgroup of a group G . If $A^+(x) > A^+(y)$ and $A^-(x) < A^-(y)$, for some x and y in G , then

$$(i) \quad A^+(xy) = A^+(y) = A^+(yx) \text{ and } (ii) \quad A^-(xy) = A^-(y) = A^-(yx).$$

Proof: It is trivial.

2.5 Theorem: Let $A = \langle A^+, A^- \rangle$ be a bipolar-valued fuzzy subgroup of a finite group G , then $o(A)$ divides $o(G)$.

Proof: Let A be a bipolar-valued fuzzy subgroup of a finite group G with e as its identity element. Clearly $H = \{x \in G / A^+(x) = A^+(e) \text{ and } A^-(x) = A^-(e)\}$ is a subgroup of the group G . By Lagrange's theorem $o(H) \mid o(G)$.

Hence by the definition of the order of the bipolar-valued fuzzy subgroup of the group G , we have $o(A) \mid o(G)$.

2.6 Theorem: Let $A = \langle A^+, A^- \rangle$ and $B = \langle B^+, B^- \rangle$ be two bipolar-valued fuzzy subsets of an abelian group G . Then A and B are conjugate bipolar-valued fuzzy subsets of the group G if and only if $A = B$.

Proof: Let A and B be conjugate bipolar-valued fuzzy subsets of group G , then for some $y \in G$, we have $A^+(x) = B^+(y^{-1}xy) = B^+(y^{-1}yx) = B^+(ex) = B^+(x)$.

Therefore $A^+(x) = B^+(x)$.

$$\text{And, } A^-(x) = B^-(y^{-1}xy) = B^-(y^{-1}yx) = B^-(ex) = B^-(x).$$

Therefore $A^-(x) = B^-(x)$. Hence $A = B$.

Conversely if $A = B$ then for the identity element e of group G ,

$$\text{we have } A^+(x) = B^+(e^{-1}xe) \text{ and } A^-(x) = B^-(e^{-1}xe) \text{ for every } x \in G.$$

Hence A and B are conjugate bipolar-valued fuzzy subsets of the group G .

2.7 Theorem: If $A = \langle A^+, A^- \rangle$ and $B = \langle B^+, B^- \rangle$ are conjugate bipolar-valued fuzzy subgroups of the group G , then $o(A) = o(B)$.

Proof: Let A and B are conjugate bipolar-valued fuzzy subgroups of the group G .

Now, $o(A) = \text{order of } \{ x \in G / A^+(x) = A^+(e) \text{ and } A^-(x) = A^-(e) \}$

$$= \text{order of } \{ x \in G / B^+(y^{-1}xy) = B^+(y^{-1}ey) \text{ and } B^-(y^{-1}xy) = B^-(y^{-1}ey) \}$$

$$= \text{order of } \{ x \in G / B^+(x) = B^+(e) \text{ and } B^-(x) = B^-(e) \}$$

$$= o(B). \text{ Hence } o(A) = o(B).$$

2.8 Theorem: Let $A = \langle A^+, A^- \rangle$ be a bipolar-valued fuzzy normal subgroup of a group G . Then for any y in G we have $A^+(yxy^{-1}) = A^+(y^{-1}xy)$ and $A^-(yxy^{-1}) = A^-(y^{-1}xy)$, for every $x \in G$.

Proof: Let A be a bipolar-valued fuzzy normal subgroup of a group G .

For any y in G . Then we have, $A^+(yxy^{-1}) = A^+(x) = A^+(xyy^{-1}) = A^+(y^{-1}xy)$.

Therefore $A^+(yxy^{-1}) = A^+(y^{-1}xy)$.

And, $A^-(yxy^{-1}) = A^-(x) = A^-(xyy^{-1}) = A^-(y^{-1}xy)$.

Therefore $A^-(yxy^{-1}) = A^-(y^{-1}xy)$.

2.9 Theorem: A bipolar-valued fuzzy subgroup $A = \langle A^+, A^- \rangle$ of a group G is normalized if and only if $A^+(e) = 1$ and $A^-(e) = 0$, where e is the identity element of the group G .

Proof: If A is normalized then there exists $x \in G$ such that $A^+(x) = 1$ and $A^-(x) = 0$, but by properties of a bipolar-valued fuzzy subgroup A of the group G , $A^+(x) \leq A^+(e)$ and $A^-(x) \geq A^-(e)$ for every $x \in G$.

since $A^+(x) = 1$ and $A^-(x) = 0$ and $A^+(x) \leq A^+(e)$ and $A^-(x) \geq A^-(e)$.

Therefore $1 \leq A^+(e)$ and $0 \geq A^-(e)$. But $1 \geq A^+(e)$ and $0 \leq A^-(e)$.

Hence $A^+(e) = 1$ and $A^-(e) = 0$.

Conversely if $A^+(e) = 1$ and $A^-(e) = 0$, then by the definition of normalized bipolar-valued fuzzy subset A is normalized.

2.10 Theorem: If $A = \langle A^+, A^- \rangle$ is a bipolar-valued fuzzy subgroup of a group G , then for any a in G the bipolar-valued fuzzy middle coset aAa^{-1} of G is also a bipolar-valued fuzzy subgroup of a group G .

Proof: Let A is a bipolar-valued fuzzy subgroup of a group G and a in G . To prove $aAa^{-1} = \langle aA^+a^{-1}, aA^-a^{-1} \rangle$ is a bipolar-valued fuzzy subgroup of G . Let x and y in G .

Then $(aA^+a^{-1})(xy^{-1}) = A^+(a^{-1}xy^{-1}a)$

$$= A^+(a^{-1}xaa^{-1}y^{-1}a)$$

$$= A^+(a^{-1}xa(a^{-1}ya)^{-1})$$

$$\geq \min \{ A^+(a^{-1}xa), A^+(a^{-1}ya) \}$$

$$= \min \{ (aA^+a^{-1})(x), (aA^+a^{-1})(y) \}.$$

Therefore $(aA^+a^{-1})(xy^{-1}) \geq \min \{ (aA^+a^{-1})(x), (aA^+a^{-1})(y) \}.$

And $(aA^-a^{-1})(xy^{-1}) = A^-(a^{-1}xy^{-1}a)$

$$= A^-(a^{-1}xaa^{-1}y^{-1}a)$$

$$\begin{aligned}
&= A^-(a^{-1}xa(a^{-1}ya)^{-1}) \\
&\leq \max \{ A^-(a^{-1}xa), A^-(a^{-1}ya) \} \\
&= \max \{ (a A^-a^{-1})(x), (a A^-a^{-1})(y) \}.
\end{aligned}$$

Therefore $(a A^-a^{-1})(xy^{-1}) \leq \max \{ (a A^-a^{-1})(x), (a A^-a^{-1})(y) \}$. Hence aAa^{-1} is a bipolar-valued fuzzy subgroup of a group G .

2.11 Theorem: Let $A = \langle A^+, A^- \rangle$ be a bipolar-valued fuzzy subgroup of a group G and aAa^{-1} be a bipolar-valued fuzzy middle coset of the group G , then $o(aAa^{-1}) = o(A)$, for any $a \in G$.

Proof: Let A be a bipolar-valued fuzzy subgroup of a group G and $a \in G$. By Theorem 2.10, the bipolar-valued fuzzy middle coset aAa^{-1} is a bipolar-valued fuzzy subgroup of a group G . Further by the definition of a bipolar-valued fuzzy middle coset of the group G we have $(a A^+a^{-1})(x) = A^+(a^{-1}xa)$ and $(a A^-a^{-1})(x) = A^-(a^{-1}xa)$, for every x in G .

Hence for any a in G , A and aAa^{-1} are conjugate bipolar-valued fuzzy subgroup of the group G as there exists $a \in G$ such that $(a A^+a^{-1})(x) = A^+(a^{-1}xa)$ and $(a A^-a^{-1})(x) = A^-(a^{-1}xa)$ for every x in G . By Theorem 2.6, $o(aAa^{-1}) = o(A)$ for any a in G .

2.12 Theorem: Let $A = \langle A^+, A^- \rangle$ be a bipolar-valued fuzzy subgroup of a group G and $B = \langle B^+, B^- \rangle$ be a bipolar-valued fuzzy subset of a group G . If A and B are conjugate bipolar-valued fuzzy subsets of the group G then B is a bipolar-valued fuzzy subgroup of a group G .

Proof: Let A be a bipolar-valued fuzzy subgroup of a group G and B be a bipolar-valued fuzzy subset of a group G . And let A and B are conjugate bipolar-valued fuzzy subsets of the group G . To prove B is a bipolar-valued fuzzy subgroup of the group G .

Let x and y in G . Then xy^{-1} in G .

Now, $B^+(xy^{-1}) = A^+(g^{-1}xy^{-1}g) = A^+(g^{-1}xgg^{-1}y^{-1}g) = A^+(g^{-1}xg(g^{-1}yg)^{-1}) \geq \min \{ A^+(g^{-1}xg), A^+(g^{-1}yg) \} = \min \{ B^+(x), B^+(y) \}$. Therefore $B^+(xy^{-1}) \geq \min \{ B^+(x), B^+(y) \}$.

And $B^-(xy^{-1}) = A^-(g^{-1}xy^{-1}g) = A^-(g^{-1}xgg^{-1}y^{-1}g) = A^-(g^{-1}xg(g^{-1}yg)^{-1}) \leq \max \{ A^-(g^{-1}xg), A^-(g^{-1}yg) \} = \max \{ B^-(x), B^-(y) \}$.

Therefore $B^-(xy^{-1}) \leq \max \{ B^-(x), B^-(y) \}$. Hence B is a bipolar-valued fuzzy subgroup of the group G .

2.13 Theorem: Let a bipolar-valued fuzzy subgroup $A = \langle A^+, A^- \rangle$ of a group G be conjugate to a bipolar-valued fuzzy subgroup $M = \langle M^+, M^- \rangle$ of G and a bipolar-valued fuzzy subgroup $B = \langle B^+, B^- \rangle$ of a group H be conjugate to a bipolar-valued fuzzy subgroup $N = \langle N^+, N^- \rangle$ of H . Then a bipolar-valued fuzzy subgroup $A \times B = \langle (A \times B)^+, (A \times B)^- \rangle$ of a group $G \times H$ is conjugate to a bipolar-valued fuzzy subgroup $M \times N = \langle (M \times N)^+, (M \times N)^- \rangle$ of $G \times H$.

Proof: Let A and B be bipolar-valued fuzzy subgroups of the groups G and H . Let x, x^{-1} and f be in G and y, y^{-1} and g be in H . Then $(x, y), (x^{-1}, y^{-1})$ and (f, g) are in $G \times H$.

$$\begin{aligned}
\text{Now, } (A \times B)^+(f, g) &= \min \{ A^+(f), B^+(g) \} = \min \{ M^+(xfx^{-1}), N^+(ygy^{-1}) \} \\
&= (M \times N)^+(xfx^{-1}, ygy^{-1}) = (M \times N)^+[(x, y)(f, g)(x^{-1}, y^{-1})] \\
&= (M \times N)^+[(x, y)(f, g)(x, y)^{-1}].
\end{aligned}$$

Therefore, $(A \times B)^+(f, g) = (M \times N)^+[(x, y)(f, g)(x, y)^{-1}]$.

$$\begin{aligned}
\text{And, } (A \times B)^-(f, g) &= \max \{ A^-(f), B^-(g) \} = \max \{ M^-(xfx^{-1}), N^-(ygy^{-1}) \} \\
&= (M \times N)^-(xfx^{-1}, ygy^{-1}) = (M \times N)^-[(x, y)(f, g)(x^{-1}, y^{-1})] \\
&= (M \times N)^-[(x, y)(f, g)(x, y)^{-1}].
\end{aligned}$$

Therefore, $(A \times B)^-(f, g) = (M \times N)^-[(x, y)(f, g)(x, y)^{-1}]$.

Hence a bipolar-valued fuzzy subgroup $A \times B$ of a group $G \times H$ is conjugate to a bipolar-valued fuzzy subgroup $M \times N$ of $G \times H$.

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