A CONJECTURE ABOUT FUNCTIONS

K. PRASEN

Grade-12, Sri Chaitanya IIT Academy, Ram nagar, Visakhapatnam, India

EMAIL: rrmath28@gmail.com

ABSTRACT. In [1] Julien Laurendeau stated the conjecture on functions. In this paper we will address the same and we conclude the conjecture.

1. INTRODUCTION

As we know that conjecture is a proposition that is unproven [2]. Karl Popper pioneered the use of the term conjecture in scientific philosophy. Let us state the Julien Laurendeau’s conjecture.

There are no solutions to the following equation:

\[ (f(x) + g(x) + h(x)) + (f(x) + g(x))h(x) = 2. \]  \hspace{1cm} (1)

Knowing that all mentioned functions are non-linear and that:

\[ f(x) = \frac{f(x) + (g(x) - h(x))f(x)}{2}, \]
\[ g(x) = \frac{g(x) + (h(x) - f(x))g(x)}{4}, \]
\[ h(x) = \frac{h(x) + (f(x) - g(x))h(x)}{6}. \]  \hspace{1cm} (2)

With \( f(x) \neq 0, g(x) \neq 0, h(x) \neq 0 \) for all \( x \in R \).

Let us move to the next section to conclude this conjecture.

2. CONCLUDING THE CONJECTURE

Let us prove the conjecture in case wise/part wise.

**Part-I:** At least one has to be zero.

\[ K_A = \{ x \in R | f(x) \neq 0 \} \]
\[ K_B = \{ x \in R | g(x) \neq 0 \} \]
\[ K_C = \{ x \in R | h(x) \neq 0 \} \]

For time being consider \( f(x) = A, \) \( g(x) = B \) and \( h(x) = C \), then our (1) and (2) becomes;

\[ (A + B + C) + (A + B)C = 2 \]
\[ (i)2A = A + (B - C)A \]
\[ (ii)4B = B + (C - A)B \]
\[ (iii)6C = C + (A - B)C \]  \hspace{1cm} (3)

With \( A \neq 0, B \neq 0 \) and \( C \neq 0 \).
For condition-(i):

\[ 2A = A + (B - C)A; \forall x \in R \]
\[ \Leftrightarrow A = (B - C)A; \forall x \in R \]
\[ \Rightarrow 1 = B - C; \forall x \in K_A \]

For condition-(ii):

\[ 4B = B + (C - A)B; \forall x \in R \]
\[ \Leftrightarrow 3B = (C - A)B; \forall x \in R \]
\[ \Rightarrow 3 = C - A; \forall x \in K_B \]

For condition-(iii):

\[ 6C = C + (A - B)C; \forall x \in R \]
\[ \Leftrightarrow 5C = (A - B)C; \forall x \in R \]
\[ \Rightarrow 5 = A - B; \forall x \in K_C \]

\[ \forall x \in (K_A \cap K_B \cap K_C): C = B - 1 \]

Clearly, \[ \Rightarrow 3 = (B - 1) - A \]
\[ \Rightarrow 5 = A - B \]

\[ B = 4 + A \]
\[ \Rightarrow 5 = A - (4 + A) = -4, \]

This is contradiction \( \Rightarrow (K_A \cap K_B \cap K_C) = \emptyset \).

i.e., there is no \( x \in R \) such that \( f(x) \neq 0, g(x) \neq 0 \) and \( h(x) \neq 0 \).

**Part-II:** Let us check the individual cases

**Case – I:**

\[ \{ x \in R \mid C(x) = 0 \} \]

Condition is \( A + B = 2 \)

\[ 2A = A + AB \]
\[ \Leftrightarrow A = AB \]
\[ 4B = B - AB \]
\[ \Leftrightarrow -3B = AB \]
\[ A = -3B \]
\[ A + B = -3B + B \]
\[ -3B + B = -2B = 2 \]
\[ \Leftrightarrow B = -1 \]
\[ \Rightarrow A = 3 \]

**Case – II:**

\[ \{ x \in R \mid B(x) = 0 \} \]

Condition is \( A + C + AC - 2 = 0 \)

\[ 2A = A - AC \]
\[ \Leftrightarrow -A = AC \]
\[ 6C = C + AC \]
\[ \Leftrightarrow 5C = AC \]
\[ \Rightarrow -A = 5C \]
\[ \Leftrightarrow A = -5C \]
Now we can simplify the conditions: $A + C + AC -2 = 0$

\[
\Leftrightarrow -5C + C + (-5C)C - 2 = 0 \\
\Leftrightarrow -5C^2 - 4C - 2 = 0
\]

i.e., $C$ has no solution in $R$.

Hence $\{x \in R \mid B(x) = 0\} = \emptyset$.

**Case – III:**

\[\{x \in R \mid A(x) = 0\}\]

Condition is $B + C + BC - 2 = 0$

\[
4B = B + BC \\
\Leftrightarrow 3B = BC \\
6C = C - BC \\
\Leftrightarrow 5C = -BC \\
\Leftrightarrow -5C = BC \\
3B = -5C \\
\Leftrightarrow B = -5/3 C
\]

Now by simplifying the condition, we get;

\[
\frac{-5}{3} C + C + \left(\frac{-5}{3} C\right)C - 2 = 0 \\
\frac{-5}{3} C^2 + \left(\frac{-2}{3} C\right) - 2 = 0 \\
\frac{5}{3} C^2 + \left(\frac{2}{3} C\right) + 2 = 0
\]

\[
\Leftrightarrow C_{1,2} = \frac{1}{2(-5)} \left(-4 \pm \sqrt{16 - 4(-5)(-2)}\right) \\
= \frac{3}{10} \left(-\frac{2}{3} \pm \frac{4}{9} - \frac{40}{3}\right)
\]

i.e., $C$ has no solution in $R$.

Hence $\{x \in R \mid A(x) = 0\} = \emptyset$

Finally...

\[
\forall x \in R, f'(x) = 0 \text{ or } g(x) = 0 \text{ or } h(x) = 0 \\
\forall x \in R : f'(x) \neq 0 \text{ and } g(x) \neq 0 \\
\forall x \in R : h'(x) = 0 \\
\Rightarrow h = 0 \Leftrightarrow C = 0 \\
\Rightarrow contradiction
\]

Therefore, there is no such real functions $f, g, h$.
3. SHORT PROOF
We have from all deductions, the following:
Form (i), we have; $2A = A + (B - C)A; \forall x \in R$

$$\Rightarrow 2A - A + (B - C)A = 0$$
$$\Rightarrow A(2 - (1 + (B - C))) = 0$$
$$\Rightarrow A(1 - (B - C)) = 0$$

similarly we can find the others as shown below:

$$A(1 - (B - C)) = 0$$
$$B(3 - (C - A)) = 0 \ldots (*)$$
$$C(5 - (A - B)) = 0$$

Since $A \neq 0, B \neq 0 \& C \neq 0$,
The (*) becomes by adding;

\[
\begin{align*}
1 - (B - C) &= 0 \\
3 - (C - A) &= 0 \quad \text{or} \quad 3 = (C - A) \ldots (+) \\
5 - (A - B) &= 0 \quad 5 = (A - B)
\end{align*}
\]

i.e., $9 = 0$, this is contradiction. □

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REFERENCES