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ON SEMIPRIME FINITELY GENERATED RIGHT ALTERNATIVE WEAKLY M – RINGS

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ABSTRACT: Right alternative rings, satisfying the weakly M-ring identity $(w,xy,z) = x(w,y,z)$ are studied. Any one of the following additional assumptions imply associativity: semi prime and a finite number of generators: prime with char. $\neq 2$ and minimum condition on either right ideals or on trivial left ideals, or simple and char. $\neq 2$.

INTRODUCTION:

Through out this paper R will denote a right alternative ring which satisfies the identity $(x,y,y)= 0$. In addition we impose on R the weakly M-ring identity $(w,xy,z) = x(w,y,z)$… (1)
We definethe middle nucleus M of R and the commutative center U of R as follows: $M = \{n \in R \,/\, (R,n,R) = 0\}$ and $U = \{u \in R \,/\, [R,U] = 0\}$. The associator ideal A consists of all finite sums of associators and leftmultiples of associators. As a consequence of (1) we observe that the associator ideal A of R may be described as consisting of all finite sums of associators.

Thedy [1] has studied right alternative rings. He started collecting information on two concepts in a right alternative algebra R, the sub module M generated by all associators (x,x,y) and a new nucleus N_β. He delt mainly with results on simple right alternative algebras and shown that a simple right alternative algebra with char. $\neq 2$ is the alternative, hence associative or a Cayley algebra over its center. Hentzel [5] has shown that a simple locally (-1,1) nill ring is associator and Hentzel with Smith [4] have shown that a semi prime locally (-1,1) ring with minimal condition on right ideals is alternator. Skosyrskii [7 & 8] has studied the right alternative rings which are prime with minimum condition on right ideals on trivial left ideals. Using the results in this paper we study right alternative rings satisfying weakly M-ring identity and shows that if the ring R is semi prime and has a finite number of generators then the ring is associator. Also we show that if R is prime with char. $\neq 2$ and has minimum condition on either right ideals or on trivial left ideals or simple and char. $\neq 2$ then R is associator.

MAIN SECTION:

Lemma1: If M is an ideal of R such that $MA = 0$, R semi prime and $I = M \in A$, then $I = 0$. If R is prime and not associative then $M = 0$.
Proof: For arbitrary elements $x,y,z \in R$ and $n \in M$, we have $(x,ym,z) = y(x,m,z) = 0$, using (1) and thedefinition of M. Hence M is a left ideal. Thus $(x,y,zn) = 0$, using the right alternative identity. But then the Teichmuller identity $(wx,m,z) - (w,xm,z) + (w,x,mz) = w(x,m,z) + (w,x,m)z$. Which gives $(x,mz,x) = 0$, since the other four associators are known to be zero already. Thus M is a right ideal and therefore an ideal of R. Moreover (1) implies $0 = (x,ny,z) = n(x,y,z)$ i.e., $MA = 0$. If I is as defined then $I_2 \subseteq MA = 0$ so that semi prime and not associative then $A \neq 0$ so that $M = 0$. This completes the proof of the lemma.

Lemma 2: For arbitrary elements $a,b,x,y,z \in R$, define $p = (a,(b,y,z),x)$. Then under all 12 permutations that map $\{a,b\}$ on itself and $\{x,y,z\}$ on itself, even permutations fix p while odd permutations map p on $-p$.

Proof: Since $p = (a,(by)z - b(yz),\mathrm{x}) = by(a,z,x) - b(a,yz,x) = by(a,z,x) - b.y(a,z,x) = (b,y,(a,z,x))$ using right alternativity, it follows that $p = (a,(b,y,z),x) = -\ (b,(a,z,x),y) = (b,(a,x,z),y)$. Thus p is left by the permutation $(ab)(yzx)$. Because of the right alternative identity it follows that (xz) changes p to $-p$. But then $(a,(b,x,z),x) = -\ (a,(b,z,x),x) = (b,(a,x,x),z) = 0$, using right alternativity. The linearization of this identity gives $(a,(b,x,z),y) + (a,(b,y,z),x) = 0$ implying $(a,(b,x,z),y) = -(a,(b,y,z),x)$ so that the permutation (x,y,z) changes p to $-p$. But then $p = (a,(b,y,z),x) = -(b,(a,z,x),y) = (b,(a,z,x),y) = -\ (b,(\mathrm{a},y,z),x)$. Thus the permutation (ab) also changes p to $-p$. Which shows that we have enough generators for the entire subgroup. This completes the proof of the lemma.

Definition: We define a sequence J_n in R as follows. $J_1 = A . J_{n+1} = \Sigma\ (R,R,J_n)$.

Lemma 3: The J_n are all ideals of R and satisfy $J_n \supseteq J_{n+1}$.

Proof: It follows from induction and (1) that J_n is a left ideal for $n > 1$. Since $J_1 = A$, it is an ideal of R. Then by induction it follows from the Teichmuller identity together with (1) and right alternativity that J_n is a right ideal as well. This completes the proof the lemma.

Lemma 4: If R is finitely generated then there exists a positive integer n such that $J_n = 0$.

Proof: In a finitely generated ring each element is a sum of nonassociative words in the generators. We will consider forming $p' = (a',p,x')$ where p is as in lemma 2, $p''(a'',p',x'')$ etc. until the number of x's plus 2 exceeds the number of generators. Let us then substitute nonassociative words for the $x's$, y and z. We classify each word according to the last letter in each word. Clearly there must be at least two words which end in the same last letter. By a series of permutations described in lemma 2 we may maneuver those two words in to the places originally occupied by x and z without change. But if W and W_* are two words which end in d, then $(b,w,w_*) = (b,d,d)$ followed by a sequence of left multiplications, since we can break up the words by means of repeated use of (1). Then right alternativity shows that we get zero. What we have proved for words goes over readily to sums of words. Thus if R has n generators then $J_n = 0$. This completes the proof of the lemma.

Theorem 1: If R satisfies (1), is right alternative, finitely generated and semi prime, then R must be associative.

Proof: As a result of lemma (1) it follows that $I = 0$. Using lemma (4), it follows that $J_n = 0$, for some positive integer n. We may assume that n is the least positive integer such that $J_n = 0$. Assume $n > 1$. Thus J_{n-1} is contained in M. Since J_{n-1} is clearly in A as well, we conclude it is contained in I and therefore zero. This contradicts the choice of n, unless $n = 1$. But $A = 0$ implies R is associative. This concludes the proof the theorem.

Theorem 2: If R is alternative, satisfies (1) and is semi prime then R must be associative.
Proof: In an alternative ring we have the identity $(x_2,y,z) = (y,z,x_2) = x(y,z,x) + (y,z,x)x$ [13]. On the other hand (1) implies that $(y,x_2,z) = (y,xx,z) = x(y,x,z)$. Thus by comparison we must have $(y,x,z)x = 0$, so that $(x,y,z)x = 0$ (2)
At this point we have the alternative identity $0 = -\ (x,y,z)x$ implies $(x,y,z)x = 0$ [13]. But then (1) implies $(x,y,z)x = 0$ so that using (2) we get $(x_2,y,z) = 0$.

Let N be the nucleus of R. we have shown that $x^2 \in N$, for all $x \in R$. (3)
Linearizing (3), we obtain
$xy + yx \in N$ for all $x,y \in R$. (4)
Since $M = N$ in an alternative ring then lemma (1) implies that
N is an ideal of R. (5)
However $(x,ny,z) = 0 = n(x,y,z)$, using (1). Therefore
$NA = 0$ (6)
Using the hypothesis of semi prime it follows that
$N \cap A = 0$ (7)

Since $w(x,y,z) + (x,y,z)w \in N$ as a result of (4) and obviously in A, using (7) we obtain $w(x,y,z) + (x,y,z)w = 0$, for all $w,x,y,z \in R$ … (8)

Then for $v \in R$, $0 = w.v(x,y,z) + v(x,y,z).w = w(x,vy,z) + (x,vy,z)$.w, using (1) and (8). Thus w anticommutes with $v(x,y,z)$ (9)
Starting with (x,wy,vz) we can take out the elements w and v from the associator. We do this twice, but in different order. This results in
$w.v(x,y,z) = v.w(x,y,z)$ (10)
By combining (10) with (9) we obtain $v(x,y,z).w = w(x,y,z).v$ (11)
However $w(x,y,z) = -(x,y,z)w$as a result of (8). Substituting this in to (11) results in $v(x,y,z).w = -(x,y,z)w.v$ (12)
But v anticommutes with $w(x,y,z)$ as a result of (9), so it must anticommute with $(x,y,z)w$ as a consequence of (8). Applying this in to (12) yields $v(x,y,z).w = v.(x,y,z)w$ (13)
Now (13) can be written in associator form as $(v,(x,y,z),w) = 0$ (14)
But (14) is equivalent to asserting that $(R,R,R) \subseteq N \cap A$ (15)
But now lemma (1) and (15) may be utilized to show that (R,R,R) = 0. Consequently R must be associative. This concludes the proof of the theorem.

Lemma 5: If R is right alternative, char. $\neq 2$ and satisfies (1) then it satisfies the identity $[z,(x,y,z)]=0$.
Proof: In a right alternative ring with char. $\neq 2$, $(x,y,z)z = (x,zy,z)$ [13]. Thus $[z,(x,y,z)] = z(x,y,z) - (x,y,z)z = z(x,y,z) - (x,zy,z) = (x,zy,z) - (x,zy,z) = 0$. This concludes the proof of the lemma.

Lemma 6: If R is prime and associative then R is strongly M-ring, i.e., it satisfies the identity $y.xz=x.yz$.

Proof: Through the repeated use of (1) we obtain $(a,y.xz,b) = y(a,xz,b) = (a,xz,yb) = x(a,z,yb) =x.y(a,z,b) = x(a,yz,b) = (a,x.yz,b)$. In other words $y.xz - x.yz \in M$. Since $A \neq 0$, it follows from lemma (1) that $M = 0$. This completes the proof of the lemma.

Lemma 7: If R is strongly M-ring then U is an ideal.

Proof: Note that $[xy.y] = xy.y - y.xy = x.y^2 - x.y^2= 0$ using the strongly M-ring. Linearization results in $[xy,z] = -[xz,y]$. If $u \in U$ and we let $y = u$, then $[xu,z] = 0$. Thus U is a left ideal. Since $xu = ux$, it follows that U is an ideal of R. This completes the proof of the lemma.

Lemma 8: If R is simple and not associative, then $U = 0$.

Proof: Using lemma (6) the hypothesis it follows that R strongly M-ring. Thus $0 = y.xz = x.yz = (yx)z - (y,x,z) - (xy)z - (x,y,z) = [y,x]z - (y,x,z) + (x,y,z) = [x,y]z + (y,x,z) - (x,y,z)$. Consequently $[x,y]z = (x,y,z) - (y,x,z)$. (16)
Next we use the Semi-Jacobi identity, $[xz,y] = x[z,y] + [x,y]z + 2(x,z,y) + (y,x,z)$ (17)
which holds in any right alternative ring, to transform (16). Using right alternativity it follows

that $[xz,y] = x[z,y] + (x,z,y)$. (18)
Letting $y = u \in U$ in (18) gives $(x,z,u) = 0$, Therefore $U \subseteq M$. But then lemma (1) implies that $U = 0$. This completes the proof of the lemma.

Lemma 9: If R has char. $\neq 2$ then it satisfies the identity $(xy,x,y) + (y,xy,x) + (x,y,xy) = 0$ for all $x,y \in R$.
Proof: We recall the identity $(xy,x,y) + (x,y,[x,y]) = x(y,x,y) + (x,x,y)y$ (19)
which is valid in all right alternative rings of char. $\neq 2$ [13]. However using (1) we have $(x,-yx,y) = -y(x,x,y) = y(x,x,y)$, using the right alternative identity. As a result of lemma (5) we have $[y,(x,x,y)] = 0$. Thus $(xy,x,y) + (x,y,xy) = x(y,x,y)$ (20)
Using (1) and the right alternative identity we have $x(y,x,y) = -x(y,y,x) = -(y,xy,x)$. This completes the proof of the lemma.

Lemma 10: If R has char. $\neq 2$ then it satisfies the identity $(w,x,y)(w,x,z) = 0$.
Proof: As noted before in the proof of lemma (5) we have the identity $(w,x,y)x = (w,x,xy)$. Using this,(1) and right alternativity it follows that $(w,x,y)\ (w,x,z) = (w,(w,x,y)x,z) = (w,(w,x,xy),z)$. Now using lemma (2) and char. $\neq 2$ we have $(w,(w,x,xy),z) = 0$ showing $(w,x,y)(w,x,z) = 0$. This completes the proof of the lemma.

Theorem 3: Each of the following conditions will imply that a right alternative ring R of char. $\neq 2$ which satisfies (1) must be associative:
(i) R is simple. (ii) R is prime with minimum condition on right ideals.
(iii) R is prime with minimum condition on trivial left ideals.

Proof: Defining a ring to be locally (-1,1) if the sub ring generated by any two elements is a(-1,1) ring and assuming char. $\neq 2,3$ it is shown in [3] that a simple locally (-1,1) nil ring is associative and shown in [2] that a prime locally (-1,1) ring with minimum condition on right ideals is alternative. However using lemma (9) and other facts established in this paper, it can be verified that both these results carry over to right alternative rings of char. $\neq 2$ that satisfy (1). Thus (i) and (ii) follow from [10,11] and theorem 2. If R satisfies (iii) and is not associative then there exist elements $x,y \in R$ such that $(x,y,R) \neq 0$. As a result of (1) we know that (x,y,R) is a left ideal. As a consequence of lemma (10) its square is zero. If L is a minimal left ideal contained in it then by [2] it would be a nonzero ideal which squares to zero, a contradiction. Thus R must have been associative to begin with. This concludes the proof of the theorem.

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