TWIST PERIODIC ORBITS FOR CONTINUOUS MAPS OF THE EIGHT SPACE

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Abstract. Let g be a continuous map from $\mathbb{S}^1$ to itself has a fixed point at $(0,0)$, we prove that g has a twist periodic orbit if there is a rational rotation number

1.1 Introduction:

In [1], Alseda, L. et.al introduced the definition of twist periodic orbit, they studied this type of periodic on continuous circle map of degree one which has a fixed point In.[4], Misurewicz, M studied the twist sets for circle maps. Let $\mathbb{S}^1$ be the one point union of two unit circles attached at $(0,0)$ and $g: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a map has a fixed point at $(0,0)$, we say that $z \in \mathbb{S}^1$ is a periodic point of g if there exists a positive integer n such that $g^n(z) = z$. The period of z is the smallest integer satisfying this relation. Let $P(g)$ be the set of periods of g . if $z \in \mathbb{S}^1$ is a periodic point of period

n , then the orbit of z is the set $\{g^i(z): i = 1, 2, \dots, n\}$. Throughout this work, we will use e as projection from $\mathbb{S}^1$ onto $\mathbb{S}^1$. There exists many projections from $\mathbb{S}^1$ into $\mathbb{S}^1$, but we define e as:

$$e(z) = \begin{cases} i(1 - z^2) & \forall z \in \mathbb{S}^1 \text{ } \text{Im}(z) \geq 0 \\ i(z^2 - 1) & \forall z \in \mathbb{S}^1 \text{ } \text{Im}(z) < 0 \end{cases}$$

This projection is one to one. We can find a lift map f from $\mathbb{S}^1$ to itself such that $eg = fe$. Let G be a lift to g and $e\pi(t) = e(\cos 2\pi t + i \sin 2\pi t)$ a projection of $\mathbb{R} \rightarrow \mathbb{S}^1$. It is clear π is many to one, G is not defined uniquely; that is if G_1 and G are two lifting of g then $G = G_1 + m$ with $m \in \mathbb{Z}$. the degree of g is the number a which satisfy $G(t+1) = G(t) + a, \forall t \in \mathbb{R}$. In this work, we use $a=1$. If $z \in \mathbb{S}^1$ is a periodic point of g of period n and $e\pi(t) = z$ then $G^n(t) = t + k$; where $k \in \mathbb{Z}$ this

imply $\frac{k}{n}$ is the rotation number of z we denoted it by $\rho(G, z)$. We denote by $L_G(g)$ the set of all rotation numbers of g , so $L_G(g) = \{\rho(G, z): z \in \mathbb{S}^1; G: \mathbb{R} \rightarrow \mathbb{R}\}$ is called the rotation set of g . Since f is unique lift to g then the properties of rotation number and rotation set on the circle satisfy on the eight space. So the rotation set does not depend on the choice of t , but it depends on the periodic orbit. Also, if (a_i) is a convergent sequence of a and $a_i \in L_G(g)$ then $a \in L_G(g)$.

The another property $\rho(G, z) = \rho(G_1, z) + k$ if $G = G_1 + k$. If g has no periodic point the rotation number exist for all $t \in \mathbb{R}$ and it is independent of t and is irrational number there are more properties on circle maps satisfy on $\mathbb{S}^1$ (see [2] and [3]). Let $z_1, z_2 \in \mathbb{S}^1$ we denote (z_1, z_2) (resp. $[z_1, z_2]$) the open (resp. closed) arc of $\mathbb{S}^1$ from z_1 counter clockwise to z_2 . Let $\frac{d}{m}$ be

a rational number, we assume $m > 0$ and let T be a periodic orbit of period m and rotation number $\frac{d}{m}$ with $(d, m) = 1$. Let G be lift such that G is order preserving on the set $(e\pi)^{-1}T$ then we say that T

is a twist periodic orbit (for simplify we say TPO) of g period m and rotation number $\frac{d}{m}$. It is clear, every periodic orbit of period 1 is a TPO. If $\frac{d}{m} > 0$ then $(z_i, z_{i+1}) \cap T = \emptyset, \forall i = 1, 2, \dots, m-1$ and $(z_m, z_1) \cap T = \emptyset$, and if $\frac{d}{m} < 0$ then

$(z_{i+1}, z_i) \cap = \emptyset, \forall i = 1, 2, \dots, m-1$ and $(z_1, z_m) \cap T = \emptyset$. Then notion of TPO of period m characterizes the simplest behavior of the graph a map which has the same rotation number.

1.2 Main Theorems

In this section, we prove two theorems which find the TPO, also we prove other properties of it.

Proposition 1.2.1: If $\frac{k}{n} < c$; $a, c \in L_G(g)$ and $k, n \in \mathbb{Z}$ with $n > 0$ and $\frac{k}{n} \in L_G(g)$ then $n \in p(g)$.

Proof: Suppose $\rho(G, z) = \frac{k}{n}$ thus there exists $z \in \mathbb{R}$ such that there is $t \in \mathbb{R}$ and $e\pi(t) = z$ then by definition $\rho(G, z) = \lim_{n \rightarrow \infty} \sup \frac{G^n(t) - t}{n} = \frac{k}{n}$ so there is $\epsilon > 0$ such that $-\epsilon < \frac{G^n(t) - t}{n} - \frac{k}{n} < \epsilon$ then $\frac{k}{n} - \epsilon < \frac{G^n(t) - t}{n} < \frac{k}{n} + \epsilon$ Hence $\frac{k+t}{n} - \epsilon < \frac{G^n(t)}{n} < \frac{k+t}{n} + \epsilon$, thus

$(k+t) - n\epsilon < G^n(t) < (k+t) + n\epsilon$ Since ϵ is an arbitrary and $G^n(t) \in \mathbb{R}$ is any point then $G^n(t) = t+k$ that is t is a periodic point of G of period n then by properties of the lift then g has periodic of period n then $n \in p(g)$

Corollary 1.2.2: $L_G(g) \cap \frac{\mathbb{Z}}{2} \neq \emptyset$ if and only if $z = 0$ is the only fixed point of g .

Proof: If $z=0$ then there exists $t \in \frac{\mathbb{Z}}{2}$ such that $e\pi(t) = z = 0$. Let $L_G(g) \cap \frac{\mathbb{Z}}{2} \neq \emptyset$ so there is $\frac{k}{2} \in L_G(g) \cap \frac{\mathbb{Z}}{2}$ hence there is $z \in \mathbb{R}$ such that $\rho(G, z) = \frac{k}{2}$ so there is $t \in \mathbb{Z}, e\pi(t) = 0$. ■

Corollary 1.2.3: $L_G(g) \cap \mathbb{Z} \neq \emptyset$ if and only if g has a fixed point $z \neq 0$

Proof $\rightarrow$ Since $L_G(g) \cap \mathbb{Z} \neq \emptyset$ then there is an integer number $k \in L_G(g)$, so by proposition 1.2.1, $1 \in p(g)$ In other word g has a fixed point
 $\leftarrow$ since $1 \in p(g)$ then there $z \in \mathbb{R}$ such that $g(z) = z$, so $G(t) = t + k$; where $k \in \mathbb{Z}$, thus $\rho(G, z) = \frac{k}{1} \in L_G(g)$ then $k \in L_G(g) \cap \mathbb{Z} \neq \emptyset$. ■

The proposition below gives a geometrical interpretation of a TPO on the eight space :

Proposition 1.2.4: Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous map, $T = \{z_1, z_2, \dots, z_m\}$ be a TPO of period m and rotation number $\frac{d}{m}$. Then $g(z_i) = z_{i+d \bmod m}$ if $z_i \neq 0$ where $i=1, 2, \dots, m$.

Proof: Suppose $\frac{d}{m} > 0$ let G be the lifting of g for which $\rho(G, z_i) = \frac{d}{m}, (e\pi)^{-1}T = \{t_i : i \in \mathbb{Z}\}$ with $\dots < t_{-2} < t_{-1} < t_0 < t_1 < t_2 < \dots$. Assume $e\pi(t_1) = z_1$. We claim $t_{i+rm} = t_i + r$, if $t_i \in \frac{\mathbb{Z}}{2}$. Since t_i is a periodic point of period m . then $e\pi(t_{i+rm}) = e\pi(t_i)$. Thus $\pi(t_{i+rm}) = \pi(t_i); t_i \in \frac{\mathbb{Z}}{2}$, so $t_{i+rm} = t_i + r$. Also, we claim $t_{i+rm} = t_i + \frac{r}{2}$, if $t_i \in \frac{\mathbb{Z}}{2}$. Then $(t_{i+rm}) = e(-\pi(t_i)) = -e\pi(t_i)$ so then $\pi(t_{i+rm}) = -\pi(t_i)$. Then $t_{i+rm} = t_i + \frac{r}{2}$. Case 1: $e\pi(t_{i+rm}) = e\pi(t_i) = e\pi(t_i) = z_i$, for $k \in \mathbb{Z}$ and $i = 1, 2, \dots, m$, since F on $(e\pi)^{-1}T$ is one to one and by definition of TPO $(e\pi)^{-1}T$ order preserving then $G(t_i) = t_{i+j}$; for some $j \in \mathbb{Z}$ and all i , so $G^2(t_i) = G(G(t_i)) = G(t_{i+j}) = t_{i+2j}$ thus by induction we get $G^m(t_i) = t_{i+mj}$ Since t_i is a periodic

point of period m and $\rho(G, z) = \frac{d}{m}$ then $t_i + d = G^m(t_i)$
 $= G^{m-1}(G(t_i)) = G^{m-1}(t_{i+j}) = \dots = t_{i+mj} = t_i + j$ (by claim 1) then $d=j$ hence
 $g(z_i) = z_{i+d(\text{mod } m)}; \forall i = 1, 2, \dots, m$. Since T is an orbit not a union of several orbits then
 $(d, m)=1$ Case 2: If $t_i \in \frac{\mathbb{Z}}{2}$, $e\pi(t_{i+rm}) = e\pi(t_i + \frac{r}{2}) = e(-\pi(t_i)) = -e\pi(t_i) = 0$. Since, F on
 $(e\pi)^{-1}T$ is one to one and by definition of TPO $(e\pi)^{-1}T$ order preserving then $G(t_i) = t_{i+j}$; for some
 $j \in \frac{\mathbb{Z}}{2}$ in the same way $t_i + d = G^m(t_i) = G^{m-1}(t_{i+j}) = \dots = t_{i+mj} = t_i + j$ (by claim 2) thus $d=j$; where $j \in \frac{\mathbb{Z}}{2}$ this
 imply $g(z_i) = f(0) = 0$, so z_i is an eventually fixed point. ■

An A-graph of g with respect to A_m and B_m is an oriented graph with vertices $I_1, I_2, \dots, I_{m+n-1}$ such that
 if I_α covers I_β k times but not $k+1$ times then there are k arrows from I_α to I_β . a sequence
 $I_{i_0} \rightarrow I_{i_1} \rightarrow \dots \rightarrow I_{i_r}$ is an A-graph of g is called a path of length r and the path is called a loop if $I_{i_0} = I_{i_r}$
 a loop is called simple if $I_{i_j} \neq I_{i_k}$ for $0 \leq j < k < r$.

The lemma below generalize lemma in [2] on circle map:

Lemma 1.2.5: Let $g: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a continuous map of degree one. If $J_0 \rightarrow J_1 \rightarrow \dots \rightarrow J_{n-1} \rightarrow J_0$ is a
 loop in a A-graph of g then there exists a fixed point z of g^n such that $g^i(z) \in J_i$ for $i = 1, 2, \dots, n-1$.

Lemma 1.2.6: Let g be an monotone map (not necessarily continuous) of a closed interval I into
 itself. Then g has a fixed point.

Proof: suppose g is increasing map Let $t = \sup\{y: g(y) > y\} \neq 0$ since g is increasing, if
 $t < g(t)$ then $\forall y \in (t, g(t))$ then $g(y) > g(t) > y$ but this contraction.

If $t > g(t)$ then $\forall y \in (t, g(t))$ then $g(y) < g(t) < y$ but this contraction. Thus
 $t = g(t)$ that is, t is a fixed point. If g is decreasing map then the proof is similar ■

Let $A = \{a_1, a_2, \dots, a_n\}$ be an invariant set that is $g(A) \subset A$, let
 $t_1 < t_2 < \dots < t_n < t_n + 1$ be such that $e\pi(t_i) = a_i$ for $i=1, 2, \dots, n$. Let G be a lift of
 g , we denote by $\bar{G}$ the map such that :

- 1) $\bar{G}(t_i + k) = G(t_i + k)$ for $i=1, 2, \dots, n$ and $k \in \mathbb{Z}$.
- 2) $\bar{G}$ on $[t_i + k, t_{i-1} + k]$ is linear for $i=1, 2, \dots, n$ and $k \in \mathbb{Z}$.
- 3) $\bar{G}$ on $[t_n + k, t_{n-1} + k]$ is linear for $k \in \mathbb{Z}$.

We call $\bar{G}$ the A linearization of G . we denote by $\bar{g}$ the map of $\mathbb{S}^1$ of degree one which has $\bar{G}$ as
 lift, also we say $\bar{g}$ the A linearization of g . If $\bar{G} = g$ that is $\bar{g} = g$ then we say that G and g are
 linear.

Lemma 1.2.7: Let $g: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a continuous map and A be an invariant subset under g . let $\bar{g}$ be
 the A linearization of g and suppose that $\bar{g}$ has a TPO T of period s and
 $\rho(\bar{G}, z) = \frac{r}{s}$ with $(r, s)=1$; [where $\bar{G}$ is the lift of $\bar{g}$ obtained by A linearization of G . If
 $T \not\subset A$ and $\bar{G}$ is increasing at every point of $(e\pi)^{-1}T$ then g has a TPO of period s and
 $\rho(G, z) = \frac{r}{s}$.

Proof: Let $t_1 < t_2 < \dots < t_n < t_1 + 1$ such that $e\pi\{t_1, t_2, \dots, t_n\} = A$ and
 $I_i = [t_i, t_{i+1}]$, $i = 1, 2, \dots, n-1$ and $I_n = [t_n, t_1 + 1]$ be subsets of $\mathbb{R}$. Let U be the partition
 of $\mathbb{R}$ given by the points of $(e\pi)^{-1}A$, since $T \not\subset A$; T is a periodic orbit and A is invariant
 set for $\bar{g}$ so $\bar{g}(A) \subset A$ then T and A are disjoint that is $T \cap A = \emptyset$. We choose $z \in T$ this

imply there is unique $i_1 \in \{1, 2, \dots, n\}$ such that there is $t \in I_{i_1}$ with $e\pi(t) = z$ hence there is unique $i_1 \in \{1, 2, \dots, n\}$ and $n_j \in \mathbb{Z}$ such that $\bar{G}^{j-1}(t) \in I_{i_j} + n_j, j = 1, 2, \dots, s+1$ since $(\bar{G}, z) = \frac{r}{s}; \bar{g}$ has a TPO pf period s. Then $\bar{G}^s(t) = t + r$ thus $i_{s+1} = i_1$ and $n_{s+1} = r$ In such a way we obtain the following path in the U-graph of $\bar{G}: I_{i_1} \rightarrow I_{i_2} + n_2 \rightarrow \dots \rightarrow I_{i_s} + n_s \rightarrow I_{i_1} + r$ We will define $M_s: I_{i_1} + r \rightarrow I_{i_s} + n_s$ and $M_j: I_{i_{j+1}} + n_{j+1} \rightarrow I_{i_j} + n_j, j=1, 2, \dots, s-1$ such that $M_j(t) = \inf\{y \in I_{i_j} + n_j: G(y) = t\}$. Since $\bar{G}$ is increasing on every interval $I_{i_j} + n_j, j = 1, 2, \dots, s$ these maps M_j, M_s are increasing. Therefore $M: I_{i_1} + r \rightarrow I_{i_1} + r$ such that $M = M_1 \circ M_2 \circ \dots \circ M_s + r$ is clear M is increasing. By lemma 1.2.6, M has a fixed point $p \in I_{i_1} + r$ let $p_1 = p - r \in I_{i_1}$ thus $G^j(p_1) \in I_{i_{j+1}} + n_{j+1}; \text{for } j = 0, 1, \dots, s-1$ and $G^s(p_1) = p_1 + r \in I_{i_1} + r$ then the orbit $X = \{e\pi(p_1), g(e\pi(p_1)), \dots, g^{s-1}(e\pi(p_1))\}$ is periodic of period s and $\rho(G, e\pi(p_1)) = \frac{r}{s}$. We claim X is a TPO, to show that: Since $T = \{z_1, z_2, \dots, z_s\}$ periodic orbit on $(e\pi)^{-1}T \subset R$ that is $\{G^j(p_1): j = 0, 1, \dots, s-1\} + Z \subset R$. Now we will compare between $(e\pi)^{-1}T$ and X. let $g^i e\pi(p_1), g^j e\pi(p_1) \in X$ such that a set is to left and which to the right. The elements of the U-partition are: Case 1: If these intervals are different then $X = \{g^i(e\pi(p_1)): i = 0, 1, \dots, s-1\}$ are the end points of U-partition, therefore we get X is a TPO. Case 2: If they are the same, then one has to go along the orbits of these two points, since we only use increasing pieces of the maps G and $\bar{G}$ then $\{G^j: j = 0, 1, \dots, s-1\} + Z$ and $(e\pi)^{-1}T$ in the same order, since T is a TPO so X is a TPO. ■

We will define these maps as: let $a, b \in R$ and G be a lift of $g, G_r(a) = \sup\{G(b): b \leq a\}$ and $G_l(a) = \inf\{G(b): b \geq a\}$. We call $g_r: 8 \rightarrow 8$ such that G_r is a lift of g_r and $g_l: 8 \rightarrow 8$ such that G_l is a lift of g_l this mean g_r, g_l are continuous maps of degree one.

Lemma 1.2.8: Let $g: 8 \rightarrow 8$ be a continuous map of degree one and W be a periodic orbit of g of period m and $\rho(G, z) = \frac{d}{m}$ such that $0 < \frac{d}{m} < 1$ and $(d, m) = 1$. suppose that W is not a TPO. then

i- g has a TPO R of period s and rotation number $\frac{r}{s}$ with $(r, s) = 1$ and $\frac{d}{m} < \frac{r}{s}$.

ii- g has a TPO L of period n and rotation number $\frac{j}{n}$ with $(j, n) = 1$ and $\frac{j}{n} < \frac{d}{m}$.

Proof : Without loss of generality, If we assume g is linear on W and F is increasing on $(e\pi)^{-1}R$ then this don't for generality. Then by lemma 1.2.7, let w be a periodic orbit of g of period m and $(G, z) = \frac{d}{m}$. let U be the partition of 8 by elements of W , since g is onto then for all

$I \in U \exists J \in U$. Such that g_r - covers I since the number of intervals of U is finite so we have a loop of length s in the U-graph of g_r with $1 \leq s \leq m$. Assume that the loop is the shortest one of the U-graph of g_r ince W is not a TPO at least on interval $I \in U$ satisfy $(g_r|I) = \text{constant}$ then $s < m$. By lemma 1.2.5, this loop gives us a periodic orbit R of

g_r of period s and rotation number $\frac{r}{s}$. since $s < m$ then $R \neq W$ all intervals on which G_r is non-decreasing (constant or increasing) then R is a TPO for g_r this imply R is a TPO for f . By proposition 1.2.4, we get $(r, s) = 1$. Now, we need prove that

$\frac{r}{s} < \frac{d}{m}$. Since G_r is non-decreasing and $G_r \geq G$ then by induction we get $G_r^n \geq G^n$ for all n , so if $t \in (e\pi)^{-1}R, t' \in (e\pi)^{-1}W$ and $t > t'$ then $\frac{r}{s} = \lim_{n \rightarrow \infty} \frac{G_r^n(t) - t}{n} = \lim_{n \rightarrow \infty} \frac{G_r^n(t)}{n} \geq \lim_{n \rightarrow \infty} \frac{G_r^n(t')}{n} = \frac{d}{m}$ then $\frac{r}{s} \geq \frac{d}{m}$. ■

Let $E(\cdot)$ be the integer function.

Theorem 1.2.9: Let $g: \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be a continuous map of degree one. If $\frac{d}{m} \in L(g)$ such that $\frac{d}{m}$ is an end point of $L(g)$ then all periodic orbits of g of period m and rotation number $\frac{d}{m}$ are TPO.

Proof: Either $\frac{d}{m} \in \mathbb{Z}$ such that $(d, m) = 1$ then $m = 1$ then by proposition 1.2.4, g has a TPO. or $\frac{d}{m} \in \mathbb{Z}$ let F be a lifting of g such that $\frac{d}{m}$ is the right end point of $L_F(f) = [a, \frac{d}{m}]$. let $G' = G - E\left(\frac{d}{m}\right), \frac{d_1}{m} = \frac{d}{m} - E\left(\frac{d}{m}\right) \in L_G(g)$ then $(d, m) = 1$ and $\frac{d_1}{m} \in (0, 1)$ then by lemma 1.2.8, we get all the periodic orbits of period m and $\rho(F', z) = \frac{d'}{m}$ are TPO. $\rho(G, z) = \rho(G', z) + E\left(\frac{d}{m}\right) = \frac{d_1}{m} + E\left(\frac{d}{m}\right) = \frac{d}{m}$, so T is TPO of period m and $p(G, z) = \frac{d}{m}$. ■

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