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Investigations on the Theory of Riemann Zeta Function III: A Simple Proof for a Restricted Lindelöf Hypothesis

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ABSTRACT. We create new formulas for proving Lindelöf Hypothesis from Zeta Function.

1. INTRODUCTION

In [1], we encounter that Lindelöf, in his paper [2], showed that the function $\mu\left(\frac{1}{2}\right)$ is decreasing and convex. This led him to conjecture that $\mu\left(\frac{1}{2}\right)=0$, and consequently that

$$\zeta\left(\frac{1}{2}+it\right)\leq t^{\epsilon}, \tag{1.1}$$

whatever $\epsilon>0$.

In this paper, we will demonstrate that

$$\zeta\left(\frac{1}{2}+it\right)\leq t^{\epsilon}, \tag{1.2}$$

whatsoever $\epsilon>0$ and any $t\in\mathbb{R}_{\geq 0.30611645227149686\ldots}$.

2. PRELIMINARES

In [3] we have a convergent series representation for $\zeta(s,q)$, defined when $q>-1$ and any complex $s\neq 1$, which was given by Helmut Hasse, in 1930 [4]:

$$\zeta(s,q)=\frac{1}{s-1}\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}(q+k)^{1-s}. \tag{2.1}$$

This series converges uniformly on compact subsets of the s-plane to an entire function. The inner sum may be understood to be the n*th* forward difference of q^{1-s}; i.e.,

$$\Delta^n q^{1-s}=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(q+k)^{1-s}, \tag{2.2}$$

where Δ denotes the forward difference operator. As soon, we may write

$$\begin{aligned}\zeta(s,q)&=\frac{1}{s-1}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\Delta^n q^{1-s}\\ &=\frac{1}{s-1}\frac{\log(1+\Delta)}{\Delta}q^{1-s}.\end{aligned} \tag{2.3}$$

In [5], we see that the complex exponentiation satisfies

$$(a+bi)^{c+di}=(a^2+b^2)^{(c+id)/2}e^{i(c+id)\arg(a+ib)}, \tag{2.4}$$

where $\arg(z)$ denotes the complex argument. We explicitly written in terms of real and imaginary parts, as follows

$$\begin{aligned}(a+bi)^{c+di}&=(a^2+b^2)^{c/2}\\ \times\left\{\cos\left[c\cdot\arg(a+ib)+\frac{1}{2}d\log(a^2+b^2)\right]\right.&+\left.i\sin\left[c\cdot\arg(a+ib)+\frac{1}{2}d\log(a^2+b^2)\right]\right\}.\end{aligned} \tag{2.5}$$

THEOREM 1. *Let* $\mathrm{Re}(s) > 0$ *and* $s \neq 1$, *then*

$$\zeta(\mathrm{s}) = \frac{2^{\mathrm{s}}}{2^{\mathrm{s}}-1} + \frac{\zeta\left(\mathrm{s},\frac{3}{2}\right)}{2^{\mathrm{s}}-1}, \tag{2.6}$$

where $\zeta(s)$ *is the Riemann zeta function and* $\zeta(s,a)$*is the Hurwitz zeta function.*

Proof. See [6]. □

3. LEMMAS AND THEOREMS

LEMMA 1. *For* $t \in \mathbb{R}_{\geq 0}$, *then*

$$\zeta\left(\frac{1}{2}+it,\frac{3}{2}\right) = -\frac{2t}{4t^2+1} \times \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{1/2} \cos\left[t\log\left(\frac{2k+3}{2}\right)\right]$$
$$-\frac{4t}{4t^2+1} \times \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{1/2} \sin\left[t\log\left(\frac{2k+3}{2}\right)\right]$$
$$+\frac{2i}{4t^2+1} \times \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{1/2} \sin\left[t\log\left(\frac{2k+3}{2}\right)\right]$$
$$-\frac{4it}{4t^2+1} \times \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{1/2} \cos\left[t\log\left(\frac{2k+3}{2}\right)\right], \tag{3.1}$$

where $\zeta(s,a)$ *is the Hurwitz zeta function.*

Proof. Let $s = \frac{1}{2} + it$ and $q = \frac{3}{2}$ in (2.1)

$$\zeta\left(\frac{1}{2}+it,\frac{3}{2}\right) = \frac{2}{-1+2it} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{\frac{1}{2}-it}$$
$$= \frac{2}{-1+2it} \times \left(\frac{-1-2it}{-1-2it}\right) \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{\frac{1}{2}-it}$$
$$= \frac{-2-4it}{4t^2+1} \times \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{\frac{1}{2}-it}. \tag{3.2}$$

On the other hand, we evaluate, using (2.5), that

$$\left(\frac{2k+3}{2}\right)^{\frac{1}{2}-it} = \left(\frac{2k+3}{2}\right)^{1/2} \times$$
$$\times \left\{\cos\left[\frac{1}{2}\cdot\arg\left(\frac{2k+3}{2}\right) - t\log\left(\frac{2k+3}{2}\right)\right] + i\sin\left[\frac{1}{2}\cdot\arg\left(\frac{2k+3}{2}\right) - t\log\left(\frac{2k+3}{2}\right)\right]\right\}. \tag{3.3}$$

Since $k = 0,1,2,3,\ldots$, then $\arg\left(\frac{2k+3}{2}\right) = 0$; we set this in (3.3)

$$\left(\frac{2k+3}{2}\right)^{\frac{1}{2}-it} = \left(\frac{2k+3}{2}\right)^{1/2} \times \left\{\cos\left[-t\log\left(\frac{2k+3}{2}\right)\right] + i\sin\left[-t\log\left(\frac{2k+3}{2}\right)\right]\right\}$$
$$= \left(\frac{2k+3}{2}\right)^{1/2} \times \left\{\cos\left[t\log\left(\frac{2k+3}{2}\right)\right] - i\sin\left[t\log\left(\frac{2k+3}{2}\right)\right]\right\}. \tag{3.4}$$

Substituting (3.4) in (3.2), we encounter

$$\zeta\left(\frac{1}{2}+it,\frac{3}{2}\right)=\left(\frac{-2-4it}{4t^2+1}\right)\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\times$$

$$\times\left\{\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]-i\sin\left[t\log\left(\frac{2k+3}{2}\right)\right]\right\}$$

$$=\left(\frac{-2-4it}{4t^2+1}\right)\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]$$

$$+\left(\frac{-4t+2i}{4t^2+1}\right)\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\sin\left[t\log\left(\frac{2k+3}{2}\right)\right]$$

$$=-\frac{2t}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]$$

$$-\frac{4it}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]$$

$$-\frac{4t}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\sin\left[t\log\left(\frac{2k+3}{2}\right)\right]$$

$$+\frac{2i}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\sin\left[t\log\left(\frac{2k+3}{2}\right)\right]$$

$$=-\frac{2t}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]$$

$$-\frac{4t}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\sin\left[t\log\left(\frac{2k+3}{2}\right)\right]$$

$$+\frac{2i}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\sin\left[t\log\left(\frac{2k+3}{2}\right)\right]$$

$$-\frac{4it}{4t^2+1}\times\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\cos\left[t\log\left(\frac{2k+3}{2}\right)\right].\square \tag{3.5}$$

THEOREM 1. *For* $\epsilon>0$ *and any* $t\in\mathbb{R}_{\geq 0}$, *then*

$$\zeta\left(\frac{1}{2}+it\right)\leq t^{\epsilon}. \tag{3.6}$$

Proof. Hereinafter, we will use the *reductio ad absurdum* to prove (3.6).

Step 1. We assume, by hypothesis, that

$$\zeta\left(\frac{1}{2}+it\right)>t^{\epsilon}, \tag{3.7}$$

whatsoever $\epsilon>0$ and any $t\in\mathbb{R}_{\geq 0}$. Let $s=\frac{1}{2}+it$ in (2.6)

$$\zeta\left(\frac{1}{2}+it\right)=\frac{2^{\frac{1}{2}+it}}{2^{\frac{1}{2}+it}-1}+\frac{\zeta\left(\frac{1}{2}+it,\frac{3}{2}\right)}{2^{\frac{1}{2}+it}-1}. \tag{3.8}$$

Substituting the right-hand side of (3.8) in (3.7), we obtain

$$\frac{2^{\frac{1}{2}+it}}{2^{\frac{1}{2}+it}-1}+\frac{\zeta\left(\frac{1}{2}+it,\frac{3}{2}\right)}{2^{\frac{1}{2}+it}-1}>t^{\epsilon}\Rightarrow 2^{it}+\frac{t^{\epsilon}}{\sqrt{2}}+\frac{\zeta\left(\frac{1}{2}+it,\frac{3}{2}\right)}{\sqrt{2}}>2^{it}t^{\epsilon}. \tag{3.9}$$

Step 2. We defined

$$C(t):=\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\cos\left[t\log\left(\frac{2k+3}{2}\right)\right] \tag{3.10}$$

and

$$S(t):=\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\sin\left[t\log\left(\frac{2k+3}{2}\right)\right] \tag{3.11}$$

using this in (3.1)

$$\begin{aligned}\zeta\left(\frac{1}{2}+it,\frac{3}{2}\right)&=-\frac{2t}{4t^2+1}C(t)-\frac{4t}{4t^2+1}S(t)+\frac{2i}{4t^2+1}S(t)-\frac{4it}{4t^2+1}C(t)\\&=-\frac{2t}{4t^2+1}\cdot[C(t)+2S(t)]+\frac{2i}{4t^2+1}[S(t)-2tC(t)].\end{aligned} \tag{3.12}$$

Step 3. We use (2.5) for evaluate 2^{it}, as follows

$$2^{it}=\cos(t\log 2)+i\sin(t\log 2). \tag{3.13}$$

Step 4. From (3.9), (3.12) and (3.13), we obtain

$$\cos(t\log 2)+i\sin(t\log 2)+\frac{t^{\epsilon}}{\sqrt{2}}-\frac{\sqrt{2}t}{4t^2+1}\cdot[C(t)+2S(t)]+\frac{\sqrt{2}i}{4t^2+1}[S(t)-2tC(t)]> \\ \cos(t\log 2)\cdot t^{\epsilon}+i\sin(t\log 2)\cdot t^{\epsilon}, \tag{3.14}$$

so

$$\cos(t\log 2)+\frac{t^{\epsilon}}{\sqrt{2}}-\frac{\sqrt{2}t}{4t^2+1}\cdot[C(t)+2S(t)]+i\sin(t\log 2)+\frac{\sqrt{2}i}{4t^2+1}[S(t)-2tC(t)]> \\ \cos(t\log 2)\cdot t^{\epsilon}+i\sin(t\log 2)\cdot t^{\epsilon}, \tag{3.15}$$

Step 5. We compare the real and imaginary part separately of (3.15). Therefore, for the real part, we find

$$\begin{aligned}&\cos(t\log 2)+\frac{t^{\epsilon}}{\sqrt{2}}-\frac{\sqrt{2}t}{4t^2+1}\cdot[C(t)+2S(t)]>\cos(t\log 2)\cdot t^{\epsilon}\\&\Rightarrow\cos(t\log 2)+\frac{t^{\epsilon}}{\sqrt{2}}>\cos(t\log 2)\cdot t^{\epsilon}+\frac{\sqrt{2}t}{4t^2+1}\cdot[C(t)+2S(t)]\\&\Rightarrow\frac{\cos(t\log 2)}{t^{\epsilon}}+\frac{1}{\sqrt{2}}>\cos(t\log 2)+\frac{\sqrt{2}t}{(4t^2+1)t^{\epsilon}}\cdot[C(t)+2S(t)].\end{aligned} \tag{3.16}$$

and, for the imaginary part, we encounter

$$\begin{aligned}&\sin(t\log 2)+\frac{\sqrt{2}}{4t^2+1}[S(t)-2tC(t)]>\sin(t\log 2)\cdot t^{\epsilon}\\&\Rightarrow\sin(t\log 2)+\frac{\sqrt{2}}{4t^2+1}S(t)>\sin(t\log 2)\cdot t^{\epsilon}+\frac{2t\sqrt{2}}{4t^2+1}C(t).\end{aligned} \tag{3.17}$$

Step 6. Real part. We divide the inequality (3.16) by $t^{2+\epsilon}$

$$\frac{\cos(t\log 2)}{t^{2(\epsilon+1)}}+\frac{1}{t^{2+\epsilon}\sqrt{2}}>\frac{\cos(t\log 2)}{t^{2+\epsilon}}+\frac{\sqrt{2}t}{(4t^2+1)t^{2(\epsilon+1)}}\cdot[C(t)+2S(t)]. \tag{3.18}$$

We evaluate the limit when $t\to+\infty$ of (3.18)

$$\lim_{t\to+\infty}\left[\frac{\cos(t\log 2)}{t^{2(\epsilon+1)}}+\frac{1}{t^{2+\epsilon}\sqrt{2}}\right]>\lim_{t\to+\infty}\left\{\frac{\cos(t\log 2)}{t^{2+\epsilon}}+\frac{\sqrt{2}t}{(4t^2+1)t^{2(\epsilon+1)}}\cdot[C(t)+2S(t)]\right\},$$

$$\lim_{t\to+\infty}\left[\frac{\cos(t\log 2)}{t^{2(\epsilon+1)}}\right]+\lim_{t\to+\infty}\left(\frac{1}{t^{2+\epsilon}\sqrt{2}}\right)>\lim_{t\to+\infty}\left[\frac{\cos(t\log 2)}{t^{2+\epsilon}}\right]+\lim_{t\to+\infty}\left\{\frac{\sqrt{2}t}{(4t^2+1)t^{2(\epsilon+1)}}\cdot [C(t)+2S(t)]\right\}. \tag{3.19}$$

Note 1: We calculate, for any $\epsilon>0$ and $k=0,1,2,3,\ldots$: 1.º) when $t\to+\infty$, then $\frac{\cos(t\log 2)}{t^{2(\epsilon+1)}}\to 0$; 2.º) when $t\to+\infty$, then $\frac{1}{t^{2+\epsilon}\sqrt{2}}\to 0$; 3.º) when $t\to+\infty$, then $\frac{\cos(t\log 2)}{t^{2+\epsilon}}\to 0$; 4.º) and when $t\to+\infty$, then $\frac{\sqrt{2}t}{(4t^2+1)t^{2(\epsilon+1)}}\cdot\left\{\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]+2\sin\left[t\log\left(\frac{2k+3}{2}\right)\right]\right\}\to 0$. So, our hypothesis is false, because $0\ngtr 0$; but, $0=0$. We conclude that our hypothesis for the real part is false.

Step 7. Imaginary part. We divide the inequality (3.17) by $t^{2+\epsilon}$

$$\frac{\sin(t\log 2)}{t^{2+\epsilon}}+\frac{\sqrt{2}}{(4t^2+1)t^{2+\epsilon}}S(t)>\frac{\sin(t\log 2)\cdot t^{\epsilon}}{t^{2+\epsilon}}+\frac{2t\sqrt{2}}{(4t^2+1)t^{2+\epsilon}}C(t)$$

$$\Rightarrow\frac{\sin(t\log 2)}{t^{2+\epsilon}}+\frac{\sqrt{2}}{(4t^2+1)t^{2+\epsilon}}S(t)>\frac{\sin(t\log 2)}{t^{2}}+\frac{2t\sqrt{2}}{(4t^2+1)t^{2+\epsilon}}C(t). \tag{3.20}$$

We evaluate the limit when $t\to+\infty$ of (3.20)

$$\lim_{t\to+\infty}\left[\frac{\sin(t\log 2)}{t^{2+\epsilon}}+\frac{\sqrt{2}}{(4t^2+1)t^{2+\epsilon}}S(t)\right]>\lim_{t\to+\infty}\left[\frac{\sin(t\log 2)}{t^{2}}+\frac{2t\sqrt{2}}{(4t^2+1)t^{2+\epsilon}}C(t)\right],$$

$$\lim_{t\to+\infty}\left[\frac{\sin(t\log 2)}{t^{2+\epsilon}}\right]+\lim_{t\to+\infty}\left[\frac{\sqrt{2}}{(4t^2+1)t^{2+\epsilon}}S(t)\right]>\lim_{t\to+\infty}\left[\frac{\sin(t\log 2)}{t^{2}}\right]+\lim_{t\to+\infty}\left[\frac{2t\sqrt{2}}{(4t^2+1)t^{2+\epsilon}}C(t)\right]. \tag{3.21}$$

Note 2: We calculate, for any $\epsilon>0$ and $k=0,1,2,3,\ldots$: 1.º) when $t\to+\infty$, then $\frac{\sin(t\log 2)}{t^{2+\epsilon}}\to 0$; 2.º) when $t\to+\infty$, then $\frac{\sqrt{2}}{(4t^2+1)t^{2+\epsilon}}\sin\left[t\log\left(\frac{2k+3}{2}\right)\right]\to 0$; 3.º) when $t\to+\infty$, then $\frac{\sin(t\log 2)}{t^{2}}\to 0$; 4.º) and when $t\to+\infty$, then $\frac{2t\sqrt{2}}{(4t^2+1)t^{2+\epsilon}}\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]\to 0$. So, our hypothesis is false, because $0\ngtr 0$; but, $0=0$. We conclude that our hypothesis for the imaginary part is false.

Step 8. We evaluate the function $\max_{t\in\mathbb{R}_{\geq 0},k\in\mathbb{Z}_{\geq 0}}\left\{\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]\right\}$. For $k=0$, then $\max_{t\in\mathbb{R}_{\geq 0}}\left\{\cos\left[t\log\left(\frac{3}{2}\right)\right]\right\}=1$, when $t=0$; for $k=1$, then $\max_{t\in\mathbb{R}_{\geq 0}}\left\{\cos\left[t\log\left(\frac{5}{2}\right)\right]\right\}=1$, when $t=0$; for $k=2$, then $\max_{t\in\mathbb{R}_{\geq 0}}\left\{\cos\left[t\log\left(\frac{7}{2}\right)\right]\right\}=1$, when $t=0$; and so on. We easily deduzimos que $\max_{t\in\mathbb{R}_{\geq 0},k\in\mathbb{Z}_{\geq 0}}\left\{\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]\right\}=1$, when $t=0$. So,

$\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]=:C(t)<$
$\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\max_{t\in\mathbb{R}_{\geq 0},k\in\mathbb{Z}_{\geq 0}}\left\{\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]\right\}=$
$\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2},$ (3.22)

where $\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}$ tends to 1.0095 ... very slowly. Simplifying, we find

$$1.0095\ldots>C(t) \tag{3.23}$$

On the other hand, continuando with the evaluate for the function $\min_{t\in\mathbb{R}_{\geq 0},k\in\mathbb{Z}_{\geq 0}}\left\{\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]\right\}$. For $k=0$, then $\min_{t\in\mathbb{R}_{\geq 0}}\left\{\cos\left[t\log\left(\frac{3}{2}\right)\right]\right\}=\cos(\pi)=-1$, when $t=\frac{\pi}{\log 3-\log 2}=7.74812083893\ldots$; for $k=1$, then $\min_{t\in\mathbb{R}_{\geq 0}}\left\{\cos\left[t\log\left(\frac{5}{2}\right)\right]\right\}=\cos(\pi)=-1$, when $t=\frac{\pi}{\log 5-\log 2}=3.42859809044\ldots$; for $k=2$, then $\min_{t\in\mathbb{R}_{\geq 0}}\left\{\cos\left[t\log\left(\frac{7}{2}\right)\right]\right\}=\cos(\pi)=-1$, when $t=\frac{\pi}{\log 7-\log 2}=2.50773109726\ldots$; and so on. We easily deduzimos que $\min_{t\in\mathbb{R}_{\geq 0},k\in\mathbb{Z}_{\geq 0}}\left\{\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]\right\}=\cos(\pi)=-1$, when $t=\frac{\pi}{\log(2k+3)-\log 2}$, for $k=0,1,2,3,\ldots$. So,

$\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]=:C(t)>>$
$\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}\min_{t\in\mathbb{R}_{\geq 0},k\in\mathbb{Z}_{\geq 0}}\left\{\cos\left[t\log\left(\frac{2k+3}{2}\right)\right]\right\}=$
$-\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2},$ (3.24)

where $\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{2k+3}{2}\right)^{1/2}$ tends to 1.0095 ... very slowly. Simplifying, we have

$$C(t)>-1.0095\ldots. \tag{3.25}$$

From (3.23) and (3.25), it follows that

$$-1.0095\ldots<C(t)<1.0095\ldots. \tag{3.26}$$

Step 10. From (3.16) and (3.17), we have the following system of inequalities

$$\begin{cases}\cos(t\log 2)+\frac{t^{\epsilon}}{\sqrt{2}}>\cos(t\log 2)\cdot t^{\epsilon}+\frac{\sqrt{2}t}{4t^2+1}\cdot[C(t)+2S(t)]\\ \sin(t\log 2)+\frac{\sqrt{2}S(t)}{4t^2+1}>\sin(t\log 2)\cdot t^{\epsilon}+2t\frac{\sqrt{2}C(t)}{4t^2+1}\end{cases} \tag{3.27}$$

Dividing both the members of (3.27) by t^{ϵ}, we encounter

$$\begin{cases}\frac{\cos(t\log 2)}{t^{\epsilon}}+\frac{1}{\sqrt{2}}>\cos(t\log 2)+\frac{\sqrt{2}t}{t^{\epsilon}(4t^2+1)}\cdot[C(t)+2S(t)]\\ \frac{\sin(t\log 2)}{t^{\epsilon}}+\frac{\sqrt{2}S(t)}{t^{\epsilon}(4t^2+1)}>\sin(t\log 2)+\frac{2\sqrt{2}tC(t)}{t^{\epsilon}(4t^2+1)}\end{cases} \tag{3.28}$$

From second inequality of (3.28), we find

$$\frac{\sqrt{2}S(t)}{t^{\epsilon}(4t^2+1)}>\sin(t\log 2)-\frac{\sin(t\log 2)}{t^{\epsilon}}+\frac{2\sqrt{2}tC(t)}{t^{\epsilon}(4t^2+1)}. \tag{3.29}$$

From first inequality of (3.28) and the right-hand side of (3.29), we have

$$\frac{\cos(t\log 2)}{t^{\epsilon}}+\frac{1}{\sqrt{2}}>\cos(t\log 2)+\frac{\sqrt{2}t}{t^{\epsilon}(4t^2+1)}\cdot[C(t)+2S(t)]$$

$$=\cos(t\log 2)+t\cdot\left[\frac{\sqrt{2}C(t)}{t^{\epsilon}(4t^2+1)}+2\frac{\sqrt{2}S(t)}{t^{\epsilon}(4t^2+1)}\right]$$

$$>\cos(t\log 2)+t\cdot\left[\frac{\sqrt{2}C(t)}{t^{\epsilon}(4t^2+1)}+2\sin(t\log 2)-2\frac{\sin(t\log 2)}{t^{\epsilon}}+\frac{4\sqrt{2}tC(t)}{t^{\epsilon}(4t^2+1)}\right]$$

$$=\cos(t\log 2)+2t\sin(t\log 2)-2t\frac{\sin(t\log 2)}{t^{\epsilon}}+\frac{\sqrt{2}t(4t+1)C(t)}{t^{\epsilon}(4t^2+1)}. \tag{3.30}$$

Consequently,

$$\frac{\cos(t\log 2)}{t^{\epsilon}}+\frac{1}{\sqrt{2}}>\cos(t\log 2)+2t\sin(t\log 2)-2t\frac{\sin(t\log 2)}{t^{\epsilon}}+\frac{\sqrt{2}t(4t+1)C(t)}{t^{\epsilon}(4t^2+1)}. \tag{3.31}$$

Thus,

$$\frac{t^{\epsilon}(4t^2+1)}{\sqrt{2}t(4t+1)}\left[\frac{\cos(t\log 2)}{t^{\epsilon}}-\cos(t\log 2)-2t\sin(t\log 2)+2t\frac{\sin(t\log 2)}{t^{\epsilon}}+\frac{1}{\sqrt{2}}\right]>C(t). \tag{3.32}$$

Step 10. We divide (3.32) by (3.23)

$$\frac{\cos(t\log 2)}{t^{\epsilon}}-\cos(t\log 2)-2t\sin(t\log 2)+2t\frac{\sin(t\log 2)}{t^{\epsilon}}+\frac{1}{\sqrt{2}}>1.0095\ldots\frac{\sqrt{2}t(4t+1)}{t^{\epsilon}(4t^2+1)},$$

$$\frac{\cos(t\log 2)}{t^{\epsilon}}+2t\frac{\sin(t\log 2)}{t^{\epsilon}}+\frac{1}{\sqrt{2}}>\cos(t\log 2)+2t\sin(t\log 2)+1.0095\ldots\frac{\sqrt{2}t(4t+1)}{t^{\epsilon}(4t^2+1)}. \tag{3.33}$$

Multiplying (3.33) by t^{ϵ}, we find

$$\cos(t\log 2)+2t\sin(t\log 2)>\left[\cos(t\log 2)+2t\sin(t\log 2)+1.0095\ldots\frac{\sqrt{2}t(4t+1)}{4t^2+1}-\frac{1}{\sqrt{2}}\right]t^{\epsilon}. \tag{3.34}$$

From (3.34), we deduce that

$$\cos(t\log 2)+2t\sin(t\log 2)>\left[\cos(t\log 2)+2t\sin(t\log 2)+1.0095\ldots\frac{\sqrt{2}t(4t+1)}{4t^2+1}-\frac{1}{\sqrt{2}}\right]t^{\epsilon}>\left[\cos(t\log 2)+2t\sin(t\log 2)+1.0095\ldots\frac{\sqrt{2}t(4t+1)}{4t^2+1}-\frac{1}{\sqrt{2}}\right]t^{0}=\cos(t\log 2)+2t\sin(t\log 2)+1.0095\ldots\frac{\sqrt{2}t(4t+1)}{4t^2+1}-\frac{1}{\sqrt{2}}. \tag{3.35}$$

It is easy to see that: for $1.0095\ldots\frac{\sqrt{2}t(4t+1)}{4t^2+1}\geq\frac{1}{\sqrt{2}}$, the inequality (3.35) is false.

Note 3: For any $t\in\mathbb{R}^{+}$ and $t\geq 0.30611645227149686\ldots$, then the inequality (3.40) is false. So, our hypothesis is false.

Step 11. Thus, from Notes 1, 2 and 3, we show that $\zeta\left(\frac{1}{2}+it\right)\leq t^{\epsilon}$, for whatsoever $\epsilon>0$ and any $t\in\mathbb{R}_{\geq 0.30611645227149686\ldots}$. □

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