

## Investigations on the Theory of Riemann Zeta Function III: A Simple Proof for a Restricted Lindelöf Hypothesis

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**ABSTRACT.** We create new formulas for proving Lindelöf Hypothesis from Zeta Function.

### 1. INTRODUCTION

In [1], we encounter that Lindelöf, in his paper [2], showed that the function  $\mu\left(\frac{1}{2}\right)$  is decreasing and convex. This led him to conjecture that  $\mu\left(\frac{1}{2}\right) = 0$ , and consequently that

$$\zeta\left(\frac{1}{2} + it\right) \leq t^\epsilon, \tag{1.1}$$

whatever  $\epsilon > 0$ .

In this paper, we will demonstrate that

$$\zeta\left(\frac{1}{2} + it\right) \leq t^\epsilon, \tag{1.2}$$

whatsoever  $\epsilon > 0$  and any  $t \in \mathbb{R}_{\geq 0.30611645227149686\dots}$ .

### 2. PRELIMINARES

In [3] we have a convergent series representation for  $\zeta(s, q)$ , defined when  $q > -1$  and any complex  $s \neq 1$ , which was given by Helmut Hasse, in 1930 [4]:

$$\zeta(s, q) = \frac{1}{s-1} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} (q+k)^{1-s}. \tag{2.1}$$

This series converges uniformly on compact subsets of the s-plane to an entire function. The inner sum may be understood to be the  $n$ th forward difference of  $q^{1-s}$ ; i.e.,

$$\Delta^n q^{1-s} = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} (q+k)^{1-s}, \tag{2.2}$$

where  $\Delta$  denotes the forward difference operator. As soon, we may write

$$\begin{aligned} \zeta(s, q) &= \frac{1}{s-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \Delta^n q^{1-s} \\ &= \frac{1}{s-1} \frac{\log(1+\Delta)}{\Delta} q^{1-s}. \end{aligned} \tag{2.3}$$

In [5], we see that the complex exponentiation satisfies

$$(a + bi)^{c+di} = (a^2 + b^2)^{(c+id)/2} e^{i(c+id)\arg(a+ib)}, \tag{2.4}$$

where  $\arg(z)$  denotes the complex argument. We explicitly written in terms of real and imaginary parts, as follows

$$\begin{aligned} &(a + bi)^{c+di} = (a^2 + b^2)^{c/2} \\ &\times \left\{ \cos \left[ c \cdot \arg(a + ib) + \frac{1}{2} d \log(a^2 + b^2) \right] + i \sin \left[ c \cdot \arg(a + ib) + \frac{1}{2} d \log(a^2 + b^2) \right] \right\}. \end{aligned} \tag{2.5}$$

THEOREM 1. Let  $\text{Re}(s) > 0$  and  $s \neq 1$ , then

$$\zeta(s) = \frac{2^s}{2^s - 1} + \frac{\zeta\left(s, \frac{3}{2}\right)}{2^s - 1}, \tag{2.6}$$

where  $\zeta(s)$  is the Riemann zeta function and  $\zeta(s, a)$  is the Hurwitz zeta function.

*Proof.* See [6].  $\square$

### 3. LEMMAS AND THEOREMS

LEMMA 1. For  $t \in \mathbb{R}_{\geq 0}$ , then

$$\begin{aligned} \zeta\left(\frac{1}{2} + it, \frac{3}{2}\right) &= -\frac{2t}{4t^2 + 1} \times \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{1/2} \cos\left[t \log\left(\frac{2k+3}{2}\right)\right] \\ &\quad - \frac{4t}{4t^2 + 1} \times \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{1/2} \sin\left[t \log\left(\frac{2k+3}{2}\right)\right] \\ &\quad + \frac{2i}{4t^2 + 1} \times \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{1/2} \sin\left[t \log\left(\frac{2k+3}{2}\right)\right] \\ &\quad - \frac{4it}{4t^2 + 1} \times \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{1/2} \cos\left[t \log\left(\frac{2k+3}{2}\right)\right], \end{aligned} \tag{3.1}$$

where  $\zeta(s, a)$  is the Hurwitz zeta function.

*Proof.* Let  $s = \frac{1}{2} + it$  and  $q = \frac{3}{2}$  in (2.1)

$$\begin{aligned} \zeta\left(\frac{1}{2} + it, \frac{3}{2}\right) &= \frac{2}{-1 + 2it} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{\frac{1}{2}-it} \\ &= \frac{2}{-1 + 2it} \times \left(\frac{-1 - 2it}{-1 - 2it}\right) \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{\frac{1}{2}-it} \\ &= \frac{-2-4it}{4t^2+1} \times \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{\frac{1}{2}-it}. \end{aligned} \tag{3.2}$$

On the other hand, we evaluate, using (2.5), that

$$\begin{aligned} \left(\frac{2k+3}{2}\right)^{\frac{1}{2}-it} &= \left(\frac{2k+3}{2}\right)^{1/2} \times \\ &\times \left\{ \cos\left[\frac{1}{2} \cdot \arg\left(\frac{2k+3}{2}\right) - t \log\left(\frac{2k+3}{2}\right)\right] + i \sin\left[\frac{1}{2} \cdot \arg\left(\frac{2k+3}{2}\right) - t \log\left(\frac{2k+3}{2}\right)\right] \right\}. \end{aligned} \tag{3.3}$$

Since  $k = 0, 1, 2, 3, \dots$ , then  $\arg\left(\frac{2k+3}{2}\right) = 0$ ; we set this in (3.3)

$$\begin{aligned} \left(\frac{2k+3}{2}\right)^{\frac{1}{2}-it} &= \left(\frac{2k+3}{2}\right)^{1/2} \times \left\{ \cos\left[-t \log\left(\frac{2k+3}{2}\right)\right] + i \sin\left[-t \log\left(\frac{2k+3}{2}\right)\right] \right\} \\ &= \left(\frac{2k+3}{2}\right)^{1/2} \times \left\{ \cos\left[t \log\left(\frac{2k+3}{2}\right)\right] - i \sin\left[t \log\left(\frac{2k+3}{2}\right)\right] \right\}. \end{aligned} \tag{3.4}$$

Substituting (3.4) in (3.2), we encounter

$$\begin{aligned}
\zeta\left(\frac{1}{2} + it, \frac{3}{2}\right) &= \left(\frac{-2 - 4it}{4t^2 + 1}\right) \times \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{1/2} \times \\
&\quad \times \left\{ \cos \left[ t \log \left( \frac{2k+3}{2} \right) \right] - i \sin \left[ t \log \left( \frac{2k+3}{2} \right) \right] \right\} \\
&= \left(\frac{-2 - 4it}{4t^2 + 1}\right) \times \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{1/2} \cos \left[ t \log \left( \frac{2k+3}{2} \right) \right] \\
&\quad + \left(\frac{-4t + 2i}{4t^2 + 1}\right) \times \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{1/2} \sin \left[ t \log \left( \frac{2k+3}{2} \right) \right] \\
&= -\frac{2t}{4t^2 + 1} \times \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{1/2} \cos \left[ t \log \left( \frac{2k+3}{2} \right) \right] \\
&\quad - \frac{4it}{4t^2 + 1} \times \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{1/2} \cos \left[ t \log \left( \frac{2k+3}{2} \right) \right] \\
&\quad - \frac{4t}{4t^2 + 1} \times \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{1/2} \sin \left[ t \log \left( \frac{2k+3}{2} \right) \right] \\
&\quad + \frac{2i}{4t^2 + 1} \times \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{1/2} \sin \left[ t \log \left( \frac{2k+3}{2} \right) \right] \\
&= -\frac{2t}{4t^2 + 1} \times \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{1/2} \cos \left[ t \log \left( \frac{2k+3}{2} \right) \right] \\
&\quad - \frac{4t}{4t^2 + 1} \times \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{1/2} \sin \left[ t \log \left( \frac{2k+3}{2} \right) \right] \\
&\quad + \frac{2i}{4t^2 + 1} \times \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{1/2} \sin \left[ t \log \left( \frac{2k+3}{2} \right) \right] \\
&\quad - \frac{4it}{4t^2 + 1} \times \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{1/2} \cos \left[ t \log \left( \frac{2k+3}{2} \right) \right]. \quad \square
\end{aligned} \tag{3.5}$$

**THEOREM 1.** For  $\epsilon > 0$  and any  $t \in \mathbb{R}_{\geq 0}$ , then

$$\zeta\left(\frac{1}{2} + it\right) \leq t^\epsilon. \tag{3.6}$$

*Proof.* Hereinafter, we will use the *reductio ad absurdum* to prove (3.6).

Step 1. We assume, by hypothesis, that

$$\zeta\left(\frac{1}{2} + it\right) > t^\epsilon, \tag{3.7}$$

whatsoever  $\epsilon > 0$  and any  $t \in \mathbb{R}_{\geq 0}$ . Let  $s = \frac{1}{2} + it$  in (2.6)

$$\zeta\left(\frac{1}{2} + it\right) = \frac{2^{\frac{1}{2}+it}}{2^{\frac{1}{2}+it}-1} + \frac{\zeta\left(\frac{1}{2}+it, \frac{3}{2}\right)}{2^{\frac{1}{2}+it}-1}. \tag{3.8}$$

Substituting the right-hand side of (3.8) in (3.7), we obtain

$$\frac{2^{\frac{1}{2}+it}}{2^{\frac{1}{2}+it}-1} + \frac{\zeta\left(\frac{1}{2}+it, \frac{3}{2}\right)}{2^{\frac{1}{2}+it}-1} > t^\epsilon \Rightarrow 2^{it} + \frac{t^\epsilon}{\sqrt{2}} + \frac{\zeta\left(\frac{1}{2}+it, \frac{3}{2}\right)}{\sqrt{2}} > 2^{it}t^\epsilon. \tag{3.9}$$

Step 2. We defined

$$C(t) := \sum_{n=0}^\infty \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{1/2} \cos\left[t \log\left(\frac{2k+3}{2}\right)\right] \tag{3.10}$$

and

$$S(t) := \sum_{n=0}^\infty \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{1/2} \sin\left[t \log\left(\frac{2k+3}{2}\right)\right] \tag{3.11}$$

using this in (3.1)

$$\begin{aligned} \zeta\left(\frac{1}{2} + it, \frac{3}{2}\right) &= -\frac{2t}{4t^2+1}C(t) - \frac{4t}{4t^2+1}S(t) + \frac{2i}{4t^2+1}S(t) - \frac{4it}{4t^2+1}C(t) \\ &= -\frac{2t}{4t^2+1} \cdot [C(t) + 2S(t)] + \frac{2i}{4t^2+1} [S(t) - 2tC(t)]. \end{aligned} \tag{3.12}$$

Step 3. We use (2.5) for evaluate  $2^{it}$ , as follows

$$2^{it} = \cos(t \log 2) + i \sin(t \log 2). \tag{3.13}$$

Step 4. From (3.9), (3.12) and (3.13), we obtain

$$\begin{aligned} \cos(t \log 2) + i \sin(t \log 2) + \frac{t^\epsilon}{\sqrt{2}} - \frac{\sqrt{2}t}{4t^2+1} \cdot [C(t) + 2S(t)] + \frac{\sqrt{2}i}{4t^2+1} [S(t) - 2tC(t)] > \\ \cos(t \log 2) \cdot t^\epsilon + i \sin(t \log 2) \cdot t^\epsilon, \end{aligned} \tag{3.14}$$

so

$$\begin{aligned} \cos(t \log 2) + \frac{t^\epsilon}{\sqrt{2}} - \frac{\sqrt{2}t}{4t^2+1} \cdot [C(t) + 2S(t)] + i \sin(t \log 2) + \frac{\sqrt{2}i}{4t^2+1} [S(t) - 2tC(t)] > \\ \cos(t \log 2) \cdot t^\epsilon + i \sin(t \log 2) \cdot t^\epsilon, \end{aligned} \tag{3.15}$$

Step 5. We compare the real and imaginary part separately of (3.15). Therefore, for the real part, we find

$$\begin{aligned} \cos(t \log 2) + \frac{t^\epsilon}{\sqrt{2}} - \frac{\sqrt{2}t}{4t^2+1} \cdot [C(t) + 2S(t)] > \cos(t \log 2) \cdot t^\epsilon \\ \Rightarrow \cos(t \log 2) + \frac{t^\epsilon}{\sqrt{2}} > \cos(t \log 2) \cdot t^\epsilon + \frac{\sqrt{2}t}{4t^2+1} \cdot [C(t) + 2S(t)] \\ \Rightarrow \frac{\cos(t \log 2)}{t^\epsilon} + \frac{1}{\sqrt{2}} > \cos(t \log 2) + \frac{\sqrt{2}t}{(4t^2+1)t^\epsilon} \cdot [C(t) + 2S(t)]. \end{aligned} \tag{3.16}$$

and, for the imaginary part, we encounter

$$\begin{aligned} \sin(t \log 2) + \frac{\sqrt{2}}{4t^2+1} [S(t) - 2tC(t)] > \sin(t \log 2) \cdot t^\epsilon \\ \Rightarrow \sin(t \log 2) + \frac{\sqrt{2}}{4t^2+1} S(t) > \sin(t \log 2) \cdot t^\epsilon + \frac{2t\sqrt{2}}{4t^2+1} C(t). \end{aligned} \tag{3.17}$$

Step 6. Real part. We divide the inequality (3.16) by  $t^{2+\epsilon}$

$$\frac{\cos(t \log 2)}{t^{2(\epsilon+1)}} + \frac{1}{t^{2+\epsilon}\sqrt{2}} > \frac{\cos(t \log 2)}{t^{2+\epsilon}} + \frac{\sqrt{2}t}{(4t^2+1)t^{2(\epsilon+1)}} \cdot [C(t) + 2S(t)]. \quad (3.18)$$

We evaluate the limit when  $t \rightarrow +\infty$  of (3.18)

$$\begin{aligned} \lim_{t \rightarrow +\infty} \left[ \frac{\cos(t \log 2)}{t^{2(\epsilon+1)}} + \frac{1}{t^{2+\epsilon}\sqrt{2}} \right] &> \lim_{t \rightarrow +\infty} \left\{ \frac{\cos(t \log 2)}{t^{2+\epsilon}} + \frac{\sqrt{2}t}{(4t^2+1)t^{2(\epsilon+1)}} \cdot [C(t) + 2S(t)] \right\}, \\ \lim_{t \rightarrow +\infty} \left[ \frac{\cos(t \log 2)}{t^{2(\epsilon+1)}} \right] + \lim_{t \rightarrow +\infty} \left( \frac{1}{t^{2+\epsilon}\sqrt{2}} \right) &> \lim_{t \rightarrow +\infty} \left[ \frac{\cos(t \log 2)}{t^{2+\epsilon}} \right] + \lim_{t \rightarrow +\infty} \left\{ \frac{\sqrt{2}t}{(4t^2+1)t^{2(\epsilon+1)}} \cdot [C(t) + 2S(t)] \right\}. \end{aligned} \quad (3.19)$$

Note 1: We calculate, for any  $\epsilon > 0$  and  $k = 0, 1, 2, 3, \dots$ : 1.º) when  $t \rightarrow +\infty$ , then  $\frac{\cos(t \log 2)}{t^{2(\epsilon+1)}} \rightarrow 0$ ; 2.º) when  $t \rightarrow +\infty$ , then  $\frac{1}{t^{2+\epsilon}\sqrt{2}} \rightarrow 0$ ; 3.º) when  $t \rightarrow +\infty$ , then  $\frac{\cos(t \log 2)}{t^{2+\epsilon}} \rightarrow 0$ ; 4.º) and when  $t \rightarrow +\infty$ , then  $\frac{\sqrt{2}t}{(4t^2+1)t^{2(\epsilon+1)}} \cdot \left\{ \cos \left[ t \log \left( \frac{2k+3}{2} \right) \right] + 2 \sin \left[ t \log \left( \frac{2k+3}{2} \right) \right] \right\} \rightarrow 0$ . So, our hypothesis is false, because  $0 \neq 0$ ; but,  $0 = 0$ . We conclude that our hypothesis for the real part is false.

Step 7. Imaginary part. We divide the inequality (3.17) by  $t^{2+\epsilon}$

$$\begin{aligned} \frac{\sin(t \log 2)}{t^{2+\epsilon}} + \frac{\sqrt{2}}{(4t^2+1)t^{2+\epsilon}} S(t) &> \frac{\sin(t \log 2) \cdot t^\epsilon}{t^{2+\epsilon}} + \frac{2t\sqrt{2}}{(4t^2+1)t^{2+\epsilon}} C(t) \\ \Rightarrow \frac{\sin(t \log 2)}{t^{2+\epsilon}} + \frac{\sqrt{2}}{(4t^2+1)t^{2+\epsilon}} S(t) &> \frac{\sin(t \log 2)}{t^2} + \frac{2t\sqrt{2}}{(4t^2+1)t^{2+\epsilon}} C(t). \end{aligned} \quad (3.20)$$

We evaluate the limit when  $t \rightarrow +\infty$  of (3.20)

$$\begin{aligned} \lim_{t \rightarrow +\infty} \left[ \frac{\sin(t \log 2)}{t^{2+\epsilon}} + \frac{\sqrt{2}}{(4t^2+1)t^{2+\epsilon}} S(t) \right] &> \lim_{t \rightarrow +\infty} \left[ \frac{\sin(t \log 2)}{t^2} + \frac{2t\sqrt{2}}{(4t^2+1)t^{2+\epsilon}} C(t) \right], \\ \lim_{t \rightarrow +\infty} \left[ \frac{\sin(t \log 2)}{t^{2+\epsilon}} \right] + \lim_{t \rightarrow +\infty} \left[ \frac{\sqrt{2}}{(4t^2+1)t^{2+\epsilon}} S(t) \right] &> \lim_{t \rightarrow +\infty} \left[ \frac{\sin(t \log 2)}{t^2} \right] + \\ \lim_{t \rightarrow +\infty} \left[ \frac{2t\sqrt{2}}{(4t^2+1)t^{2+\epsilon}} C(t) \right]. \end{aligned} \quad (3.21)$$

Note 2: We calculate, for any  $\epsilon > 0$  and  $k = 0, 1, 2, 3, \dots$ : 1.º) when  $t \rightarrow +\infty$ , then  $\frac{\sin(t \log 2)}{t^{2+\epsilon}} \rightarrow 0$ ; 2.º) when  $t \rightarrow +\infty$ , then  $\frac{\sqrt{2}}{(4t^2+1)t^{2+\epsilon}} \sin \left[ t \log \left( \frac{2k+3}{2} \right) \right] \rightarrow 0$ ; 3.º) when  $t \rightarrow +\infty$ , then  $\frac{\sin(t \log 2)}{t^2} \rightarrow 0$ ; 4.º) and when  $t \rightarrow +\infty$ , then  $\frac{2t\sqrt{2}}{(4t^2+1)t^{2+\epsilon}} \cos \left[ t \log \left( \frac{2k+3}{2} \right) \right] \rightarrow 0$ . So, our hypothesis is false, because  $0 \neq 0$ ; but,  $0 = 0$ . We conclude that our hypothesis for the imaginary part is false.

Step 8. We evaluate the function  $\max_{t \in \mathbb{R}_{\geq 0}, k \in \mathbb{Z}_{\geq 0}} \left\{ \cos \left[ t \log \left( \frac{2k+3}{2} \right) \right] \right\}$ . For  $k = 0$ , then  $\max_{t \in \mathbb{R}_{\geq 0}} \left\{ \cos \left[ t \log \left( \frac{3}{2} \right) \right] \right\} = 1$ , when  $t = 0$ ; for  $k = 1$ , then  $\max_{t \in \mathbb{R}_{\geq 0}} \left\{ \cos \left[ t \log \left( \frac{5}{2} \right) \right] \right\} = 1$ , when  $t = 0$ ; for  $k = 2$ , then  $\max_{t \in \mathbb{R}_{\geq 0}} \left\{ \cos \left[ t \log \left( \frac{7}{2} \right) \right] \right\} = 1$ , when  $t = 0$ ; and so on. We easily deduzimos que  $\max_{t \in \mathbb{R}_{\geq 0}, k \in \mathbb{Z}_{\geq 0}} \left\{ \cos \left[ t \log \left( \frac{2k+3}{2} \right) \right] \right\} = 1$ , when  $t = 0$ . So,

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{1/2} \cos \left[ t \log \left(\frac{2k+3}{2}\right) \right] =: C(t) < \\ & \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{1/2} \max_{t \in \mathbb{R}_{\geq 0}, k \in \mathbb{Z}_{\geq 0}} \left\{ \cos \left[ t \log \left(\frac{2k+3}{2}\right) \right] \right\} = \\ & \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{1/2}, \end{aligned} \tag{3.22}$$

where  $\sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{1/2}$  tends to 1.0095 ... very slowly. Simplifying, we find

$$1.0095 \dots > C(t) \tag{3.23}$$

On the other hand, continuando with the evaluate for the function  $\min_{t \in \mathbb{R}_{\geq 0}, k \in \mathbb{Z}_{\geq 0}} \left\{ \cos \left[ t \log \left(\frac{2k+3}{2}\right) \right] \right\}$ . For  $k = 0$ , then  $\min_{t \in \mathbb{R}_{\geq 0}} \left\{ \cos \left[ t \log \left(\frac{3}{2}\right) \right] \right\} = \cos(\pi) = -1$ , when  $t = \frac{\pi}{\log 3 - \log 2} = 7.74812083893 \dots$ ; for  $k = 1$ , then  $\min_{t \in \mathbb{R}_{\geq 0}} \left\{ \cos \left[ t \log \left(\frac{5}{2}\right) \right] \right\} = \cos(\pi) = -1$ , when  $t = \frac{\pi}{\log 5 - \log 2} = 3.42859809044 \dots$ ; for  $k = 2$ , then  $\min_{t \in \mathbb{R}_{\geq 0}} \left\{ \cos \left[ t \log \left(\frac{7}{2}\right) \right] \right\} = \cos(\pi) = -1$ , when  $t = \frac{\pi}{\log 7 - \log 2} = 2.50773109726 \dots$ ; and so on. We easily deduzimos que  $\min_{t \in \mathbb{R}_{\geq 0}, k \in \mathbb{Z}_{\geq 0}} \left\{ \cos \left[ t \log \left(\frac{2k+3}{2}\right) \right] \right\} = \cos(\pi) = -1$ , when  $t = \frac{\pi}{\log(2k+3) - \log 2}$ , for  $k = 0, 1, 2, 3, \dots$ . So,

$$\begin{aligned} & \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{1/2} \cos \left[ t \log \left(\frac{2k+3}{2}\right) \right] =: C(t) >> \\ & \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{1/2} \min_{t \in \mathbb{R}_{\geq 0}, k \in \mathbb{Z}_{\geq 0}} \left\{ \cos \left[ t \log \left(\frac{2k+3}{2}\right) \right] \right\} = \\ & - \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{1/2}, \end{aligned} \tag{3.24}$$

where  $\sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{2k+3}{2}\right)^{1/2}$  tends to 1.0095 ... very slowly. Simplifying, we have

$$C(t) > -1.0095 \dots \tag{3.25}$$

From (3.23) and (3.25), it follows that

$$-1.0095 \dots < C(t) < 1.0095 \dots \tag{3.26}$$

Step 10. From (3.16) and (3.17), we have the following system of inequalities

$$\begin{cases} \cos(t \log 2) + \frac{t^\epsilon}{\sqrt{2}} > \cos(t \log 2) \cdot t^\epsilon + \frac{\sqrt{2}t}{4t^2+1} \cdot [C(t) + 2S(t)] \\ \sin(t \log 2) + \frac{\sqrt{2}S(t)}{4t^2+1} > \sin(t \log 2) \cdot t^\epsilon + 2t \frac{\sqrt{2}C(t)}{4t^2+1} \end{cases} \tag{3.27}$$

Dividing both the members of (3.27) by  $t^\epsilon$ , we encounter

$$\begin{cases} \frac{\cos(t \log 2)}{t^\epsilon} + \frac{1}{\sqrt{2}} > \cos(t \log 2) + \frac{\sqrt{2}t}{t^\epsilon(4t^2+1)} \cdot [C(t) + 2S(t)] \\ \frac{\sin(t \log 2)}{t^\epsilon} + \frac{\sqrt{2}S(t)}{t^\epsilon(4t^2+1)} > \sin(t \log 2) + \frac{2\sqrt{2}tC(t)}{t^\epsilon(4t^2+1)} \end{cases} \tag{3.28}$$

From second inequality of (3.28), we find

$$\frac{\sqrt{2}S(t)}{t^\epsilon(4t^2+1)} > \sin(t \log 2) - \frac{\sin(t \log 2)}{t^\epsilon} + \frac{2\sqrt{2}tC(t)}{t^\epsilon(4t^2+1)}. \tag{3.29}$$

From first inequality of (3.28) and the right-hand side of (3.29), we have

$$\begin{aligned}
& \frac{\cos(t \log 2)}{t^\epsilon} + \frac{1}{\sqrt{2}} > \cos(t \log 2) + \frac{\sqrt{2}t}{t^\epsilon(4t^2 + 1)} \cdot [C(t) + 2S(t)] \\
& = \cos(t \log 2) + t \cdot \left[ \frac{\sqrt{2}C(t)}{t^\epsilon(4t^2 + 1)} + 2 \frac{\sqrt{2}S(t)}{t^\epsilon(4t^2 + 1)} \right] \\
& > \cos(t \log 2) + t \cdot \left[ \frac{\sqrt{2}C(t)}{t^\epsilon(4t^2 + 1)} + 2 \sin(t \log 2) - 2 \frac{\sin(t \log 2)}{t^\epsilon} + \frac{4\sqrt{2}tC(t)}{t^\epsilon(4t^2 + 1)} \right] \\
& = \cos(t \log 2) + 2t \sin(t \log 2) - 2t \frac{\sin(t \log 2)}{t^\epsilon} + \frac{\sqrt{2}t(4t+1)C(t)}{t^\epsilon(4t^2+1)}. \quad (3.30)
\end{aligned}$$

Consequently,

$$\frac{\cos(t \log 2)}{t^\epsilon} + \frac{1}{\sqrt{2}} > \cos(t \log 2) + 2t \sin(t \log 2) - 2t \frac{\sin(t \log 2)}{t^\epsilon} + \frac{\sqrt{2}t(4t+1)C(t)}{t^\epsilon(4t^2+1)}. \quad (3.31)$$

Thus,

$$\frac{t^\epsilon(4t+1)}{\sqrt{2}t(4t+1)} \left[ \frac{\cos(t \log 2)}{t^\epsilon} - \cos(t \log 2) - 2t \sin(t \log 2) + 2t \frac{\sin(t \log 2)}{t^\epsilon} + \frac{1}{\sqrt{2}} \right] > C(t). \quad (3.32)$$

Step 10. We divide (3.32) by (3.23)

$$\begin{aligned}
& \frac{\cos(t \log 2)}{t^\epsilon} - \cos(t \log 2) - 2t \sin(t \log 2) + 2t \frac{\sin(t \log 2)}{t^\epsilon} + \frac{1}{\sqrt{2}} > 1.0095 \dots \frac{\sqrt{2}t(4t+1)}{t^\epsilon(4t^2+1)}, \\
& \frac{\cos(t \log 2)}{t^\epsilon} + 2t \frac{\sin(t \log 2)}{t^\epsilon} + \frac{1}{\sqrt{2}} > \cos(t \log 2) + 2t \sin(t \log 2) + 1.0095 \dots \frac{\sqrt{2}t(4t+1)}{t^\epsilon(4t^2+1)}. \quad (3.33)
\end{aligned}$$

Multiplying (3.33) by  $t^\epsilon$ , we find

$$\cos(t \log 2) + 2t \sin(t \log 2) > \left[ \cos(t \log 2) + 2t \sin(t \log 2) + 1.0095 \dots \frac{\sqrt{2}t(4t+1)}{4t^2+1} - \frac{1}{\sqrt{2}} \right] t^\epsilon. \quad (3.34)$$

From (3.34), we deduce that

$$\begin{aligned}
& \cos(t \log 2) + 2t \sin(t \log 2) > \left[ \cos(t \log 2) + 2t \sin(t \log 2) + 1.0095 \dots \frac{\sqrt{2}t(4t+1)}{4t^2+1} - \frac{1}{\sqrt{2}} \right] t^\epsilon > \\
& \left[ \cos(t \log 2) + 2t \sin(t \log 2) + 1.0095 \dots \frac{\sqrt{2}t(4t+1)}{4t^2+1} - \frac{1}{\sqrt{2}} \right] t^0 = \cos(t \log 2) + 2t \sin(t \log 2) + \\
& 1.0095 \dots \frac{\sqrt{2}t(4t+1)}{4t^2+1} - \frac{1}{\sqrt{2}} \quad (3.35)
\end{aligned}$$

It is easy to see that: for  $1.0095 \dots \frac{\sqrt{2}t(4t+1)}{4t^2+1} \geq \frac{1}{\sqrt{2}}$  the inequality (3.35) is false.

Note 3: For any  $t \in \mathbb{R}^+$  and  $t \geq 0.30611645227149686 \dots$ , then the inequality (3.40) is false. So, our hypothesis is false.

Step 11. Thus, from Notes 1, 2 and 3, we show that  $\zeta\left(\frac{1}{2} + it\right) \leq t^\epsilon$ , for whatsoever  $\epsilon > 0$  and any  $t \in \mathbb{R}_{\geq 0.30611645227149686 \dots}$ .  $\square$

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