Modified three-step iteration process with errors for three Local strongly pseudo contractive operators

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Abstract. The purpose of this paper is to introduce more general modified two step and three step Ishikawa iterative with errors for local strongly pseudo contractive and local strongly accretive mappings which is much more general than the important class of strongly pseudo contractive and strongly accretive mappings. Also, we study the convergence of this iteration for locally mappings in the framework of Banach space. The results presented in this paper improve, generalize of the results of Mogbademn and OlaleauRafq, Yuguang and Fang and others.

Introduction and preliminaries:

Let X be a real Banach spaces, X^* be the dual space on X . The normalized duality mapping $J: X \rightarrow 2^{X^*}$ is defined by

$$J(x) = \{f \in X^*: \langle x, f \rangle = \|x\| \|f\|, \|x\| = \|f\|\} \text{ for all } x \in X$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if X is a uniformly smooth Banach space, then J is single-valued and it is uniformly continuous on any bounded subset of X . In the sequel we shall denote single-valued normalized duality mapping by j . By means of the normalized duality mapping by J . The symbol J and $F(T)$ the identity mapping on X and the set of all fixed points of T respectively.

Let us recall the following three iteration processes due to Ishikawa[1], Mann[2] and Xu [3]. Let K be a non-empty convex subset of an arbitrary normed linear space X and $T: K \rightarrow K$ be an operator.

i. For any given $x_0 \in K$ the sequence defined by $\langle x_n \rangle$

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T y_n \\ y_n &= (1 - \beta_n)x_n + \beta_n T x_n, \quad n \geq 0 \end{aligned} \quad (1.1)$$

Is called the Ishikawa iteration sequence where $\langle \alpha_n \rangle, \langle \beta_n \rangle$ are real sequences in $[0, 1]$ satisfying appropriate conditions.

ii. In particular if $\beta_n = 0$ for all $n \geq 0$ then the sequence $\langle x_n \rangle$ defined by:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_1 y_n, \quad n \geq 0$$

Is called the Mann iteration sequence.

iii. For any $x_0 \in K$ the sequence $\langle x_n \rangle$ defined by:

$$\begin{aligned} x_{n+1} &= a_n x_n + b_n T y_n + c_n u_n \\ y_n &= a'_n x_n + b'_n T y_n + c'_n v_n, \quad n \geq 0. \end{aligned} \quad (1.2)$$

Where $\langle u_n \rangle, \langle v_n \rangle$ are arbitrary bounded sequences in K and $\langle a_n \rangle, \langle b_n \rangle, \langle c_n \rangle, \langle a'_n \rangle, \langle b'_n \rangle, \langle c'_n \rangle$ are real sequences in $[0, 1]$ such that $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$ for all $n \geq 0$ is called the Ishikawa iteration sequences with errors.

iv. If, with the same notations and definitions as in (iii), $b'_n = c'_n = 0$ for all $n \geq 0$, then the sequence $\langle x_n \rangle$ now defined by:

$$x_{n+1} = a_n x_n + b_n T y_n + c_n u_n, \quad n \geq 0$$

Is called the Mann iteration sequences with errors.

It is clear that the Ishikawa and Mann iteration sequences are all special cases of the Ishikawa and Mann iteration sequences with errors respectively.

Now Let, $T_1, T_2 : K \rightarrow K$ be two mappings. For any given $x_0 \in K$ the more general modified two step iteration $\langle x_n \rangle$ defined by:

$$\begin{aligned} x_{n+1} &= a_n x_n + b_n T_1 y_n + c_n u_n \\ y_n &= a'_n x_n + b'_n T_2 y_n + c'_n v_n, n \geq 0 \end{aligned} \quad (1.3)$$

Where the read sequence as in (1.2)

It is that the iteration schemes (1.1) – (1.2) are special cases of (1.3).

Noor[4] gave the following three- step iteration process for solving non-linear operator equations in real Banach space, let K be a nonempty closed convex subset of X and $T_1, T_2 : K \rightarrow K$ be mapping.

For an arbitrary $x_0 \in K$ be mapping. For an arbitrary the sequence $\langle x_n \rangle$ defined by:

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n) x_n + \alpha_n T_1 y_n \\ y_n &= (1 - \beta_n) x_n + \beta_n T_2 z_n \\ z_n &= (1 - \gamma_n) x_n + \gamma_n T_3 x_n, \quad n \geq 0 \end{aligned} \quad (1.4)$$

Where $\langle \alpha_n \rangle, \langle \beta_n \rangle, \langle \gamma_n \rangle$ are three sequences in $[0,1]$, is called the three-step iteration or (Noor iteration).

Rafiq[7] introduced the following new type of iteration, the modified three-step iteration $\langle x_n \rangle$ is defined by:

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n) x_n + \alpha_n T y_n \\ y_n &= (1 - \beta_n) x_n + \beta_n T z_n \\ z_n &= (1 - \gamma_n) x_n + \gamma_n T_3 x_n, \quad n \geq 0 \end{aligned} \quad (1.5)$$

Where $\langle \alpha_n \rangle, \langle \beta_n \rangle, \langle \gamma_n \rangle$ are real sequences in $[0,1]$.

It is clear that the iteration schemes (1.4) is special case of (1.5), we define the more general modified three - step iteration process with errors by:

$$\begin{aligned} x_{n+1} &= a_n x_n + b_n T_1 y_n + c_n u_n \\ y_n &= a'_n x_n + b'_n T_2 z_n + c'_n v_n \\ z_n &= a''_n x_n + b''_n T_3 y_n + c''_n w_n, n \geq 0 \end{aligned}$$

Where $\langle u_n \rangle, \langle v_n \rangle$ and $\langle w_n \rangle$ are arbitrary bounded sequences in K and $\langle a_n \rangle, \langle b_n \rangle, \langle c_n \rangle, \langle a'_n \rangle, \langle b'_n \rangle, \langle c'_n \rangle, \langle a''_n \rangle, \langle b''_n \rangle$ and $\langle c''_n \rangle$ are real sequences in $[0,1]$ satisfying some conditions.

Now, we introduce local strongly pseudo –contractive (local strongly accretive) operators as follows.

Definition(1.1)[6],[7]:-

An operator T with domain $D(T)$ and rang $R(T)$ in X is called:-1- Pseudo contractive if for all $r > 0$, the inequality

$$\|x - y\| \leq \| (1 + r)(x - y) - r(Tx - Ty) \| \quad (1.6)$$

2- Accretive if for all $r > 0$ the inequality

$$\|x - y\| \leq \|x - y + r(T_x - T_y)\| \quad (1.7)$$

holds for each pair of points $x, y \in D(T)$.

An operator T is called strongly pseudo-contractive (strongly accretive) if there exists a real number $k \in (0,1)$ such that

($T - KI$) is pseudo-contractive (accretive).

3- Local strongly pseudo-contractive if for each $x \in D(T)$ there exists $t_x > 1$ such that for all $y \in D(T)$ and $r > 0$

$$\|x - y\| \leq \|(1+r)(x - y) - r t_x (T_x - T_y)\| \quad (1.8)$$

4- local strongly accretive if for given $x \in D(T)$ there exists $k_x \in (0,1)$ such that for each $y \in D(T)$ there is

$$j(x - y) \in J(x - y) \text{ satisfying} \\ \langle T_x - T_y, j(x - y) \rangle \geq k_x \|x - y\|^2 \quad (1.9)$$

5- strongly pseudo-contractive (strongly accretive) if it is local strongly pseudo-contractive (local strongly accretive) and $t_x = t$ ($k_x = k$) and ($k_x = k$) independent of $x \in D(T)$

Remark (1-2):-[7]

1. Each strongly pseudo contractive operator is local strongly pseudo-contractive and each strongly accretive operator is local strongly accretive.

2. T is local strongly pseudo-contractive if and only if $(I - T)$ is local strongly accretive and $k_x = 1 - \frac{1}{t_x}$, where t_x and k_x are the constants appearing in (1.8) and (1.9) respectively.

3. If T is local strongly accretive then $(T - K_x I)$ accretive.

Lemma (1.3)[5]:-

Let X be a real Banach space and $J: X \rightarrow 2^{X^*}$ be the normalized duality mapping. Then, for any $x, y \in X$

$$\|x - y\|^2 \leq \|x\|^2 + 2 \langle y, j(x + y) \rangle, \quad \forall j(x + y) \in J(x + y)$$

Lemma (1.4) [8] :-

Let $\{\alpha_n\}$ be a non-negative sequence which satisfied the following inequality

$$\alpha_{n+1} \leq (1 - \lambda_n) \alpha_n + \delta_n$$

$$\sum_{n=1}^{\infty} \lambda_n = \infty \text{ and Where } \lambda_n \in (0,1), \forall n \in N, \text{ and } \delta_n = o(\lambda_n)$$

$$\text{Then } \lim_{n \rightarrow \infty} \alpha_n = 0$$

our purpose in this paper to prove that the modified three-step iteration process with error for three local strongly pseudo-contractive operators strongly convergence to fixed points. The results presented in this paper generalize the corresponding Main Results in [4],[5],[9],[10],[11],[12], and others

Theorem (2.1):-

Let X be a uniformly smooth Banach space and $T_1, T_2 : X \rightarrow X$ be a local strongly accretive mapping suppose that there exists a solution of the equation $T_i x = f$ ($i = 1, 2$) for some $f \in X$ define $H_i : X \rightarrow X$ by $H_i x = f + x - T_i x$ suppose that $R(X)$ is bounded.

Let $\{x_n\} \in X$ the two step iteration sequence with errors $\{x_n\}$ defined by:

$$x_{n+1} = a_n x_n + b_n H_1 y_n + c_n u_n \quad (1)$$

$$y_n = a'_n x_n + b'_n H_2 x_n + c'_n v_n \quad (2)$$

Where $\langle u_n \rangle$ and $\langle v_n \rangle$ are two bounded sequence in X and $\langle a_n \rangle, \langle b_n \rangle, \langle \hat{b}_n \rangle, \langle c_n \rangle$ and $\langle \hat{c}_n \rangle$ are real sequences in $[0,1]$ such that $a_n + b_n + c_n = \hat{a}_n + \hat{b}_n + \hat{c}_n = 1$ satisfying the conditions:-

$$\text{i) } \lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \hat{b}_n = 0$$

$$\text{ii) } \sum_{n=0}^{\infty} b_n = \infty,$$

$$\text{iii) } c_n \leq b_n, \hat{c}_n \leq \hat{b}_n.$$

Then the sequence $\langle x_n \rangle$ converges strongly to the unique fixed point of the equation $T_i x = f$

Proof:-

Let $T_i w = f$ so that w is a fixed point of H_i , since T_i is local strongly accretive mapping, it follows from definition of H_i ($i=1,2$) that

$$\langle H_i x - H_i y, j(x - y) \rangle \leq \|x - y\|^2 - \|x - y\|^2 \quad (3)$$

Setting $y=w$, we have

$$\langle H_i x - H_i w, j(x - w) \rangle \leq \|x - w\|^2 - \|x - w\|^2 \quad (4)$$

If z is a fixed point of H_i , then (4) with $x=z$ implies $w=z$, we prove that $\langle x_n \rangle$ and $\langle y_n \rangle$ are bounded.

Let $\sup \{ \|H_i x - H_i w\| + \|H_i y - w\| : n \geq 0 \} + \|x_0 - w\| = A_i$

$\sup \{ \|u_n\| + \|v_n\| : n \geq 0 \} = B, M_i = A_i + B$ for all $i=1,2$

and $M = \sup \{M_1, M_2\}$

by (1) and (iii) we get

$$\begin{aligned} \|x_{n+1} - w\| &\leq a_n \|x_n - w\| + b_n \|H_1 y_n - w\| + c_n \|u_n\| \\ &\leq a_n \|x_n - w\| + b_n A_1 + b_n B \\ &\leq a_n \|x_n - w\| + b_n M_1 \\ &\leq a_n \|x_n - w\| + b_n M \end{aligned} \quad (7)$$

Now, from (2) and (iii), we get

$$\begin{aligned} \|y_n - w\| &\leq \hat{a}_n \|x_n - w\| + \hat{b}_n \|H_2 x_n - w\| + \hat{c}_n \|v_n\| \\ &\leq \hat{a}_n \|x_n - w\| + \hat{b}_n A_2 + \hat{b}_n B \\ \|y_n - w\| &\leq \hat{a}_n \|x_n - w\| + \hat{b}_n M_2 \\ &\leq \hat{a}_n \|x_n - w\| + \hat{b}_n M \end{aligned} \quad (6)$$

Now, we show by induction that $\|x_n - w\| \leq M$ (7)

For all $n \geq 0$ for $n=0$ we have $\|x_0 - w\| \leq A_i \leq M_i \leq M$

Assume now that $\|x_n - w\| \leq M$ for some $n \geq 0$. then by (5)

$$\begin{aligned} \|x_{n+1} - w\| &\leq a_n \|x_n - w\| + b_n M \\ &\leq a_n M + b_n M \\ &= (a_n + b_n)M = (1 - c_n)M \leq M \end{aligned}$$

Therefore; the inequality (7) holds

Substituting (7) into (6), we get

$$\|y_n - w\| \leq M \quad (8)$$

From (6), we have

$$\begin{aligned}\|y_n - w\|^2 &\leq \hat{a}_n^2 \|x_n - w\|^2 + 2\hat{a}_n \hat{b}_n M \|x_n - w\| + \hat{b}_n^2 M^2 \\ \text{Since } \hat{a}_n &\leq 1 \text{ and } \|x_n - w\| \leq M, \text{ we get} \\ \|y_n - w\|^2 &\leq \|x_n - w\|^2 + 2\hat{b}_n M^2 + \hat{b}_n^2 M^2 \\ &= \|x_n - w\|^2 + 3\hat{b}_n M^2\end{aligned}\quad (9)$$

Using Lemma(1.3), we get

$$\begin{aligned}\|x_{n+1} - w\|^2 &\leq \|a_n(x_n - w) + c_n u_n + b_n(H_1 y_n - w)\|^2 \\ &\leq \|a_n(x_n - w) + c_n u_n\|^2 + 2b_n \langle H_1 y_n - w, j(x_{n+1} - w) \rangle \\ &\leq a_n^2 \|x_n - w\|^2 + 2a_n c_n \|u_n\| \|x_n - w\| + c_n^2 \|u_n\|^2 + 2b_n \\ &\quad \langle H_1 y_n - w, j(y_n - w) \rangle + 2b_n \langle H_1 y_n - w, j(x_{n+1} - w) - j(y_n - w) \rangle \\ &\quad >\end{aligned}$$

Hence, using definition of local strongly accretive and definition of M , we get

$$\begin{aligned}\|x_{n+1} - w\|^2 &\leq \|x_n - w\|^2 - 2b_n \|x_n - w\|^2 + b_n^2 \|x_n - w\|^2 + 2a_n c_n M^2 + \\ &\quad c_n^2 M^2 + 2b_n \|y_n - w\|^2 - 2b_n k_x \|y_n - w\|^2 + 2b_n e_n \\ \text{Where } e_n &= \langle H_1 y_n - w, j(x_{n+1} - w) - j(y_n - w) \rangle\end{aligned}\quad (10)$$

By (7), (9), $c_n \leq b_n$ and $-2a_n c_n + c_n^2 \leq 0$, we obtain

$$\begin{aligned}\|x_{n+1} - w\|^2 &\leq \|x_n - w\|^2 - 2b_n M^2 + b_n^2 M^2 + 2a_n c_n M^2 - c_n^2 M^2 + 2c_n^2 M + \\ &\quad 2b_n(M^2 + 2\hat{b}_n M^2) - 2b_n k_x \|y_n - w\|^2 + 2b_n e_n \\ &\leq \|x_n - w\|^2 - b_n M^2 + b_n^2 M^2 - (-2a_n c_n - c_n^2)M^2 + 2c_n - b_n M^2 + 2b_n M^2 + \\ &\quad 4b_n \cdot b_n M^2 - 2b_n k_x \|y_n - w\|^2 + 2b_n e_n \\ &= \|x_n - w\|^2 - 2b_n k_x \|y_n - w\|^2 + b_n \lambda_n\end{aligned}\quad (11)$$

Where

$$\lambda_n = (b_n + 2c_n + 4\hat{b}_n)M^2 + 2e_n$$

First, we show that $c_n \rightarrow 0$ as $n \rightarrow \infty$.

From (1) and (2) we get

$$\|x_{n+1} - y_n\| \leq \|(a_n - \hat{a}_n)(x_n - w) + b_n(H_1 y_n - w) - b_n(H_2 x_n - w) + c_n u_n - \hat{c}_n \hat{v}_n\|$$

$$\leq (a_n - \hat{a}_n) \|x_n - w\| + b_n \|H_1 y_n - w\| + \hat{b}_n \|H_2 x_n - w\| + c_n \|u_n\| + \hat{c}_n \|v_n\|$$

$$\leq (1 - b_n - c_n - 1 + \hat{b}_n + \hat{c}_n) \|x_n - w\| + b_n \|H_1 y_n - w\| + \hat{b}_n \|H_2 x_n - w\| + b_n \|u_n\| + \hat{b}_n \|v_n\|$$

By (7) and definition of M , we get

$$\begin{aligned}\|x_{n+1} - y_n\| &\leq 2(b_n + \hat{b}_n)M + (b_n + \hat{b}_n)M + (b_n + \hat{b}_n)M \text{ i.e.,} \\ \|x_{n+1} - y_n\| &\leq 4(b_n + \hat{b}_n)M\end{aligned}\quad (12)$$

Therefore, $\|x_{n+1} - w - (y_n - w)\| \rightarrow 0$ as $n \rightarrow \infty$.

Since $\langle x_{n+1} - y_n \rangle$, $\langle y_n - w \rangle$ and $\langle H_1 y_n - w \rangle$ are bounded and j is uniformly continuous on any bounded subset of X we have

$$j(x_{n+1} - w) - j(y_n - w) \rightarrow 0 \text{ and } \langle e_n \rangle \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Thus, $\lim_{n \rightarrow \infty} \lambda_n = 0$

Let $\delta = \inf \{\|y_n - w\|^2 : n \geq 0\}$

We prove that $\delta = 0$

Assume the contrary, $\delta > 0$,

Then $\|y_n - w\|^2 \geq \delta > 0$ for all $n \geq 0$, hence

$k_x(\|y_n - w\|^2) \geq k_x(\delta) > 0$ where $k_x \in (0,1)$

Thus from (11)

$$\|x_{n+1} - w\|^2 \leq \|x_n - w\|^2 - b_n k_x(\delta) - b_n [k_x(\delta) - \lambda_n] \quad (13)$$

Since $\lim_{n \rightarrow \infty} \lambda_n = 0$, there exists a positive integer n_0

Such that $\lambda_n \leq k_x(\delta)$ for all $n \geq n_0$

Therefore, from (13), we have

$$\|x_{n+1} - w\|^2 \leq \|x_n - w\|^2 - b_n k_x(\delta) \quad \text{or} \\ b_n k_x(\delta) \leq \|x_n - w\|^2 - \|x_{n+1} - w\|^2 \quad \text{for all } n \geq 0$$

Hence,

$$k_x(\delta) \sum_{j=n_0}^n b_j = \|x_{n_0} - w\|^2 - \|x_{n+1} - w\|^2 \leq \|x_{n_0} - w\|^2.$$

Which implies $\sum_{n=0}^{\infty} b_n < \infty$ contradicting (ii). Therefore $\delta = 0$ from definition of δ , there exists a subsequence of $\langle \|y_n - w\| \rangle$ which we will denote by $\langle \|y_i - w\| \rangle$ such that

$$\lim_{n \rightarrow \infty} \|y_i - w\| = 0 \quad (14)$$

$$\|x_n - w\|^2 \leq \|y_n - w + (\hat{b}_n + \hat{c}_n)(x_n - w) - \hat{b}_n(H_2 y_n - w) - \hat{c}_n v_n\|^2 \leq \|y_n - w\|^2 + (\hat{b}_n + \hat{c}_n) \|x_n - w\| - \hat{b}_n \|H_2 x_n - w\| + \hat{c}_n \|v_n\|.$$

Since $\hat{c}_n \leq \hat{b}_n$ and by definition A_i , B and M , we get

$$\|x_n - w\|^2 \leq \|y_n - w\| + 2 \hat{b}_n M \quad \text{for all } n \geq 0 \quad (15)$$

Thus

$$\lim_{j \rightarrow \infty} \|x_j - w\| = 0 \quad (16)$$

Now, let $\epsilon > 0$ be arbitrary, since $\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \lambda_n = 0$ and $\lim_{n \rightarrow \infty} \lambda_n = 0$ there exists a positive integer. no.

Such that

$$b_n \leq \frac{\epsilon}{12M}, \quad \hat{b}_n \leq \frac{\epsilon}{12M}, \quad \lambda_n \leq k_x\left(\frac{\epsilon}{3}\right) \quad \text{for all } n \geq N_0$$

From (16), there exists $k \geq N_0$ such that

$$\|x_k - w\| < \epsilon \quad (17)$$

$$\text{We prove by induction that } \|x_{k+n} - w\| < \epsilon \text{ for all } n \geq 0 \quad (18)$$

Suppose that (18) holds for some $n \geq 0$ and that

For $n=0$, we see that (18) holds by (17)

$$\begin{aligned} \|x_{k+n+1} - w\| &\geq \epsilon. \text{ then by (12), we get} \\ \epsilon &\leq \|x_{k+n+1} - w\| = \|y_{k+n} - w + x_{k+n+1} - y_{k+n}\| \\ &\leq \|y_{k+n} - w\| + \|x_{k+n+1} - y_{k+n}\| \\ &\leq \|y_{k+n} - w\| + 3(b_{k+n} + \hat{b}_{k+n})M \\ &\leq \|y_{k+n} - w\| + \frac{\epsilon}{2} \end{aligned}$$

Hence, $\|y_{k+n} - w\| \geq \frac{\epsilon}{2}$

From (11), we get

$$\epsilon^2 \leq \|x_{k+n+1} - w\|^2 \leq \|x_{k+n} - w\|^2 - 2b_{k+n} k_x \left(\frac{\epsilon}{3}\right) + b_{k+n} k_x$$

$\leq \|x_{k+n} - w\|^2 < \epsilon^2$, which is a contradiction

Thus we proved (18), hence, $\lim_{n \rightarrow \infty} \|x_n - w\| = 0$

Theorem (2.2):-

Let X be a uniformly smooth Banach space, let k be a non empty bounded closed subset of X and $T_1, T_2: K \rightarrow K$ be local strongly pseudo-contractive mappings. Let w be a fixed point of T_i ($i=1,2$) and let $x_0 \in k$ the Ishikawa iteration sequence $\langle x_n \rangle$ be defined by

$$\begin{aligned} x_{n+1} &= a_n x_n + b_n T_1 y_n + c_n u_n \\ y_n &= \hat{a}_n x_n + \hat{b}_n T_2 y_n + \hat{c}_n v_n, \quad n \geq 0 \end{aligned}$$

Where $\langle u_n \rangle$ and $\langle v_n \rangle \subset K$, $\langle b_n \rangle, \langle \hat{b}_n \rangle, \langle c_n \rangle, \langle \hat{c}_n \rangle$ are sequences as in theorem (2.1). then $\langle x_n \rangle$ converges strongly to the unique fixed point of T_i .

Proof:-

Obviously $\langle x_n \rangle$ and $\langle y_n \rangle$ are both contained in K and therefore, bounded

Since T_i is local strongly pseudo-contractive, then $(I - T_i)$ is local strongly accretive for all ($i=1,2$) put $y=w$ and ($T_i = H_i$), we get (7) the proof of theorem (2.1) follows.

Now, we establish the convergence of more general modified three-step to the unique solution of uniformly continuous and locally operators in arbitrary Banach space.

Theorem (2.3):-

Let X be a uniformly smooth Banach space, let k be a non empty bounded closed subset of X ($i=1,2,3$) and T_i ($i=1,2,3$) is a local strongly pseudo-contractive mappings of k and $\bigcap_{i=1}^3 F(T_i) \neq \emptyset$.

Define sequence $\langle x_n \rangle$ iteratively for $x_1 \in k$ by

$$\begin{aligned} x_{n+1} &= a_n x_n + b_n T_1 y_n + c_n u_n \\ y_n &= \hat{a}_n x_n + \hat{b}_n T_2 z_n + \hat{c}_n v_n \\ z_n &= \acute{a}_n x_n + \acute{b}_n T_3 y_n + \acute{c}_n w_n \end{aligned}$$

Where $\langle u_n \rangle, \langle v_n \rangle$ and $\langle w_n \rangle$ are bounded sequence in k and

$\langle a_n \rangle, \langle \hat{a}_n \rangle, \langle \acute{a}_n \rangle, \langle b_n \rangle, \langle \hat{b}_n \rangle, \langle \acute{b}_n \rangle,$

$\langle c_n \rangle, \langle \hat{c}_n \rangle$ and $\langle \acute{c}_n \rangle$ are real sequence in $[0,1]$ such that

$a_n + b_n + c_n = \hat{a}_n + \hat{b}_n + \hat{c}_n = \acute{a}_n + \acute{b}_n + \acute{c}_n = 1$ and satisfying the following:

$$\cdot \sum b_n = \infty$$

$$\cdot \lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \hat{b}_n = \lim_{n \rightarrow \infty} \acute{b}_n = 0$$

$$\cdot \alpha_n = b_n + c_n, \beta_n = \hat{b}_n + \hat{c}_n, \gamma_n = \acute{b}_n + \acute{c}_n \text{ and } \lim_{n \rightarrow \infty} \frac{1}{1 + k_x \alpha_n} = 0$$

For all $k_x \in (0,1)$ Then the sequence $\langle x_n \rangle$ converges strongly to the unique a fixed point of T_i

Proof:-

Since $\bigcap_{i=1}^3 F(T_i) \neq \emptyset$ it follows from (1.8) that $\bigcap_{i=1}^3 F(T_i)$ is singleton say q . The operators T_i is local strongly pseudo contractive implies that $(I - T_i)$ is local strongly accretive and therefore $(I - T_i) - k_x I = I - T_i - k_x I$ ($i = 1, 2, 3$) is accretive. Hence, for all $r > 0$ and $k_x \in (0, 1)$, we have:

$$\|x - y\| \leq \|x - y + r[(I - T_i - k_x I)x - (I - T_i - k_x I)y]\|$$

From our hypothesis, we obtain the following

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + b_n T_1 y_n + c_n u_n \\ &= x_n - \alpha_n x_n + b_n T_1 y_n + c_n u_n \end{aligned}$$

Now, we have

$$\begin{aligned} x_n &= x_{n+1} + \alpha_n x_n - b_n T_1 y_n - c_n u_n \\ x_n &= x_{n+1} + \alpha_n x_{n+1} - \alpha_n x_{n+1} + \alpha_n x_n - b_n T_1 y_n - c_n u_n \\ &= (1 + \alpha_n)x_{n+1} + \alpha_n (I - T_i - k_x I)x_{n+1} - \alpha_n (I - T_i - k_x I)x_{n+1} \\ &\quad - \alpha_n (x_{n+1} - x_n) - b_n T_1 y_n - c_n u_n \\ &= (1 + \alpha_n)x_{n+1} + \alpha_n (I - T_i - k_x I)x_{n+1} - \alpha_n (I - k_x I)x_{n+1} \\ &\quad + \alpha_n T_i x_{n+1} + \alpha_n (x_n - x_{n+1}) - b_n T_1 y_n - c_n u_n \end{aligned} \quad (19)$$

Since q is a fixed point of T_i , then

$$q = (1 + \alpha_n)q + \alpha_n (I - T_i - k_x I)q - \alpha_n (I - k_x I)q \quad (20)$$

Subtracting (20) from (19) we obtain

$$\begin{aligned} x_n - q &= (1 + \alpha_n) \left[(x_{n+1} - q) + \frac{\alpha_n}{1 + \alpha_n} \{ (I - T_i - k_x I)x_{n+1} - (I - T_i - k_x I)q \} \right] - \\ &\quad \alpha_n (I - k_x I)(x_{n+1} - q) + [\alpha_n (T_i - I)x_{n+1} - b_n T_1 y_n] + [\alpha_n x_n - c_n u_n] \\ \|x_n - q\| &= \|(1 + \alpha_n) \left[(x_{n+1} - q) + \frac{\alpha_n}{1 + \alpha_n} \{ (I - T_i - k_x I)x_{n+1} - (I - T_i - k_x I)q \} \right] - \\ &\quad \alpha_n (I - k_x I)(x_{n+1} - q) + [\alpha_n (T_i - I)x_{n+1} - b_n T_1 y_n] + [\alpha_n x_n - c_n u_n]\| \\ \|x_n - q\| &= (1 + \alpha_n) \left\| (x_{n+1} - q) + \frac{\alpha_n}{1 + \alpha_n} \{ (I - T_i - k_x I)x_{n+1} - (I - T_i - k_x I)q \} \right\| \\ &\quad - \alpha_n (1 - k_x) \|x_{n+1} - q\| - \alpha_n \| (T_i - I)x_{n+1} - c_n u_n \| - \\ &\quad - \alpha_n (1 - k_x) \|x_{n+1} - q\| - \alpha_n \| (T_i - I)x_{n+1} - c_n u_n \| - \\ &\quad \| \alpha_n x_n - b_n T_1 y_n \| \end{aligned} \quad (21)$$

Since T_i is local strongly pseudo contractive, then (21) yields

$$\begin{aligned} \|x_n - q\| &\geq (1 + \alpha_n) \|x_{n+1} - q\| - \alpha_n (I - k_x) \|x_{n+1} - q\| - \alpha_n \| (T_i - I)x_{n+1} - c_n u_n \| \\ &\quad - \alpha_n \| \alpha_n x_n - b_n T_1 y_n \| \\ &= (1 - k_x \alpha_n) \|x_{n+1} - q\| - \alpha_n \| (T_i - I)x_{n+1} - c_n u_n \| - \alpha_n \| \alpha_n x_n - b_n T_1 y_n \| \\ \|x_n - q\| &\leq \frac{1}{1 + k_x \alpha_n} [\|x_n - q\| + \alpha_n \| (T_i - I)x_{n+1} - c_n u_n \| + \alpha_n \| \alpha_n x_n - b_n T_1 y_n \|] \end{aligned}$$

Since T is uniformly continuous on the bounded set k there exists a positive real number $M < \infty$ such that

$$\|x_{n+1} - q\| \leq \frac{1}{1 + \alpha_n k_x} \|x_n - q\| + \frac{1}{1 + \alpha_n k_x} M \quad (22)$$

Now, put $\delta_n = \frac{1}{1+\alpha_n k_x}$, $\sigma_n = \delta_n M$ and $q_n = \|x_n - q\|$

Thus (22) reduces to

$$q_{n+1} \leq \delta_n q_n + \sigma_n$$

Since $0 \leq \delta_n \leq 1$, $\lim_{n \rightarrow \infty} \delta_n = 0$ and $\lim_{n \rightarrow \infty} \sigma_n = 0$. Therefore by Lemma (1.4), we have $\lim_{n \rightarrow \infty} q_n = 0$

Which implies that the sequence $\langle x_n \rangle$ converges strongly to q corollary (2.4).

Corollary(2.4):-

Let X be a real arbitrary Banach space and K be a nonempty closed bounded and convex subset of X . T_i local strongly pseudo contractive self mapping of K and uniformly continuous such that $F(T) \neq \emptyset$ and the sequence $\langle x_n \rangle$ iteratively for $x_1 \in K$ defined by

$$x_{n+1} = a_n x_n + b_n T y_n + c_n u_n$$

$$y_n = \acute{a}_n x_n + \acute{b}_n T z_n + \acute{c}_n v_n$$

$$z_n = \acute{\acute{a}}_n x_n + \acute{\acute{b}}_n T_3 x_n + \acute{\acute{c}}_n w_n$$

Where $\langle u_n \rangle, \langle v_n \rangle$ and $\langle w_n \rangle$ are bounded sequence in K and $\langle a_n \rangle, \langle \acute{a}_n \rangle, \langle \acute{\acute{a}}_n \rangle, \langle b_n \rangle, \langle \acute{b}_n \rangle, \langle \acute{\acute{b}}_n \rangle, \langle c_n \rangle, \langle \acute{c}_n \rangle$ and $\langle \acute{\acute{c}}_n \rangle$ are real sequences in $[0,1]$ such that

$a_n + b_n + c_n = \acute{a}_n + \acute{b}_n + \acute{c}_n = \acute{\acute{a}}_n + \acute{\acute{b}}_n + \acute{\acute{c}}_n = 1$ and satisfying the following conditions:

$$1- \sum b_n = \infty$$

$$2- \lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \acute{b}_n = 0 = \lim_{n \rightarrow \infty} \acute{\acute{b}}_n = 0$$

$$3- \alpha_n = b_n + c_n, \beta_n = \acute{b}_n + \acute{c}_n, \gamma_n = \acute{\acute{b}}_n + \acute{\acute{c}}_n \text{ and } \lim_{n \rightarrow \infty} \frac{1}{1+k_x \alpha_n} = 0$$

for all $k_x \in (0,1)$. Then the sequence $\langle x_n \rangle$ converges strongly to the unique fixed point of T_i

Proof:-

If $T_1=T_2=T_3=T$ in theorem(2.3), then corollary(2.4) follows immediately.

Remark (2.6):-

Theorems (2.1) and (2.2) are generalization of Yuguang and Fang [12] and Rafiq [13] via replace Mann iteration process with errors for strongly pseudo contractive (strongly accretive) by more general modified two-step iterative with errors for local strongly pseudo contractive (local strongly accretive) operators respectively.

Remark (2.7):-

Theorems (2.3) and (2.5) are generalization of

- 1- Results of Noor [4] and Imoru [10], [11] via replace three-step iteration of strongly pseudo contractive (strongly accretive) by a more general modified three-step iterative of local strongly pseudo contractive (local strongly accretive) respectively.
- 2- Results of Xue and Fan[8] and Rafiq [5] via replace modified three-step iterative scheme by a more general modified three-step iterative scheme with errors of local strongly pseudo contractive (strongly accretive) operators.
- 3- Results of mohbademu and Olalean[9], via replace strongly pseudo contractive (local strongly accretive) by local strongly pseudo contractive (local strongly accretive) operators respectively.

Theorem (2.5):-

Let X be a real arbitrary Banach space and K be a nonempty closed bounded and convex subset of X .

T_i is local strongly accretive self mapping of K ($i=1,2,3$), uniformly continuous and $\bigcap_{i=1}^3 F(T_i) \neq \emptyset$. define a mapping $R_i: K \rightarrow K$ by $R_i x = x - T_i x + f$, for some $f \in X$, consider the sequence defined by:

For arbitrary $x_1 \in K$

$$x_{n+1} = a_n x_n + b_n R_1 y_n + c_n u_n$$

$$y_n = \acute{a}_n x_n + \acute{b}_n R_2 z_n + \acute{c}_n v_n$$

$$z_n = \acute{\acute{a}}_n x_n + \acute{\acute{b}}_n R_3 x_n + \acute{\acute{c}}_n w_n$$

Where $\langle u_n \rangle, \langle v_n \rangle$ and $\langle w_n \rangle$ are bounded sequence in K and $\langle a_n \rangle, \langle \acute{a}_n \rangle, \langle \acute{\acute{a}}_n \rangle, \langle b_n \rangle, \langle \acute{b}_n \rangle, \langle \acute{\acute{b}}_n \rangle, \langle c_n \rangle, \langle \acute{c}_n \rangle$ and $\langle \acute{\acute{c}}_n \rangle$ are real sequences in $[0,1]$ such that

$a_n + b_n + c_n = \acute{a}_n + \acute{b}_n + \acute{c}_n = \acute{\acute{a}}_n + \acute{\acute{b}}_n + \acute{\acute{c}}_n = 1$ and satisfying the following conditions:

$$1- \sum b_n = \infty$$

$$2- \lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \acute{b}_n = \lim_{n \rightarrow \infty} \acute{\acute{b}}_n = 0$$

$$3- \alpha_n = b_n + c_n, \beta_n = \acute{b}_n + \acute{c}_n, \gamma_n = \acute{\acute{b}}_n + \acute{\acute{c}}_n \text{ and } \lim_{n \rightarrow \infty} \frac{1}{1+k_x \alpha_n} = 0$$

For all $k_x \in (0,1)$. Then the sequence $\langle x_n \rangle$ converges strongly to the unique solution of the equation $T_i x = f$.

Proof:-

It follows from definition of local strongly accretive mappings, that for given x , there exists $k_x \in (0,1)$ such that

$$\langle T_i x - T_i y, j(x - y) \rangle \geq k_x \|x - y\|^2 \text{ for all } y \in X$$

We observe that R_i, T_i are uniformly continuous and for any given $f \in K$.

$$(I - R_i)x = x - f + T_i x - x = T_i x - f$$

Which implies that

$$\langle (I - R_i)x - (I - R_i)y, j(x - y) \rangle \geq k_x \|x - y\|^2$$

That is $(I - R_i)$ is local strongly accretive. Thus R_i is local strongly pseudo contractive. thus theorem(2.5) follows from theorem(2.3)

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