

## Twin primes and Sophie Germain's prime numbers

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**Abstract:** For two millennia, the prime numbers have continued to fascinate mathematicians. Indeed, a conjecture which dates back to this period states that the number of twin primes is infinite. In 1949 Clement showed a theorem on twin primes. In a recent article, I proved a corollary of Clement's theorem [1].

In this paper, I will prove shortly the link between twin primes and Sophie Germain's prime numbers.

### Introduction:

In Mathematics we say that  $(p, q)$  form a pair of twin primes if  $q = p + 2$ . The couple  $(2, 3)$  is the only pair of consecutive primes.

Omitting the pair  $(2, 3)$ , 2 is the smallest possible distance between two primes, twin primes are two and two consecutive odd numbers.

Any pair of twin primes (with the exception of the couple  $(3, 5)$ ) is of the form  $(6n - 1, 6n + 1)$  for some integer  $n$ . Indeed, any set of three consecutive natural numbers has at least a multiple of 2 (possibly two) and one multiple of 3, these two are confused between multiple two twin primes.

It is possible to show that, for any integer, the pair  $(m, m + 2)$  consists of twin primes if and only if:

$$4[(m-1)!+1]+m \equiv 0 \pmod{m(m+2)}$$

This characterization of factorial and modular twin primes was discovered by P. A. Clement in 1949 [3].

The series of reciprocals of twin primes converges to Brun's constant, unlike the series of reciprocals of prime numbers. This property was demonstrated by Viggo Brun in 1919 [4].

The twin prime conjecture states that there are infinitely many twin primes. In other words, there are infinitely many primes  $p$  such that  $(p + 2)$  is also prime.

In 1940, Paul Erdős proved the existence of a constant  $c < 1$  and infinitely many primes  $p$  such that:

$$p' - p < c \ln(p) \text{ where } p' \text{ denotes the number immediately following the first } p.$$

This result was improved several times, in 1986, Helmut Maier showed a constant  $c < 0.25$  could be reached.

In 2003, Daniel Goldston and Cem Yildirim have shown that, for all  $c > 0$ , there are infinitely many primes  $p$  such that  $p' - p < c \ln(p)$ .

In 1966, Chen Jingrun demonstrated the existence of infinitely many primes  $p$  such that  $p + 2$  is the product of at most two prime factors (such a number, product of at most two prime factors, 2 is said -almost-first).

His approach was that of the theory of the screen, he used to treat similarly the twin prime conjecture and Goldbach's Conjecture.

In a recent article, I proved a corollary of Clement's theorem.

A prime number  $p$  is called a Sophie Germain prime number if  $2p + 1$  is also a prime number. They were served due to the demonstration of Sophie Germain about the truth of Fermat's last theorem for such primes. It was conjectured that there are infinitely many primes of Sophie Germain, but, as the twin prime conjecture, this had not yet been demonstrated.

The first few Sophie Germain primes are 2, 3, 5, 11, 23, 29, 41, 53, 83, 89, 113, 131,

In this paper, I will prove shortly the link between twin primes and Sophie Germain's prime numbers.

### Theorem 1: Theorem Wilson

Statement: An integer  $p$  strictly greater than 1, is a prime number if and only if divides  $(p - 1)! + 1$ , that is to say if and only if:  $(p-1)!+1 \equiv 0 \pmod{p}$

### Theorem 2: Theorem Clement

For any integer, the pair  $(m, m + 2)$  consists of twin primes if:

$$4[(m-1)!+1]+m \equiv 0 \pmod{m(m+2)}$$

### Theorem 3: Corollary of Clement's theorem[1]

For any integer, the pair  $(m, m + 2)$  consists of twin primes if:  $m(m+2)$  divides  $((m-2)(m-1)! - 2)$ .

### Theorem 4:

For a positive odd integer  $p$  and for any two distinct odd primes  $p_1$  and  $p_2$  with  $p_1+p_2-p = 1$ , then  $(p-p_1)!(p-p_2)! \equiv -1 \pmod{p}$   $\iff p$  is prime[2].

### Conjecture 1: Twin prime conjecture

There are infinitely many twin primes.

### Conjecture 2:

A prime number  $p$  is called a Sophie Germain prime number if  $2p + 1$  is also a prime number.

### Theorem 5: Corollary of theorem 4

Given  $p$  such that  $(2*p+1)$  is prime and,  $p$  and  $(p+2)$  are twin primes.  $p$  is a Sophie Germain prime number and:

$$((p-1)!*(p+1)! + 1) \equiv 0 \pmod{(2*p+1)}$$

Proofs of theorem 6:

Given  $p$  such that  $(2*p+1)$  is prime and,  $p$  and  $(p+2)$  are twin primes.  $p$  is a Sophie Germain prime number.

According to theorem 4:

$$(((2*p+1) - p)! * ((2*p+1) - (p + 2))!) \equiv -1 \pmod{(2*p+1)}$$

$$((p+1)! * (p-1)!) \equiv -1 \pmod{(2*p+1)}$$

$$((p-1)! * (p+1)! + 1) = 0 \text{ modulo } (2 * p + 1)$$

$$(p * (p+1) * ((p-1)!)^2 + 1) = 0 \text{ modulo } (2 * p + 1)$$

$(p-1)$  is pair

Now I put two new notions: fact pair and fact odd fact pair and fact odd will be noted respectively  $\xi(p-1)$  and  $\mu(p-1)$  with:

$$\xi(p-1)! = (p-1) * (p-3) * (p-5) * (p-7) * (p-9) \dots * 8 * 6 * 4 * 2$$

$$\mu(p-1)! = (p-2) * (p-4) * (p-6) * (p-8) * (p-10) \dots * 9 * 7 * 5 * 3 * 1$$

$$(p * (p+1) * (\xi(p-1)!)^2 * (\mu(p-1)!)^2 + 1) = 0 \text{ modulo } (2 * p + 1)$$

We can find an integer  $k$  such that:

$$\mu(p-1)! = k * 2^p$$

$$(\mu(p-1)!)^2 = k^2 * 2^{2 * p}$$

$$(p * (p+1) * (\xi(p-1)!)^2 * k^2 * 2^{2 * p} + 1) = 0 \text{ modulo } (2 * p + 1)$$

### References:

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- [5] Wikipedia, Sophie Germain primes.