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A New Hilbert-type Integral Inequality with the Homogeneous Kernel of Real Degree Form and the Integral in Whole Plane

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Abstract: In this paper,we build a new Hilbert's inequality with the homogeneous kernel of real order and the integral in whole plane. The equivalent inequality is considered. The best constant factor is calculated using Ψ f unction.

1 INTRODUCTION

If $f(x),g(x)\geq 0$, such that $0<\int_0^\infty f^2(x)\mathrm{d}x<\infty \quad 0<\int_0^\infty g^2(x)\mathrm{d}x<\infty$; then [1]

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y<\pi\left(\int_0^\infty f^2(x)\mathrm{d}x\right)^{1/2}\left(\int_0^\infty g^q(x)\mathrm{d}x\right)^{1/2}, \tag{1.1}$$

where the constant factor π is the best possible. Inequality (1.1) is well-known as Hilbert's integral inequality,which has been extended by Hardy-Riesz as [2]:

If $p>1, 1/p+1/q=1 \quad f(x),g(x)\geq 0$; such that $0<\int_0^\infty f^p(x)\mathrm{d}x<\infty$, and $0<\int_0^\infty g^q(x)\mathrm{d}x<\infty$;

then we have the following Hardy-Hilbert's integral inequality:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y<\frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty f^p(x)\mathrm{d}x\right)^{1/p}\left(\int_0^\infty g^q(x)\mathrm{d}x\right)^{1/q}; \tag{1.2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ also is the best possible.

Hilbert's inequality attracts some attention in recent years.Actually, inequalities (1.1)and(1.2) have many generalizations and variations. (1.1) has been strengthened by Yang and others (including double series inequalities) [3,4,6-21].

In 2008, Zitian Xie and Zheng Zeng gave a new Hilbert-type Inequality [4] as follows

If $a>0,b>0,c>0, p>1, 1/p+1/q=1 \quad f(x),g(x)\geq 0$; such that

$0<\int_0^\infty f^{-1-p/2}(x)\mathrm{d}x<\infty$ and $0<\int_0^\infty g^{-1-q/2}(x)\mathrm{d}x<\infty$ then

$$< K\left(\int_0^\infty f^{-1-p/2}(x)\mathrm{d}x\right)^{1/p}\left(\int_0^\infty g^{-1-q/2}(x)\mathrm{d}x\right)^{1/q}; \tag{1.3}$$

where the constant factor $K=\frac{\pi}{(a+b)(a+c)(c+b)}$ is the best possible.

In 2010,Jianhua Xhong and Bicheng Yang gave a new Hilbert-type Inequality [5] as follows :

Assume that

$$\lambda, p>0(p\neq 1), r>1,\ 1/p+1/q=1, 1/r+1/s=1, \phi(x)=x^{p(1-\frac{\lambda}{r})-1}, \varphi(x)=x^{q(1-\frac{\lambda}{s})-1}, x\in(0,\infty),$$

$$K=\Gamma(\beta+1)\sum_{k=0}^{\infty}(-1)^k\binom{\alpha-\lambda}{k}\left[\frac{1}{(k+\lambda/r)^{\beta+1}}+\frac{1}{(k+\lambda/s)^{\beta+1}}\right], \text{and } f,g\geq 0,$$

$0<\|f\|_{p,\phi}:=\left\{\int_0^{\infty} x^{p(1-\lambda/r)-1} f^p(x)\mathrm{d}x\right\}^{1/p}<\infty,\ 0<\|g\|_{q,\varphi}<\infty$ then

(1) for $p>1$ we have the following equivalent inequalities:

$$\int_0^{\infty}\int_0^{\infty}\frac{|\ln(x/y)|^{\beta} f(x)g(y)}{|x-y|(\max\{|x|,|y|\})^{\alpha}}\mathrm{d}x\mathrm{d}y<K\|f\|_{p,\phi}\|g\|_{q,\varphi}$$

(2)For $0<p<1$ the reverse of (1.5) with the best constant factor K .

$$\int_0^{\infty}\int_0^{\infty}\frac{|\ln(x/y)|^{\beta} f(x)g(y)}{|x-y|(\max\{|x|,|y|\})^{\alpha}}\mathrm{d}x\mathrm{d}y>K\|f\|_{p,\phi}\|g\|_{q,\varphi}$$

The main purpose of this paper is to build a new Hilbert-type inequality with the homogeneous kernel of real order and the integral in whole plane, by estimating the weight function using Ψ function.The equivalent inequality is considered

We knew that (in this paper,γ is the Euler's constant.)

$$\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}=-\gamma+\sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+z}\right),\psi(1)=-\gamma,\psi\left(\frac{1}{2}\right)=-\gamma-2\ln 2.$$

Recent XIE Zitin and ZHOU Qinghua prove that the expression of the Ψ -function admits a finite expression in elementary function for rational number z,and prove that[6]

$$\psi\left(\frac{a}{b}\right)=\Gamma'\left(\frac{a}{b}\right)/\Gamma\left(\frac{a}{b}\right)=-\ln b-\gamma-\ln 2-\frac{\pi}{2}\cot\frac{a\pi}{b}+\sum_{k=1}^{b-1}\cos\frac{2ka\pi}{b}\ln\sin\frac{k\pi}{b}$$

and have

$$\psi\left(\frac{1}{3}\right)=-\gamma-\ln 3-\ln 2-\frac{\pi}{2}\cot\frac{\pi}{3}+\cos\frac{2\pi}{3}\ln\sin\frac{\pi}{3}+\cos\frac{4\pi}{3}\ln\sin\frac{2\pi}{3}$$
$$=-\gamma-\frac{3}{2}\ln 3-\frac{\pi}{2\sqrt{3}};$$

$$\psi\left(\frac{2}{3}\right)=-\gamma-\frac{3}{2}\ln 3+\frac{\pi}{2\sqrt{3}};$$

$$\psi\left(\frac{1}{4}\right)=-\gamma-3\ln 2-\frac{\pi}{2};$$

$$\psi\left(\frac{3}{4}\right)=-\gamma-3\ln 2+\frac{\pi}{2};$$

$$\psi\left(\frac{1}{6}\right)=-\gamma-2\ln 2-\frac{\sqrt{2}\pi}{2};$$

$$\psi\left(\frac{5}{6}\right)=-\gamma-2\ln 2+\frac{\sqrt{2}\pi}{2};$$

$$\psi\left(\frac{1}{5}\right)=-\gamma-\frac{5\ln 5}{4}-\frac{\sqrt{5}}{2}\ln(\sqrt{5}-1)-\frac{\sqrt{5}}{2}\ln 2-\frac{\pi}{40}(5+3\sqrt{5})\sqrt{10-2\sqrt{5}};$$

$$\psi\left(\frac{4}{5}\right)=-\gamma-\frac{5\ln 5}{4}-\frac{\sqrt{5}}{2}\ln(\sqrt{5}-1)-\frac{\sqrt{5}}{2}\ln 2+\frac{\pi}{40}(5+3\sqrt{5})\sqrt{10-2\sqrt{5}}$$

In the following, we always suppose that:

$1/p+1/q=1, p>1, \min\{a\lambda+b\mu, a\mu+b\lambda\}>-1, a\mu+b\lambda\neq 0, a\lambda+b\mu\neq 0, \mu>0, \lambda>0.$

$a+b=1$.

2 SOME LEMMAS

We start by introducing some Lemmas.

Lemma 2.1 If $s>0, r\neq 0, r>-s$, then

$$1)\ \int_0^1 x^{r-1}\ln(1-x^s)dx = -\frac{1}{r}[\gamma+\psi(\frac{r+s}{s})]$$ (2.1)

$$2)\ \int_0^1 x^{r-1}\ln(1+x^s)dx = \frac{1}{r}\ln 2-\frac{1}{2r}[\psi(\frac{r+2s}{2s})-\psi(\frac{r+s}{2s})]$$

Proof. we obtain,'

$$1)\ -\int_0^1 x^{r-1}\ln(1-x^s)dx = \int_0^1 x^{r-1}\sum_{l=1}^{\infty}\frac{x^{ls}}{l}dx$$

$$=\sum_{l=1}^{\infty}\int_0^1\frac{x^{ls+r-1}}{l}dx=\sum_{l=1}^{\infty}\frac{1}{l(r+ls)}$$

$$=\frac{1}{r}\sum_{l=1}^{\infty}(\frac{1}{l}-\frac{1}{l+r/s})$$

$$=\frac{1}{r}[\gamma+\psi(\frac{r+s}{s})]$$

$$2)\int_0^1 x^{r-1}\ln(1+x^s)dx=\int_0^1 x^{r-1}\sum_{l=1}^{\infty}(-1)^{l-1}\frac{x^{ls}}{l}dx$$

$$=\sum_{l=1}^{\infty}\int_0^1(-1)^{l-1}\frac{x^{ls+r-1}}{l}dx=\sum_{l=1}^{\infty}\frac{(-1)^{l-1}}{l(r+ls)}$$

$$=\lim_{N\to\infty}\left\{\frac{1}{r}\sum_{l=1}^{2N+1}(-1)^{l-1}\frac{1}{l}-\frac{1}{2r}\left[\sum_{n=0}^{N}\left(\frac{1}{n+1}-\frac{1}{n+\frac{r+2s}{2s}}\right)-\sum_{n=0}^{N}\left(\frac{1}{n+1}-\frac{1}{n+\frac{r+s}{2s}}\right)\right]\right\}$$

$$=\frac{1}{r}\ln 2-\frac{1}{2r}[\psi(\frac{r+2s}{2s})-\psi(\frac{r+s}{2s})]$$

The lemma is proved.

In particular,if $r>-1,\ r\neq 0$, then

$$\int_0^1 u^{r-1}\ln\frac{u^2+u+1}{u^2-u+1}du$$

$$=\int_0^1 u^{r-1}\ln(1-u^3)du-\int_0^1 u^{r-1}\ln(1-u)du-\left[\int_0^1 u^{r-1}\ln(1+u^3)du-\int_0^1 u^{r-1}\ln(1+u)du\right]$$

$$=\frac{1}{r}\left[\psi(r+1)-\psi(\frac{r+3}{3})\right]+\frac{1}{2r}\left[\psi(\frac{r+6}{6})-\psi(\frac{r+3}{6})-\psi(\frac{r+2}{2})+\psi(\frac{r+1}{2})\right]$$

$$(\text{using } \psi(x+1)=\psi(x)+\frac{1}{x})$$

$$=\frac{1}{r}\left[\psi(r)-\psi(\frac{r}{3})\right]+\frac{1}{2r}\left[\psi(\frac{r}{6})-\psi(\frac{r+3}{6})-\psi(\frac{r}{2})+\psi(\frac{r+1}{2})\right]$$

$$=\frac{1}{r}\left[\psi(r)-\psi(\frac{r}{3})\right]+\frac{1}{2r}\left[2\psi(\frac{r}{6})-\psi(\frac{r}{6})-\psi(\frac{r+3}{6})-2\psi(\frac{r}{2})+\psi(\frac{r}{2})+\psi(\frac{r+1}{2})\right]$$

$$(\text{using } \psi(x+\frac{1}{2})+\psi(x)=2\psi(2x)-\psi(\frac{r+1}{2})$$

$$=\frac{2}{r}\left[\psi(r)-\psi(\frac{r}{3})\right]+\frac{1}{r}\left[(\frac{r}{6})-\psi(\frac{r}{2})\right]$$ (2.2)

Lemma 2.2 Define the weight functions as follow:

$$w(x) := \int_{-\infty}^{\infty} \frac{|x|^{a(\mu-\lambda)}}{|y|^{1-b(\mu-\lambda)}} \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| dy,$$

$$\tilde{w}(y) := \int_{-\infty}^{\infty} \frac{|y|^{b(\mu-\lambda)}}{|x|^{1-a(\mu-\lambda)}} \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| dx,$$

Then

$$w(x) = \tilde{w}(y)$$
$$= \frac{2}{a\lambda+b\mu}\left[\psi(a\lambda+b\mu) - \psi\left(\frac{a\lambda+b\mu}{3}\right)\right] + \frac{1}{a\lambda+b\mu}\left[\psi\left(\frac{a\lambda+b\mu}{6}\right) - \psi\left(\frac{a\lambda+b\mu}{2}\right)\right]$$
$$+ \frac{2}{a\mu+b\lambda}\left[\psi(a\mu+b\lambda) - \psi\left(\frac{a\mu+b\lambda}{3}\right)\right] + \frac{1}{a\lambda+b\mu}\left[\psi\left(\frac{a\mu+b\lambda}{6}\right) - \psi\left(\frac{a\mu+b\lambda}{2}\right)\right].$$
$$:= k \tag{2.3}$$

Proof We only prove that $w(x) = k$ for $x \in (-\infty, 0)$

Using lemma 2.1,setting y = ux ,and y=-ux

$$w(x) := \int_{-\infty}^{0} \frac{(-x)^{a(\mu-\lambda)}}{(-y)^{1-b(\mu-\lambda)}} \frac{(\min\{(-x),(-y)\})^{\lambda}}{(\max\{(-x),(-y)\})^{\mu}} \ln \frac{x^2+xy+y^2}{x^2+y^2} dy$$
$$+ \int_{0}^{\infty} \frac{(-x)^{a(\mu-\lambda)}}{y^{1-b(\mu-\lambda)}} \frac{(\min\{(-x),y\})^{\lambda}}{(\max\{(-x),y\})^{\mu}} \ln \frac{x^2+y^2}{x^2+xy+y^2} dy$$
$$:= w_1 + w_2$$

Then

$$w_1 = \int_0^1 u^{-1+a\lambda+b\mu} \ln \frac{u^2+u+1}{u^2+1} du + \int_1^{\infty} u^{-1-a\lambda-b\mu} \ln \frac{u^2+u+1}{u^2+1} du$$
$$= \int_0^1 u^{-1+a\lambda+b\mu} \ln \frac{u^2+u+1}{u^2+1} du + \int_0^1 u^{-1+a\mu+b\lambda} \ln \frac{u^2+u+1}{u^2+1} du$$
$$w_2 = \int_0^1 u^{-1+a\lambda+b\mu} \ln \frac{u^2+1}{u^2-u+1} du + \int_1^{\infty} u^{-1-a\lambda-b\mu} \ln \frac{u^2+1}{u^2-u+1} du$$
$$= \int_0^1 u^{-1+a\lambda+b\mu} \ln \frac{u^2+1}{u^2-u+1} du + \int_0^1 u^{-1+a\mu+b\lambda} \ln \frac{u^2+1}{u^2-u+1} du$$

And

$$w = w_1 + w_2 = \int_0^1 u^{-1+a\lambda+b\mu} \ln \frac{u^2+u+1}{u^2-u+1} du + \int_0^1 u^{-1+a\mu+b\lambda} \ln \frac{u^2+u+1}{u^2-u+1} du$$
$$= \frac{2}{a\lambda+b\mu}\left[\psi(a\lambda+b\mu) - \psi\left(\frac{a\lambda+b\mu}{3}\right)\right] + \frac{1}{a\lambda+b\mu}\left[\psi\left(\frac{a\lambda+b\mu}{6}\right) - \psi\left(\frac{a\lambda+b\mu}{2}\right)\right]$$
$$+ \frac{2}{a\mu+b\lambda}\left[\psi(a\mu+b\lambda) - \psi\left(\frac{a\mu+b\lambda}{3}\right)\right] + \frac{1}{a\lambda+b\mu}\left[\psi\left(\frac{a\mu+b\lambda}{6}\right) - \psi\left(\frac{a\mu+b\lambda}{2}\right)\right].$$
$$= k$$

Similarly,setting $x = y/u$,and $x = -y/u$

$$\tilde{w}(y) = \int_{-\infty}^{0} \frac{(-y)^{b(\mu-\lambda)}}{(-x)^{1-a(\mu-\lambda)}} \frac{(\min\{(-x),(-y)\})^{\lambda}}{(\max\{(-x),(-y)\})^{\mu}} \ln \frac{x^2+xy+y^2}{x^2+y^2} dx$$
$$+ \int_{0}^{\infty} \frac{y^{b(\mu-\lambda)}}{(-x)^{1-a(\mu-\lambda)}} \frac{(\min\{(-x),y\})^{\lambda}}{(\max\{(-x),y\})^{\mu}} \ln \frac{x^2+y^2}{x^2+xy+y^2} dx$$

$$= \int_{-\infty}^{0} \frac{(-y)^{b(\mu-\lambda)}}{(-y/u)^{1-a(\mu-\lambda)}} \frac{(\min\{(-y/u),(-y)\})^{\lambda}}{(\max\{(-y/u),(-y)\})^{\mu}} \ln \frac{(y/u)^2 + (y/u)y + y^2}{(y/u)^2 + y^2} d(y/u)$$

$$+ \int_{0}^{\infty} \frac{y^{b(\mu-\lambda)}}{(y/u)^{1-a(\mu-\lambda)}} \frac{(\min\{(y/u),y\})^{\lambda}}{(\max\{(y/u),y\})^{\mu}} \ln \frac{(-y/u)^2 + y^2}{(-y/u)^2 + (-y/u)y + y^2} d(-y/u)$$

$$= \int_{0}^{\infty} u^{-1+b(\mu-\lambda)} \frac{(\min\{1,u\})^{\lambda}}{(\max\{1,u\})^{\mu}} \ln \frac{u^2+u+1}{u^2+1} du + \int_{0}^{\infty} u^{-1+b(\mu-\lambda)} \frac{(\min\{1,u\})^{\lambda}}{(\max\{1,u\})^{\mu}} \ln \frac{u^2+1}{u^2-u+1} du$$

$$= w_1 + w_2 = k \tag{2.4}$$

and the lemma is proved.

Lemma 2.3 For ε > 0; and $\min\{a\mu + b\lambda - 2\varepsilon/q, a\lambda + b\mu - 2\varepsilon/q\} > -1$, define both functions, $\tilde{f}, \tilde{g}$ as follow:

$$\tilde{f}(x) = \begin{cases} x^{a(\mu-\lambda)-1-2\varepsilon/p}, & \text{if} \quad x \in (1,\infty), \\ 0, & \text{if} \quad x \in [-1,1], \\ (-x)^{a(\mu-\lambda)-1-2\varepsilon/p}, & \text{if} \quad x \in (-\infty,-1), \end{cases}$$

$$\tilde{g}(x) = \begin{cases} x^{b(\mu-\lambda)-1-2\varepsilon/q}, & \text{if} \quad x \in (1,\infty), \\ 0, & \text{if} \quad x \in [-1,1], \\ (-x)^{b(\mu-\lambda)-1-2\varepsilon/q}, & \text{if} \quad x \in (-\infty,-1), \end{cases}$$

Then

$$I(\varepsilon) := \varepsilon \left\{ \int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} \tilde{f}^{p}(x) dx \right\}^{1/p} \left\{ \int_{-\infty}^{\infty} |x|^{q[1-b(\mu-\lambda)]-1} \tilde{g}^{q}(x) dx \right\}^{1/q} = 1; \tag{2.5}$$

$$\tilde{I}(\varepsilon) := \varepsilon \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(x)\tilde{g}(y) \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2 + xy + y^2}{x^2 + y^2} \right| dxdy \to k(\varepsilon \to 0^{+}) \tag{2.6}$$

Proof Easily

$$I(\varepsilon) = \varepsilon \left\{ 2\int_{1}^{\infty} x^{-1} x^{-2\varepsilon} dx \right\}^{1/p} \left\{ 2\int_{1}^{\infty} x^{-1} x^{-2\varepsilon} dx \right\}^{1/q} = 1$$

Let y-Y, using $\tilde{f}(-x) = \tilde{f}(x)$, $\tilde{g}(-x) = \tilde{g}(x)$ and

$$\tilde{f}(-x) \int_{-\infty}^{\infty} \tilde{g}(y) \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2 - xy + y^2}{x^2 + y^2} \right| dy$$

$$= \tilde{f}(x) \int_{-\infty}^{\infty} \tilde{g}(Y) \frac{(\min\{|x|,|Y|\})^{\lambda}}{(\max\{|x|,|Y|\})^{\mu}} \left| \ln \frac{x^2 + xY + Y^2}{x^2 + Y^2} \right| dY$$

we have that $\tilde{f}(x) \int_{-\infty}^{\infty} \tilde{g}(y) \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2 + xy + y^2}{x^2 + y^2} \right| dy$ is an even function on x, ,then

$$\tilde{I}(\varepsilon) := 2\varepsilon \int_{-\infty}^{\infty} \tilde{f}(x) \left(\int_{-\infty}^{\infty} \tilde{g}(y) \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2 + xy + y^2}{x^2 + y^2} \right| dy \right) dx$$

$$= 2\varepsilon \left[\int_{1}^{\infty} x^{a(\mu-\lambda)-1-2\varepsilon/p} \left(\int_{-\infty}^{-1} (-y)^{b(\mu-\lambda)-1-2\varepsilon/q} \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2 + xy + y^2}{x^2 + y^2} \right| dy \right) dx \right.$$

$$\left. + \int_{1}^{\infty} x^{a(\mu-\lambda)-1-2\varepsilon/p} \left(\int_{1}^{\infty} y^{b(\mu-\lambda)-1-2\varepsilon/q} \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2 + xy + y^2}{x^2 + y^2} \right| dy \right) dx \right]$$

$$:= I_1 + I_2$$

Setting y = tx then

$$I_1 = 2\varepsilon \int_1^{\infty} x^{a(\mu-\lambda)-1-2\varepsilon/p} \left(\int_1^{\infty} y^{b(\mu-\lambda)-1-2\varepsilon/q} \frac{(\min\{x,y\})^{\lambda}}{(\max\{x,y\})^{\mu}} \ln \frac{x^2+y^2}{x^2-xy+y^2} dy \right) dx$$

$$= 2\varepsilon \int_1^{\infty} x^{-1-2\varepsilon} \left(\int_{1/x}^{\infty} t^{b(\mu-\lambda)-1-2\varepsilon/q} \frac{(\min\{1,t\})^{\lambda}}{(\max\{1,t\})^{\mu}} \ln \frac{t^2+1^2}{t^2-t+1^2} dt \right) dx$$

$$= 2\varepsilon \int_1^{\infty} x^{-1-2\varepsilon} \left(\int_1^{\infty} t^{b(\mu-\lambda)-1-2\varepsilon/q} \frac{(\min\{1,t\})^{\lambda}}{(\max\{1,t\})^{\mu}} \ln \frac{t^2+1^2}{t^2-t+1^2} dt \right) dx$$

$$+ 2\varepsilon \int_1^{\infty} x^{-1-2\varepsilon} \left(\int_{1/x}^{1} t^{b(\mu-\lambda)-1-2\varepsilon/q} \frac{(\min\{1,t\})^{\lambda}}{(\max\{1,t\})^{\mu}} \ln \frac{t^2+1^2}{t^2-t+1^2} dt \right) dx$$

$$= \int_1^{\infty} t^{b(\mu-\lambda)-1-2\varepsilon/q} \frac{(\min\{1,t\})^{\lambda}}{(\max\{1,t\})^{\mu}} \ln \frac{t^2+1^2}{t^2-t+1^2} dt$$

$$+ 2\varepsilon \int_0^{1} t^{b(\mu-\lambda)-1-2\varepsilon/q} \frac{(\min\{1,t\})^{\lambda}}{(\max\{1,t\})^{\mu}} \ln \frac{t^2+1^2}{t^2-t+1^2} \left(\int_{1/t}^{\infty} x^{-1-2\varepsilon} dx \right) dt$$

$$= \int_1^{\infty} t^{b(\mu-\lambda)-1-2\varepsilon/q} \frac{(\min\{1,t\})^{\lambda}}{(\max\{1,t\})^{\mu}} \ln \frac{t^2+1^2}{t^2-t+1^2} dt$$

$$+ \int_0^{1} t^{b(\mu-\lambda)-1-2\varepsilon/p} \frac{(\min\{1,t\})^{\lambda}}{(\max\{1,t\})^{\mu}} \ln \frac{t^2+1^2}{t^2-t+1^2} dt$$

$$= \int_0^{\infty} t^{b(\mu-\lambda)-1-2\varepsilon/q} \frac{(\min\{1,t\})^{\lambda}}{(\max\{1,t\})^{\mu}} \ln \frac{t^2+1^2}{t^2-t+1^2} dt$$

$$+ \int_0^{1} (t^{2\varepsilon/p} - t^{-2\varepsilon/q}) t^{b(\mu-\lambda)-1} \frac{(\min\{1,t\})^{\lambda}}{(\max\{1,t\})^{\mu}} \ln \frac{t^2+1^2}{t^2-t+1^2} dt$$

$$= \int_0^{\infty} t^{b(\mu-\lambda)-1-2\varepsilon/q} \frac{(\min\{1,t\})^{\lambda}}{(\max\{1,t\})^{\mu}} \ln \frac{t^2+1^2}{t^2-t+1^2} dt + \eta_1(\varepsilon) \left(\lim_{\varepsilon \to 0^+} \eta_1(\varepsilon) = 0 \right)$$

Similarly

$$I_2 = 2\varepsilon \int_1^{\infty} x^{a(\mu-\lambda)-1-2\varepsilon/p} \left(\int_1^{\infty} y^{b(\mu-\lambda)-1-2\varepsilon/q} \frac{(\min\{x,y\})^{\lambda}}{(\max\{x,y\})^{\mu}} \ln \frac{x^2+xy+y^2}{x^2+y^2} dy \right) dx$$

$$= \int_0^{\infty} t^{b(\mu-\lambda)-1-2\varepsilon/q} \frac{(\min\{1,t\})^{\lambda}}{(\max\{1,t\})^{\mu}} \ln \frac{t^2+t+1^2}{t^2+1^2} dt + \eta_2(\varepsilon), \left(\lim_{\varepsilon \to 0^+} \eta_2(\varepsilon) = 0 \right)$$

by lemma 2.2 we have

$$\tilde{I}(\varepsilon) = I_1 + I_2$$

$$= \frac{2}{a\lambda + b\mu - 2\varepsilon/q} \left[\psi(a\lambda + b\mu - 2\varepsilon/q) - \psi\left(\frac{a\lambda + b\mu - 2\varepsilon/q}{3}\right) \right]$$

$$+ \frac{1}{a\lambda + b\mu - 2\varepsilon/q} \left[\psi\left(\frac{a\lambda + b\mu - 2\varepsilon/q}{6}\right) - \psi\left(\frac{a\lambda + b\mu - 2\varepsilon/q}{2}\right) \right]$$

$$+ \frac{2}{a\mu + b\lambda - 2\varepsilon/q} \left[\psi(a\mu + b\lambda - 2\varepsilon/q) - \psi\left(\frac{a\mu + b\lambda - 2\varepsilon/q}{3}\right) \right]$$

$$+ \frac{1}{a\mu + b\lambda - 2\varepsilon/q} \left[\psi\left(\frac{a\mu + b\lambda - 2\varepsilon/q}{6}\right) - \psi\left(\frac{a\mu + b\lambda - 2\varepsilon/q}{2}\right) \right] + \eta_1(\varepsilon) + \eta_2(\varepsilon)$$

we know that $\psi(x)$ is a continuous function,then $\lim_{\varepsilon \to 0^+} \tilde{I}(\varepsilon) = k$

The lemma is proved.

Lemma 2.4 If f(x) is a nonnegative measurable function, and $0 < \int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x)dx < \infty$

Then

$$J := \int_{-\infty}^{\infty} |y|^{pb(\mu-\lambda)-1} \left(\int_{-\infty}^{\infty} f(x) \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| dx \right)^p dy$$

$$\leq k^p \int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x)dx \qquad (2.7)$$

Proof By lemma 2.2,we find

$$\left(\int_{-\infty}^{\infty} f(x) \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| dx \right)^p$$

$$= \left(\int_{-\infty}^{\infty} \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| \left(\frac{|x|^{[1-a(\mu-\lambda)]/q}}{|y|^{[1-b(\mu-\lambda)]/p}} f(x) \right) \left(\frac{|y|^{[1-b(\mu-\lambda)]/p}}{|x|^{[1-a(\mu-\lambda)]/q}} \right) dx \right)^p$$

$$\leq \int_{-\infty}^{\infty} \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| \frac{|x|^{[1-a(\mu-\lambda)](p-1)}}{|y|^{1-b(\mu-\lambda)}} f^p(x)dx$$

$$\times \left(\int_{-\infty}^{\infty} \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| \frac{|y|^{[1-b(\mu-\lambda)](q-1)}}{|x|^{1-a(\mu-\lambda)}} dx \right)^{p-1}$$

$$= k^{p-1} |y|^{1-pb(\mu-\lambda)} \int_{-\infty}^{\infty} \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| \frac{|x|^{[1-a(\mu-\lambda)](p-1)}}{|y|^{1-b(\mu-\lambda)}} f^p(x)dx$$

$$J \leq k^{p-1} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| \frac{|x|^{[1-a(\mu-\lambda)](p-1)}}{|y|^{1-b(\mu-\lambda)}} f^p(x)dx \right] dy$$

$$= k^{p-1} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| \frac{|x|^{[1-a(\mu-\lambda)](p-1)}}{|y|^{1-b(\mu-\lambda)}} dy \right] f^p(x)dx$$

$$= k^p \int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x)dx$$

3 MAIN RESULTS

Theorem 3.1 If p [illegible] both functions, f(x) and g(x) ,are nonnegative measurable functions, and satisfy [illegible] $0 < \int_{-\infty}^{\infty} |y|^{q[1-b(\mu-\lambda)]-1} g^q(x)dx < \infty$ then $0 < \int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x)dx < \infty$

$$I^* := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(y) \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| dxdy$$

$$< k \left(\int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x)dx \right)^{1/p} \left(\int_{-\infty}^{\infty} |x|^{q[1-b(\mu-\lambda)]-1} g^q(x)dx \right)^{1/q} \qquad (3.1)$$

And

$$J = \int_{-\infty}^{\infty} |y|^{pb(\mu-\lambda)-1} \left(\int_{-\infty}^{\infty} f(x) \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| dx \right)^p dy$$

$$< k^p \int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x)dx \qquad (3.2)$$

Inequalities (3.1)and (3.2) are equivalent,and where the constant factors k and kp arethe best possible.

Proof If there exist a $y \in (-\infty,0)\cup(0,\infty)$,such that (2.7) takes the form of equality, then there exists constants M and N, such that they are not all zero, and

$$M\frac{|x|^{[1-a(\mu-\lambda)](p-1)}}{|y|^{1-b(\mu-\lambda)}} f^{p}(x) = N\frac{|y|^{[1-b(\mu-\lambda)](q-1)}}{|x|^{1-a(\mu-\lambda)}}$$ a.e. In $(-\infty,\infty)$

Hence, there exists a constant C, such that

$$M|x|^{p[1-a(\mu-\lambda)]} f^{p}(x) = N|y|^{q[1-b(\mu-\lambda)]} = C$$ a.e. In $(-\infty,\infty)$

It means that M = 0. In fact, if $M \neq 0$, then

$$|x|^{p[1-a(\mu-\lambda)]-1} f^{p}(x) = \frac{C}{M|x|}$$ a.e. In $(-\infty,\infty)$

which contradicts the fact that $0<\int_{-\infty}^{\infty}|x|^{p[1-a(\mu-\lambda)]-1} f^{p}(x)dx<\infty$ In the same way, we claim that N = 0: This is too a contradiction and hence by (2.7), we have (3.2). By Holder's inequality with weight and (3.2), we have,

$$I^{*} := \int_{-\infty}^{\infty}\left[|y|^{b(\mu-\lambda)-\frac{1}{p}}\int_{-\infty}^{\infty} f(x)\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}}\left|\ln\frac{x^{2}+xy+y^{2}}{x^{2}+y^{2}}\right|dx\right]\left[|y|^{-b(\mu-\lambda)+\frac{1}{p}} g(y)\right]dy$$

$$\leq J^{1/p}\left(\int_{-\infty}^{\infty}|y|^{q[1-b(\mu-\lambda)]-1} g^{q}(y)dy\right)^{1/q} \quad (3.3)$$

Using (3.2),we have (3.1).
Setting

$$g(y) = |y|^{pb(\mu-\lambda)-1}\left(\int_{-\infty}^{\infty} f(x)\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}}\left|\ln\frac{x^{2}+xy+y^{2}}{x^{2}+y^{2}}\right|dx\right)^{p-1}$$

Then

$J = \int_{-\infty}^{\infty}|y|^{q[1-b(\mu-\lambda)]-1} g^{q}(y)dy$ by (2.?)we have $J<\infty$ if J = 0 then (3.2) is proved;

If $0<J<\infty$ we obtain

$$0<\int_{-\infty}^{\infty}|y|^{q[1-b(\mu-\lambda)]-1} g^{q}(y)dy = J = I^{*}$$

$$< k\left(\int_{-\infty}^{\infty}|x|^{p[1-a(\mu-\lambda)]-1} f^{p}(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{q[1-b(\mu-\lambda)]-1} g^{q}(x)dx\right)^{1/q}$$

$$\left(\int_{-\infty}^{\infty}|x|^{q[1-b(\mu-\lambda)]-1} g^{q}(x)dx\right)^{1/p} = J^{1/p} < k\left(\int_{-\infty}^{\infty}|x|^{p[1-a(\mu-\lambda)]-1} f^{p}(x)dx\right)^{1/p}$$

Inequalities (3.1)and (3.2) are equivalent.
If the constant factor k in (3.1) is not the best possible, then there exists a positive h (with h < k),such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(y)\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}}\left|\ln\frac{x^{2}+xy+y^{2}}{x^{2}+y^{2}}\right|dxdy$$

$$< h\left(\int_{-\infty}^{\infty}|x|^{p[1-a(\mu-\lambda)]-1} f^{p}(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{q[1-b(\mu-\lambda)]-1} g^{q}(x)dx\right)^{1/q}$$

For $\varepsilon>0$ by (2.5),using lemma 2.3 ,we have

$$k+o(1)<\varepsilon h\left(\int_{-\infty}^{\infty}|x|^{p[1-a(\mu-\lambda)]-1}\tilde{f}^{p}(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{q[1-b(\mu-\lambda)]-1}\tilde{g}^{q}(x)dx\right)^{1/q} = h \quad (3.4)$$

Hence we find, k +o(1) < h: For $\varepsilon \to 0^{+}$ it follows that $k \leq h$ which contradicts the fact that h < k .
Hence the constant k in (3.1) is the best possible.
Thus we complete the prove of the theorem.

Theorem 3.2 If 1 > p > 0; both functions, f(x) and g(x) ,are nonnegative measurable functions, and satisfy

$0<\int_{-\infty}^{\infty}|x|^{p[1-a(\mu-\lambda)]-1}f^{p}(x)dx<\infty$ and $0<\int_{-\infty}^{\infty}|x|^{q[1-b(\mu-\lambda)]-1}g^{q}(x)dx<\infty$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x)g(y)\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}\right|dxdy$$
$$>k\left(\int_{-\infty}^{\infty}|x|^{p[1-a(\mu-\lambda)]-1}f^{p}(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{q[1-b(\mu-\lambda)]-1}g^{q}(x)dx\right)^{1/q} \tag{3.5}$$

$$J>k^{p}\int_{-\infty}^{\infty}|x|^{p[1-a(\mu-\lambda)]-1}f^{p}(x)dx \tag{3.6}$$

And

$$L:=\int_{-\infty}^{\infty}|x|^{qa(\mu-\lambda)-1}\left(\int_{-\infty}^{\infty}\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}g(y)\right|dy\right)^{q}dx$$
$$<k^{q}\int_{-\infty}^{\infty}|y|^{q[1-b(\mu-\lambda)]-1}g^{q}(y)dy \tag{3.7}$$

Inequalities (3.5),(3.6)and (3.7) are equivalent,and where the constant factors k, k^p and k^q are the best possible

Proof By the reverse Holder's inequality and the same way, we can obtain the reverseforms of (2.7)and (3.3).And then we deduce the (3.5),by the some way,we obtain (3.6).

Setting g(y) as the theorem 1,we obtain $J>0$,if $J=\infty$,then we have (3.6),if $0<J<\infty$,by (3.5)

$$\int_{-\infty}^{\infty}|y|^{q[1-b(\mu-\lambda)]-1}g^{q}(y)dy=J=I^{*}$$
$$>k\left(\int_{-\infty}^{\infty}|x|^{p[1-a(\mu-\lambda)]-1}f^{p}(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{q[1-b(\mu-\lambda)]-1}g^{q}(x)dx\right)^{1/q}$$

and we have (3.6),and inequalities (3.5)and (3.6) are equivalent.

Setting

$$[g(x)]_n=\begin{cases}\frac{1}{n} & \text{if } g(x)<\frac{1}{n},\\ g(x) & \text{if } \frac{1}{n}\le g(x)\le n\\ n & \text{if } g(x)>n\end{cases}$$

$E_n=\left[-n,-\frac{1}{n}\right]\cup\left[\frac{1}{n},n\right]$ then $\exists n_0\in\mathbb{N}$, such that $n>n_0$; we have

$\int_{E_n}|y|^{q[1-b(\mu-\lambda)]-1}g^{q}(y)dy>0,$

And

$$[f(x)]_n=|x|^{qa(\mu-\lambda)-1}\left(\int_{E_n}[g(x)]_n\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}\right|dy\right)^{q-1}$$

$$[L(x)]_n=\int_{E_n}|x|^{qa(\mu-\lambda)-1}\left(\int_{E_n}[g(x)]_n\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}\right|dy\right)^{q}dx$$

Then $\exists n_0\in\mathbb{N}$, such that $n>n_0$; we have

$\int_{E_n}|y|^{q[1-b(\mu-\lambda)]-1}g^{q}(y)dy<\infty,$

In particular, from (4.1) we get the following particular cases:

1)If $\lambda+\mu=4$; then $k=2\left[\psi(2)-\psi(\frac{2}{3})\right]+\frac{2}{3}\left[\psi(\frac{1}{3})-\psi(1)\right]=2+\frac{3\ln 3}{2}-\frac{\sqrt{3}\pi}{2}$, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(y)\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{4-\lambda}}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}\right|dxdy$$

$$<\left(2+\frac{3\ln 3}{2}-\frac{\sqrt{3}\pi}{2}\right)\left(\int_{-\infty}^{\infty}|x|^{p(\lambda-1)-1}f^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{q(\lambda-1)-1}g^q(x)dx\right)^{1/q} \tag{4.2}$$

2) If $\lambda+\mu=3$; then $k=\frac{8}{3}\left[\psi(\frac{3}{2})-\psi(\frac{1}{2})\right]+\frac{4}{3}\left[\psi(\frac{1}{4})-\psi(\frac{3}{4})\right]=\frac{4}{3}(4-\pi)$, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(y)\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{4-\lambda}}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}\right|dxdy$$

$$<\left(2+\frac{3\ln 3}{2}-\frac{\sqrt{3}\pi}{2}\right)\left(\int_{-\infty}^{\infty}|x|^{p(\lambda-1)-1}f^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{q(\lambda-1)-1}g^q(x)dx\right)^{1/q} \tag{4.3}$$

3) If $\lambda+\mu=2$; then $k=4\left[\psi(1)-\psi(\frac{1}{3})\right]+2\left[\psi(\frac{1}{6})-\psi(\frac{1}{2})\right]=6\ln 3+\frac{2\pi}{\sqrt{3}}-\sqrt{2}\pi$, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(y)\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{2-\lambda}}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}\right|dxdy$$

$$<\left(6\ln 3+\frac{2\pi}{\sqrt{3}}-\sqrt{2}\pi\right)\left(\int_{-\infty}^{\infty}|x|^{p\lambda-1}f^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{q\lambda-1}g^q(x)dx\right)^{1/q} \tag{4.4}$$

B) Let $\lambda=\mu$ in (3.1),then we have a 1ntegral 1nequality with the homogeneous kernel of 0 degree form as follows:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(y)\left(\frac{\min\{|x|,|y|\}}{\max\{|x|,|y|\}}\right)^{\lambda}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}\right|dxdy$$

$$<k(\lambda)\left(\int_{-\infty}^{\infty}|x|^{p-1}f^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{q-1}g^q(x)dx\right)^{1/q} \tag{4.5}$$

There

$$k(\lambda)=\frac{4}{\lambda}\left[\psi(\lambda)-\psi(\frac{\lambda}{3})\right]+\frac{2}{\lambda}\left[\psi(\frac{\lambda}{6})-\psi(\frac{\lambda}{2})\right]$$

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