

A New Hilbert-type Integral Inequality with the Homogeneous Kernel of Real Degree Form and the Integral in Whole Plane

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Abstract: In this paper, we build a new Hilbert's inequality with the homogeneous kernel of real order and the integral in whole plane. The equivalent inequality is considered. The best constant factor is calculated using Ψ function.

1 INTRODUCTION

If $f(x), g(x) \geq 0$, such that $0 < \int_0^\infty f^2(x) dx < \infty$ $0 < \int_0^\infty g^2(x) dx < \infty$; then [1]

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \pi \left(\int_0^\infty f^2(x) dx \right)^{1/2} \left(\int_0^\infty g^2(x) dx \right)^{1/2}. \quad (1.1)$$

where the constant factor π is the best possible. Inequality (1.1) is well-known as Hilbert's integral inequality, which has been extended by Hardy-Riesz as [2]:

If $p > 1, 1/p + 1/q = 1$ $f(x), g(x) \geq 0$; such that $0 < \int_0^\infty f^p(x) dx < \infty$, and $0 < \int_0^\infty g^q(x) dx < \infty$; then we have the following Hardy-Hilbert's integral inequality:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(x) dx \right)^{1/p} \left(\int_0^\infty g^q(x) dx \right)^{1/q}; \quad (1.2)$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ also is the best possible.

Hilbert's inequality attracts some attention in recent years. Actually, inequalities (1.1) and (1.2) have many generalizations and variations. (1.1) has been strengthened by Yang and others (including double series inequalities). [3,4,6-21].

In 2008, Zitian Xie and Zheng Zeng gave a new Hilbert-type Inequality [4] as follows

If $a > 0, b > 0, c > 0; p > 1, 1/p + 1/q = 1$ $f(x), g(x) \geq 0$; such that

$$0 < \int_0^\infty f^{-1-p/2}(x) dx < \infty \text{ and } 0 < \int_0^\infty g^{-1-q/2}(x) dx < \infty \text{ then} \\ < K \left(\int_0^\infty f^{-1-p/2}(x) dx \right)^{1/p} \left(\int_0^\infty g^{-1-q/2}(x) dx \right)^{1/q}; \quad (1.3)$$

$$K = \frac{\pi}{(a+b)(a+c)(c+b)}$$

where the constant factor $\frac{\pi}{(a+b)(a+c)(c+b)}$ is the best possible.

In 2010, Jianhua Xiong and Bicheng Yang gave a new Hilbert-type Inequality [5] as follows :

Assume that

$$\lambda, p > 0 (p \neq 1), r > 1, 1/p + 1/q = 1, 1/r + 1/s = 1, \phi(x) = x^{p(1-\frac{\lambda}{r})-1}, \varphi(x) = x^{q(1-\frac{\lambda}{s})-1}, x \in (0, \infty),$$

$$K = \Gamma(\beta + 1) \sum_{k=0}^{\infty} (-1)^k \binom{\alpha - \lambda}{k} \left[\frac{1}{(k + \lambda/r)^{\beta+1}} + \frac{1}{(k + \lambda/s)^{\beta+1}} \right], \text{ and } f, g \geq 0,$$

$0 < \|f\|_{p,\phi} := \left\{ \int_0^\infty x^{p(1-\lambda/r)-1} f^p(x) dx \right\}^{1/p} < \infty, 0 < \|g\|_{q,\varphi} < \infty$ then

(1) for $p > 1$ we have the following equivalent inequalities:

$$\int_0^\infty \int_0^\infty \frac{|\ln(x/y)|^\beta f(x)g(y)}{|x-y|(\max\{|x|,|y|\})^\alpha} dx dy < K \|f\|_{p,\phi} \|g\|_{q,\varphi}$$

(2) For $0 < p < 1$ the reverse of (1.5) with the best constant factor K .

$$\int_0^\infty \int_0^\infty \frac{|\ln(x/y)|^\beta f(x)g(y)}{|x-y|(\max\{|x|,|y|\})^\alpha} dx dy > K \|f\|_{p,\phi} \|g\|_{q,\varphi}$$

The main purpose of this paper is to build a new Hilbert-type inequality with the homogeneous kernel of real order and the integral in whole plane, by estimating the weight function using Ψ function. The equivalent inequality is considered

We knew that (in this paper, γ is the Euler's constant.)

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = -\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+z} \right), \psi(1) = -\gamma, \psi\left(\frac{1}{2}\right) = -\gamma - 2 \ln 2.$$

Recent XIE Zitin and ZHOU Qinghua prove that the expression of the Ψ -function admits a finite expression in elementary function for rational number z , and prove that [6]

$$\psi\left(\frac{a}{b}\right) = \Gamma'\left(\frac{a}{b}\right) / \Gamma\left(\frac{a}{b}\right) = -\ln b - \gamma - \ln 2 - \frac{\pi}{2} \cot \frac{a\pi}{b} + \sum_{k=1}^{b-1} \cos \frac{2ka\pi}{b} \ln \sin \frac{k\pi}{b}$$

and have

$$\begin{aligned} \psi\left(\frac{1}{3}\right) &= -\gamma - \ln 3 - \ln 2 - \frac{\pi}{2} \cot \frac{\pi}{3} + \cos \frac{2\pi}{3} \ln \sin \frac{\pi}{3} + \cos \frac{4\pi}{3} \ln \sin \frac{2\pi}{3} \\ &= -\gamma - \frac{3}{2} \ln 3 - \frac{\pi}{2\sqrt{3}}; \end{aligned}$$

$$\psi\left(\frac{2}{3}\right) = -\gamma - \frac{3}{2} \ln 3 + \frac{\pi}{2\sqrt{3}};$$

$$\psi\left(\frac{1}{4}\right) = -\gamma - 3 \ln 2 - \frac{\pi}{2};$$

$$\psi\left(\frac{3}{4}\right) = -\gamma - 3 \ln 2 + \frac{\pi}{2};$$

$$\psi\left(\frac{1}{6}\right) = -\gamma - 2 \ln 2 - \frac{\sqrt{2}\pi}{2};$$

$$\psi\left(\frac{5}{6}\right) = -\gamma - 2 \ln 2 + \frac{\sqrt{2}\pi}{2};$$

$$\psi\left(\frac{1}{5}\right) = -\gamma - \frac{5 \ln 5}{4} - \frac{\sqrt{5}}{2} \ln(\sqrt{5}-1) - \frac{\sqrt{5}}{2} \ln 2 - \frac{\pi}{40} (5+3\sqrt{5}) \sqrt{10-2\sqrt{5}};$$

$$\psi\left(\frac{4}{5}\right) = -\gamma - \frac{5 \ln 5}{4} - \frac{\sqrt{5}}{2} \ln(\sqrt{5}-1) - \frac{\sqrt{5}}{2} \ln 2 + \frac{\pi}{40} (5+3\sqrt{5}) \sqrt{10-2\sqrt{5}}$$

In the following, we always suppose that:

$$1/p + 1/q = 1, p > 1, \min\{a\lambda + b\mu, a\mu + b\lambda\} > -1, a\mu + b\lambda \neq 0, a\lambda + b\mu \neq 0, \mu > 0, \lambda > 0.$$

$$a + b = 1.$$

2 SOME LEMMAS

We start by introducing some Lemmas.

Lemma 2.1 If $s > 0, r \neq 0, r > -s$, then

$$\begin{aligned}
 1) \int_0^1 x^{r-1} \ln(1-x^s) dx &= -\frac{1}{r} \left[\gamma + \psi \left(\frac{r+s}{s} \right) \right] \\
 2) \int_0^1 x^{r-1} \ln(1+x^s) dx &= \frac{1}{r} \ln 2 - \frac{1}{2r} \left[\psi \left(\frac{r+2s}{2s} \right) - \psi \left(\frac{r+s}{2s} \right) \right]
 \end{aligned}
 \tag{2.1}$$

Proof. we obtain,

$$\begin{aligned}
 1) -\int_0^1 x^{r-1} \ln(1-x^s) dx &= \int_0^1 x^{r-1} \sum_{l=1}^{\infty} \frac{x^{ls}}{l} dx \\
 &= \sum_{l=1}^{\infty} \int_0^1 \frac{x^{ls+r-1}}{l} dx = \sum_{l=1}^{\infty} \frac{1}{l(r+ls)} \\
 &= \frac{1}{r} \sum_{l=1}^{\infty} \left(\frac{1}{l} - \frac{1}{l+r/s} \right) \\
 &= \frac{1}{r} \left[\gamma + \psi \left(\frac{r+s}{s} \right) \right] \\
 2) \int_0^1 x^{r-1} \ln(1+x^s) dx &= \int_0^1 x^{r-1} \sum_{l=1}^{\infty} (-1)^{l-1} \frac{x^{ls}}{l} dx \\
 &= \sum_{l=1}^{\infty} \int_0^1 (-1)^{l-1} \frac{x^{ls+r-1}}{l} dx = \sum_{l=1}^{\infty} \frac{(-1)^{l-1}}{l(r+ls)} \\
 &= \lim_{N \rightarrow \infty} \left\{ \frac{1}{r} \sum_{l=1}^{2N+1} (-1)^{l-1} \frac{1}{l} - \frac{1}{2r} \left[\sum_{n=0}^N \left(\frac{1}{n+1} - \frac{1}{n+\frac{r+2s}{2s}} \right) - \sum_{n=0}^N \left(\frac{1}{n+1} - \frac{1}{n+\frac{r+s}{2s}} \right) \right] \right\} \\
 &= \frac{1}{r} \ln 2 - \frac{1}{2r} \left[\psi \left(\frac{r+2s}{2s} \right) - \psi \left(\frac{r+s}{2s} \right) \right]
 \end{aligned}$$

The lemma is proved.

In particular, if $r > -1$, $r \neq 0$, then

$$\begin{aligned}
 &\int_0^1 u^{r-1} \ln \frac{u^2+u+1}{u^2-u+1} du \\
 &= \int_0^1 u^{r-1} \ln(1-u^3) du - \int_0^1 u^{r-1} \ln(1-u) du - \left[\int_0^1 u^{r-1} \ln(1+u^3) du - \int_0^1 u^{r-1} \ln(1+u) du \right] \\
 &= \frac{1}{r} \left[\psi(r+1) - \psi \left(\frac{r+3}{3} \right) \right] + \frac{1}{2r} \left[\psi \left(\frac{r+6}{6} \right) - \psi \left(\frac{r+3}{6} \right) - \psi \left(\frac{r+2}{2} \right) + \psi \left(\frac{r+1}{2} \right) \right] \\
 &\quad \text{(using } \psi(x+1) = \psi(x) + \frac{1}{x} \text{)} \\
 &= \frac{1}{r} \left[\psi(r) - \psi \left(\frac{r}{3} \right) \right] + \frac{1}{2r} \left[\psi \left(\frac{r}{6} \right) - \psi \left(\frac{r+3}{6} \right) - \psi \left(\frac{r}{2} \right) + \psi \left(\frac{r+1}{2} \right) \right] \\
 &= \frac{1}{r} \left[\psi(r) - \psi \left(\frac{r}{3} \right) \right] + \frac{1}{2r} \left[2\psi \left(\frac{r}{6} \right) - \psi \left(\frac{r}{6} \right) - \psi \left(\frac{r+3}{6} \right) - 2\psi \left(\frac{r}{2} \right) + \psi \left(\frac{r}{2} \right) + \psi \left(\frac{r+1}{2} \right) \right] \\
 &\quad \text{(using } \psi \left(x + \frac{1}{2} \right) + \psi(x) = 2\psi(2x) - \psi \left(\frac{r+1}{2} \right) \text{)} \\
 &= \frac{2}{r} \left[\psi(r) - \psi \left(\frac{r}{3} \right) \right] + \frac{1}{r} \left[\psi \left(\frac{r}{6} \right) - \psi \left(\frac{r}{2} \right) \right]
 \end{aligned}
 \tag{2.2}$$

Lemma 2.2 Define the weight functions as follow:

$$w(x) := \int_{-\infty}^{\infty} \frac{|x|^{a(\mu-\lambda)}}{|y|^{1-b(\mu-\lambda)}} \frac{(\min\{|x|, |y|\})^\lambda}{(\max\{|x|, |y|\})^\mu} \left| \ln \frac{x^2 + xy + y^2}{x^2 + y^2} \right| dy,$$

$$\tilde{w}(y) := \int_{-\infty}^{\infty} \frac{|y|^{b(\mu-\lambda)}}{|x|^{1-a(\mu-\lambda)}} \frac{(\min\{|x|, |y|\})^\lambda}{(\max\{|x|, |y|\})^\mu} \left| \ln \frac{x^2 + xy + y^2}{x^2 + y^2} \right| dx,$$

Then

$$\begin{aligned} w(x) &= \tilde{w}(y) \\ &= \frac{2}{a\lambda + b\mu} \left[\psi(a\lambda + b\mu) - \psi\left(\frac{a\lambda + b\mu}{3}\right) \right] + \frac{1}{a\lambda + b\mu} \left[\psi\left(\frac{a\lambda + b\mu}{6}\right) - \psi\left(\frac{a\lambda + b\mu}{2}\right) \right] \\ &\quad + \frac{2}{a\mu + b\lambda} \left[\psi(a\mu + b\lambda) - \psi\left(\frac{a\mu + b\lambda}{3}\right) \right] + \frac{1}{a\lambda + b\mu} \left[\psi\left(\frac{a\mu + b\lambda}{6}\right) - \psi\left(\frac{a\mu + b\lambda}{2}\right) \right]. \\ &:= k \end{aligned} \tag{2.3}$$

Proof We only prove that $w(x) = k$ for $x \in (-\infty, 0)$

Using lemma 2.1, setting $y = ux$, and $y = -ux$

$$\begin{aligned} w(x) &:= \int_{-\infty}^0 \frac{(-x)^{a(\mu-\lambda)}}{(-y)^{1-b(\mu-\lambda)}} \frac{(\min\{(-x), (-y)\})^\lambda}{(\max\{(-x), (-y)\})^\mu} \ln \frac{x^2 + xy + y^2}{x^2 + y^2} dy \\ &\quad + \int_0^{\infty} \frac{(-x)^{a(\mu-\lambda)}}{y^{1-b(\mu-\lambda)}} \frac{(\min\{(-x), y\})^\lambda}{(\max\{(-x), y\})^\mu} \ln \frac{x^2 + y^2}{x^2 + xy + y^2} dy \\ &:= w_1 + w_2 \end{aligned}$$

Then

$$\begin{aligned} w_1 &= \int_0^1 u^{-1+a\lambda+b\mu} \ln \frac{u^2 + u + 1}{u^2 + 1} du + \int_1^{\infty} u^{-1-a\lambda-b\mu} \ln \frac{u^2 + u + 1}{u^2 + 1} du \\ &= \int_0^1 u^{-1+a\lambda+b\mu} \ln \frac{u^2 + u + 1}{u^2 + 1} du + \int_0^1 u^{-1+a\mu+b\lambda} \ln \frac{u^2 + u + 1}{u^2 + 1} du \\ w_2 &= \int_0^1 u^{-1+a\lambda+b\mu} \ln \frac{u^2 + 1}{u^2 - u + 1} du + \int_1^{\infty} u^{-1-a\lambda-b\mu} \ln \frac{u^2 + 1}{u^2 - u + 1} du \\ &= \int_0^1 u^{-1+a\lambda+b\mu} \ln \frac{u^2 + 1}{u^2 - u + 1} du + \int_0^1 u^{-1+a\mu+b\lambda} \ln \frac{u^2 + 1}{u^2 - u + 1} du \end{aligned}$$

And

$$\begin{aligned} w &= w_1 + w_2 = \int_0^1 u^{-1+a\lambda+b\mu} \ln \frac{u^2 + u + 1}{u^2 - u + 1} du + \int_0^1 u^{-1+a\mu+b\lambda} \ln \frac{u^2 + u + 1}{u^2 - u + 1} du \\ &= \frac{2}{a\lambda + b\mu} \left[\psi(a\lambda + b\mu) - \psi\left(\frac{a\lambda + b\mu}{3}\right) \right] + \frac{1}{a\lambda + b\mu} \left[\psi\left(\frac{a\lambda + b\mu}{6}\right) - \psi\left(\frac{a\lambda + b\mu}{2}\right) \right] \\ &\quad + \frac{2}{a\mu + b\lambda} \left[\psi(a\mu + b\lambda) - \psi\left(\frac{a\mu + b\lambda}{3}\right) \right] + \frac{1}{a\lambda + b\mu} \left[\psi\left(\frac{a\mu + b\lambda}{6}\right) - \psi\left(\frac{a\mu + b\lambda}{2}\right) \right]. \\ &= k \end{aligned}$$

Similarly, setting $x = y/u$, and $x = -y/u$

$$\begin{aligned} \tilde{w}(y) &= \int_{-\infty}^0 \frac{(-y)^{b(\mu-\lambda)}}{(-x)^{1-a(\mu-\lambda)}} \frac{(\min\{(-x), (-y)\})^\lambda}{(\max\{(-x), (-y)\})^\mu} \ln \frac{x^2 + xy + y^2}{x^2 + y^2} dx \\ &\quad + \int_0^{\infty} \frac{y^{b(\mu-\lambda)}}{(-x)^{1-a(\mu-\lambda)}} \frac{(\min\{(-x), y\})^\lambda}{(\max\{(-x), y\})^\mu} \ln \frac{x^2 + y^2}{x^2 + xy + y^2} dx \end{aligned}$$

$$\begin{aligned}
 &= \int_{-\infty}^0 \frac{(-y)^{b(\mu-\lambda)}}{(-y/u)^{1-a(\mu-\lambda)}} \frac{(\min\{(-y/u), (-y)\})^\lambda}{(\max\{(-y/u), (-y)\})^\mu} \ln \frac{(y/u)^2 + (y/u)y + y^2}{(y/u)^2 + y^2} d(y/u) \\
 &\quad + \int_0^\infty \frac{y^{b(\mu-\lambda)}}{(y/u)^{1-a(\mu-\lambda)}} \frac{(\min\{(y/u), y\})^\lambda}{(\max\{(y/u), y\})^\mu} \ln \frac{(-y/u)^2 + y^2}{(-y/u)^2 + (-y/u)y + y^2} d(-y/u) \\
 &= \int_0^\infty u^{-1+b(\mu-\lambda)} \frac{(\min\{1, u\})^\lambda}{(\max\{1, u\})^\mu} \ln \frac{u^2 + u + 1}{u^2 + 1} du + \int_0^\infty u^{-1+b(\mu-\lambda)} \frac{(\min\{1, u\})^\lambda}{(\max\{1, u\})^\mu} \ln \frac{u^2 + 1}{u^2 - u + 1} du \\
 &= w_1 + w_2 = k
 \end{aligned} \tag{2.4}$$

and the lemma is proved.

Lemma 2.3 For $\varepsilon > 0$; and $\min\{a\mu + b\lambda - 2\varepsilon/q, a\lambda + b\mu - 2\varepsilon/q\} > -1$, define both functions, \tilde{f}, \tilde{g} as follow:

$$\tilde{f}(x) = \begin{cases} x^{a(\mu-\lambda)-1-2\varepsilon/p}, & \text{if } x \in (1, \infty), \\ 0, & \text{if } x \in [-1, 1], \\ (-x)^{a(\mu-\lambda)-1-2\varepsilon/p}, & \text{if } x \in (-\infty, -1), \end{cases}$$

$$\tilde{g}(x) = \begin{cases} x^{b(\mu-\lambda)-1-2\varepsilon/q}, & \text{if } x \in (1, \infty), \\ 0, & \text{if } x \in [-1, 1], \\ (-x)^{b(\mu-\lambda)-1-2\varepsilon/q}, & \text{if } x \in (-\infty, -1), \end{cases}$$

Then

$$I(\varepsilon) := \varepsilon \left\{ \int_{-\infty}^\infty |x|^{p[1-a(\mu-\lambda)]-1} \tilde{f}^p(x) dx \right\}^{1/p} \left\{ \int_{-\infty}^\infty |x|^{q[1-b(\mu-\lambda)]-1} \tilde{g}^q(x) dx \right\}^{1/q} = 1; \tag{2.5}$$

$$\tilde{I}(\varepsilon) := \varepsilon \int_{-\infty}^\infty \int_{-\infty}^\infty \tilde{f}(x) \tilde{g}(y) \frac{(\min\{|x|, |y|\})^\lambda}{(\max\{|x|, |y|\})^\mu} \left| \ln \frac{x^2 + xy + y^2}{x^2 + y^2} \right| dx dy \rightarrow k(\varepsilon \rightarrow 0^+) \tag{2.6}$$

Proof Easily

$$I(\varepsilon) = \varepsilon \left\{ 2 \int_1^\infty x^{-1} x^{-2\varepsilon} dx \right\}^{1/p} \left\{ \int_1^\infty 2 \int_1^\infty x^{-1} x^{-2\varepsilon} dx dx \right\}^{1/q} = 1$$

Let $y=Y$, using $\tilde{f}(-x) = \tilde{f}(x)$ $\tilde{g}(-x) = \tilde{g}(x)$ and

$$\begin{aligned}
 &\tilde{f}(-x) \int_{-\infty}^\infty \tilde{g}(y) \frac{(\min\{|x|, |y|\})^\lambda}{(\max\{|x|, |y|\})^\mu} \left| \ln \frac{x^2 - xy + y^2}{x^2 + y^2} \right| dy \\
 &= \tilde{f}(x) \int_{-\infty}^\infty \tilde{g}(Y) \frac{(\min\{|x|, |Y|\})^\lambda}{(\max\{|x|, |Y|\})^\mu} \left| \ln \frac{x^2 + xY + Y^2}{x^2 + Y^2} \right| dY
 \end{aligned}$$

we have that $\tilde{f}(x) \int_{-\infty}^\infty \tilde{g}(y) \frac{(\min\{|x|, |y|\})^\lambda}{(\max\{|x|, |y|\})^\mu} \left| \ln \frac{x^2 + xy + y^2}{x^2 + y^2} \right| dy$ is an even function on x , then

$$\begin{aligned}
 \tilde{I}(\varepsilon) &:= 2\varepsilon \int_{-\infty}^\infty \tilde{f}(x) \left(\int_{-\infty}^\infty \tilde{g}(y) \frac{(\min\{|x|, |y|\})^\lambda}{(\max\{|x|, |y|\})^\mu} \left| \ln \frac{x^2 + xy + y^2}{x^2 + y^2} \right| dy \right) dx \\
 &= 2\varepsilon \left[\int_1^\infty x^{a(\mu-\lambda)-1-2\varepsilon/p} \left(\int_{-\infty}^{-1} (-y)^{b(\mu-\lambda)-1-2\varepsilon/q} \frac{(\min\{|x|, |y|\})^\lambda}{(\max\{|x|, |y|\})^\mu} \left| \ln \frac{x^2 + xy + y^2}{x^2 + y^2} \right| dy \right) dx \right. \\
 &\quad \left. + \int_1^\infty x^{a(\mu-\lambda)-1-2\varepsilon/p} \left(\int_1^\infty y^{b(\mu-\lambda)-1-2\varepsilon/q} \frac{(\min\{|x|, |y|\})^\lambda}{(\max\{|x|, |y|\})^\mu} \left| \ln \frac{x^2 + xy + y^2}{x^2 + y^2} \right| dy \right) dx \right]
 \end{aligned}$$

$$:= I_1 + I_2$$

Setting $y = tx$ then

$$\begin{aligned}
 I_1 &= 2\varepsilon \int_1^\infty x^{a(\mu-\lambda)-1-2\varepsilon/p} \left(\int_{1/x}^\infty y^{b(\mu-\lambda)-1-2\varepsilon/q} \frac{(\min\{x,y\})^\lambda}{(\max\{x,y\})^\mu} \ln \frac{x^2+y^2}{x^2-xy+y^2} dy \right) dx \\
 &= 2\varepsilon \int_1^\infty x^{-1-2\varepsilon} \left(\int_{1/x}^\infty t^{b(\mu-\lambda)-1-2\varepsilon/q} \frac{(\min\{1,t\})^\lambda}{(\max\{1,t\})^\mu} \ln \frac{t^2+1^2}{t^2-t+1^2} dt \right) dx \\
 &= 2\varepsilon \int_1^\infty x^{-1-2\varepsilon} \left(\int_1^\infty t^{b(\mu-\lambda)-1-2\varepsilon/q} \frac{(\min\{1,t\})^\lambda}{(\max\{1,t\})^\mu} \ln \frac{t^2+1^2}{t^2-t+1^2} dt \right) dx \\
 &\quad + 2\varepsilon \int_1^\infty x^{-1-2\varepsilon} \left(\int_{1/x}^1 t^{b(\mu-\lambda)-1-2\varepsilon/q} \frac{(\min\{1,t\})^\lambda}{(\max\{1,t\})^\mu} \ln \frac{t^2+1^2}{t^2-t+1^2} dt \right) dx \\
 &= \int_1^\infty t^{b(\mu-\lambda)-1-2\varepsilon/q} \frac{(\min\{1,t\})^\lambda}{(\max\{1,t\})^\mu} \ln \frac{t^2+1^2}{t^2-t+1^2} dt \\
 &\quad + 2\varepsilon \int_0^1 t^{b(\mu-\lambda)-1-2\varepsilon/q} \frac{(\min\{1,t\})^\lambda}{(\max\{1,t\})^\mu} \ln \frac{t^2+1^2}{t^2-t+1^2} \left(\int_{1/t}^\infty x^{-1-2\varepsilon} dx \right) dt \\
 &= \int_1^\infty t^{b(\mu-\lambda)-1-2\varepsilon/q} \frac{(\min\{1,t\})^\lambda}{(\max\{1,t\})^\mu} \ln \frac{t^2+1^2}{t^2-t+1^2} dt \\
 &\quad + \int_0^1 t^{b(\mu-\lambda)-1-2\varepsilon/p} \frac{(\min\{1,t\})^\lambda}{(\max\{1,t\})^\mu} \ln \frac{t^2+1^2}{t^2-t+1^2} dt \\
 &= \int_0^\infty t^{b(\mu-\lambda)-1-2\varepsilon/q} \frac{(\min\{1,t\})^\lambda}{(\max\{1,t\})^\mu} \ln \frac{t^2+1^2}{t^2-t+1^2} dt \\
 &\quad + \int_0^1 (t^{2\varepsilon/p} - t^{-2\varepsilon/q}) t^{b(\mu-\lambda)-1} \frac{(\min\{1,t\})^\lambda}{(\max\{1,t\})^\mu} \ln \frac{t^2+1^2}{t^2-t+1^2} dt \\
 &= \int_0^\infty t^{b(\mu-\lambda)-1-2\varepsilon/q} \frac{(\min\{1,t\})^\lambda}{(\max\{1,t\})^\mu} \ln \frac{t^2+1^2}{t^2-t+1^2} dt + \eta_1(\varepsilon), (\lim_{\varepsilon \rightarrow 0^+} \eta_1(\varepsilon) = 0)
 \end{aligned}$$

Similarly

$$\begin{aligned}
 I_2 &= 2\varepsilon \int_1^\infty x^{a(\mu-\lambda)-1-2\varepsilon/p} \left(\int_1^\infty y^{b(\mu-\lambda)-1-2\varepsilon/q} \frac{(\min\{x,y\})^\lambda}{(\max\{x,y\})^\mu} \ln \frac{x^2+xy+y^2}{x^2+y^2} dy \right) dx \\
 &= \int_0^\infty t^{b(\mu-\lambda)-1-2\varepsilon/q} \frac{(\min\{1,t\})^\lambda}{(\max\{1,t\})^\mu} \ln \frac{t^2+t+1^2}{t^2+1^2} dt + \eta_2(\varepsilon), (\lim_{\varepsilon \rightarrow 0^+} \eta_2(\varepsilon) = 0)
 \end{aligned}$$

by lemma 2.2, we have

$$\begin{aligned}
 \tilde{I}(\varepsilon) &= I_1 + I_2 \\
 &= \frac{2}{a\lambda + b\mu - 2\varepsilon/q} \left[\psi(a\lambda + b\mu - 2\varepsilon/q) - \psi\left(\frac{a\lambda + b\mu - 2\varepsilon/q}{3}\right) \right] \\
 &\quad + \frac{1}{a\lambda + b\mu - 2\varepsilon/q} \left[\psi\left(\frac{a\lambda + b\mu - 2\varepsilon/q}{6}\right) - \psi\left(\frac{a\lambda + b\mu - 2\varepsilon/q}{2}\right) \right] \\
 &+ \frac{2}{a\mu + b\lambda - 2\varepsilon/q} \left[\psi(a\mu + b\lambda - 2\varepsilon/q) - \psi\left(\frac{a\mu + b\lambda - 2\varepsilon/q}{3}\right) \right] \\
 &\quad + \frac{1}{a\mu + b\lambda - 2\varepsilon/q} \left[\psi\left(\frac{a\mu + b\lambda - 2\varepsilon/q}{6}\right) - \psi\left(\frac{a\mu + b\lambda - 2\varepsilon/q}{2}\right) \right] + \eta_1(\varepsilon) + \eta_2(\varepsilon)
 \end{aligned}$$

we know that $\psi(x)$ is a continuous function, then $\lim_{\varepsilon \rightarrow 0^+} \tilde{I}(\varepsilon) = k$

The lemma is proved.

Lemma 2.4 If $f(x)$ is a nonnegative measurable function, and 0

$$0 < \int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x) dx < \infty$$

Then

$$J := \int_{-\infty}^{\infty} |y|^{pb(\mu-\lambda)-1} \left(\int_{-\infty}^{\infty} f(x) \frac{(\min\{|x|,|y|\})^\lambda}{(\max\{|x|,|y|\})^\mu} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| dx \right)^p dy \leq k^p \int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x) dx \tag{2.7}$$

Proof By lemma 2.2, we find

$$\begin{aligned} & \left(\int_{-\infty}^{\infty} f(x) \frac{(\min\{|x|,|y|\})^\lambda}{(\max\{|x|,|y|\})^\mu} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| dx \right)^p \\ &= \left(\int_{-\infty}^{\infty} \frac{(\min\{|x|,|y|\})^\lambda}{(\max\{|x|,|y|\})^\mu} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| \left(\frac{|x|^{[1-a(\mu-\lambda)]/q}}{|y|^{[1-b(\mu-\lambda)]/p}} f(x) \right) \left(\frac{|y|^{[1-b(\mu-\lambda)]/p}}{|x|^{[1-a(\mu-\lambda)]/q}} dx \right)^p \right. \\ &\leq \int_{-\infty}^{\infty} \frac{(\min\{|x|,|y|\})^\lambda}{(\max\{|x|,|y|\})^\mu} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| \frac{|x|^{[1-a(\mu-\lambda)](p-1)}}{|y|^{1-b(\mu-\lambda)}} f^p(x) dx \\ &\quad \times \left(\int_{-\infty}^{\infty} \frac{(\min\{|x|,|y|\})^\lambda}{(\max\{|x|,|y|\})^\mu} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| \frac{|y|^{[1-b(\mu-\lambda)](q-1)}}{|x|^{1-a(\mu-\lambda)}} dx \right)^{p-1} \\ &= k^{p-1} |y|^{1-pb(\mu-\lambda)} \int_{-\infty}^{\infty} \frac{(\min\{|x|,|y|\})^\lambda}{(\max\{|x|,|y|\})^\mu} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| \frac{|x|^{[1-a(\mu-\lambda)](p-1)}}{|y|^{1-b(\mu-\lambda)}} f^p(x) dx \\ J &\leq k^{p-1} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \frac{(\min\{|x|,|y|\})^\lambda}{(\max\{|x|,|y|\})^\mu} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| \frac{|x|^{[1-a(\mu-\lambda)](p-1)}}{|y|^{1-b(\mu-\lambda)}} f^p(x) dx \right] dy \\ &= k^{p-1} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \frac{(\min\{|x|,|y|\})^\lambda}{(\max\{|x|,|y|\})^\mu} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| \frac{|x|^{[1-a(\mu-\lambda)](p-1)}}{|y|^{1-b(\mu-\lambda)}} dy \right] f^p(x) dx \\ &= k^p \int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x) dx \end{aligned}$$

3 MAIN RESULTS

Theorem 3.1 If $p > 1$; both functions, $f(x)$ and $g(x)$, are nonnegative measurable functions, and

satisfy and $0 < \int_{-\infty}^{\infty} |y|^{q[1-b(\mu-\lambda)]-1} g^q(x) dx < \infty$ then $0 < \int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x) dx < \infty$

$$I^* := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) g(y) \frac{(\min\{|x|,|y|\})^\lambda}{(\max\{|x|,|y|\})^\mu} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| dx dy < k \left(\int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x) dx \right)^{1/p} \left(\int_{-\infty}^{\infty} |x|^{q[1-b(\mu-\lambda)]-1} g^q(x) dx \right)^{1/q} \tag{3.1}$$

And

$$J = \int_{-\infty}^{\infty} |y|^{pb(\mu-\lambda)-1} \left(\int_{-\infty}^{\infty} f(x) \frac{(\min\{|x|,|y|\})^\lambda}{(\max\{|x|,|y|\})^\mu} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| dx \right)^p dy < k^p \int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x) dx \tag{3.2}$$

Inequalities (3.1) and (3.2) are equivalent, and where the constant factors k and k_p are the best possible.

Proof If there exist a $y \in (-\infty, 0) \cup (0, \infty)$, such that (2.7) takes the form of equality, then there exists constants M and N, such that they are not all zero, and

$$M \frac{|x|^{[1-a(\mu-\lambda)](p-1)}}{|y|^{[1-b(\mu-\lambda)](q-1)}} f^p(x) = N \frac{|y|^{[1-b(\mu-\lambda)](q-1)}}{|x|^{[1-a(\mu-\lambda)](p-1)}} \quad \text{a.e. In } (-\infty, \infty)$$

Hence, there exists a constant C, such that

$$M |x|^{p[1-a(\mu-\lambda)]} f^p(x) = N |y|^{q[1-b(\mu-\lambda)]} = C \quad \text{a.e. In } (-\infty, \infty)$$

It means that M = 0. In fact, if $M \neq 0$, then

$$|x|^{p[1-a(\mu-\lambda)]-1} f^p(x) = \frac{C}{M|x|} \quad \text{a.e. In } (-\infty, \infty)$$

which contradicts the fact that $0 < \int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x) dx < \infty$

In the same way, we claim that N = 0: This is too a contradiction and hence by (2.7), we have (3.2). By Holder's inequality with weight and (3.2), we have,

$$\begin{aligned} I^* &:= \int_{-\infty}^{\infty} \left[|y|^{b(\mu-\lambda)-\frac{1}{p}} \int_{-\infty}^{\infty} f(x) \frac{(\min\{|x|, |y|\})^\lambda}{(\max\{|x|, |y|\})^\mu} \left| \ln \frac{x^2 + xy + y^2}{x^2 + y^2} \right| dx \right] \left[|y|^{-b(\mu-\lambda)-\frac{1}{p}} g(y) \right] dy \\ &\leq J^{1/p} \left(\int_{-\infty}^{\infty} |y|^{q[1-b(\mu-\lambda)]-1} g^q(y) dy \right)^{1/q} \end{aligned} \quad (3.3)$$

Using (3.2), we have (3.1).

Setting

$$g(y) = |y|^{pb(\mu-\lambda)-1} \left(\int_{-\infty}^{\infty} f(x) \frac{(\min\{|x|, |y|\})^\lambda}{(\max\{|x|, |y|\})^\mu} \left| \ln \frac{x^2 + xy + y^2}{x^2 + y^2} \right| dx \right)^{p-1}$$

Then

$$J = \int_{-\infty}^{\infty} |y|^{q[1-b(\mu-\lambda)]-1} g^q(y) dy \quad \text{by (2.7) we have } J < \infty. \text{ if } J = 0 \text{ then (3.2) is proved;}$$

If $0 < J < \infty$ we obtain

$$\begin{aligned} 0 < \int_{-\infty}^{\infty} |y|^{q[1-b(\mu-\lambda)]-1} g^q(y) dy &= J = I^* \\ &< k \left(\int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x) dx \right)^{1/p} \left(\int_{-\infty}^{\infty} |x|^{q[1-b(\mu-\lambda)]-1} g^q(x) dx \right)^{1/q} \\ \left(\int_{-\infty}^{\infty} |x|^{q[1-b(\mu-\lambda)]-1} g^q(x) dx \right)^{1/p} &= J^{1/p} < k \left(\int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x) dx \right)^{1/p} \end{aligned}$$

Inequalities (3.1) and (3.2) are equivalent.

If the constant factor k in (3.1) is not the best possible, then there exists a positive h (with $h < k$), such that

$$\begin{aligned} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) g(y) \frac{(\min\{|x|, |y|\})^\lambda}{(\max\{|x|, |y|\})^\mu} \left| \ln \frac{x^2 + xy + y^2}{x^2 + y^2} \right| dx dy \\ < h \left(\int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x) dx \right)^{1/p} \left(\int_{-\infty}^{\infty} |x|^{q[1-b(\mu-\lambda)]-1} g^q(x) dx \right)^{1/q} \end{aligned}$$

For $\varepsilon > 0$ by (2.5), using lemma 2.3, we have

$$k + o(1) < \varepsilon h \left(\int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} \tilde{f}^p(x) dx \right)^{1/p} \left(\int_{-\infty}^{\infty} |x|^{q[1-b(\mu-\lambda)]-1} \tilde{g}^q(x) dx \right)^{1/q} = h \quad (3.4)$$

Hence we find, $k + o(1) < h$: For $\varepsilon \rightarrow 0^+$ it follows that $k \leq h$ which contradicts the fact that $h < k$.

Hence the constant k in (3.1) is the best possible.

Thus we complete the prove of the theorem.

Theorem 3.2 If $1 > p > 0$; both functions, $f(x)$ and $g(x)$, are nonnegative measurable functions, and satisfy

$$0 < \int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x) dx < \infty \text{ and } 0 < \int_{-\infty}^{\infty} |x|^{q[1-b(\mu-\lambda)]-1} g^q(x) dx < \infty$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)g(y) \frac{(\min\{|x|,|y|\})^\lambda}{(\max\{|x|,|y|\})^\mu} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| dx dy$$

$$> k \left(\int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x) dx \right)^{1/p} \left(\int_{-\infty}^{\infty} |x|^{q[1-b(\mu-\lambda)]-1} g^q(x) dx \right)^{1/q}$$
(3.5)

$$J > k^p \int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x) dx$$
(3.6)

And

$$L := \int_{-\infty}^{\infty} |x|^{qa(\mu-\lambda)-1} \left(\int_{-\infty}^{\infty} \frac{(\min\{|x|,|y|\})^\lambda}{(\max\{|x|,|y|\})^\mu} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| g(y) dy \right)^q dx$$

$$< k^q \int_{-\infty}^{\infty} |y|^{q[1-b(\mu-\lambda)]-1} g^q(y) dy$$
(3.7)

Inequalities (3.5),(3.6)and (3.7) are equivalent,and where the constant factors k, k^p and k^q are the best possible

Proof By the reverse Holder's inequality and the same way, we can obtain the reverseforms of (2.7)and (3.3).And then we deduce the (3.5),by the some way,we obtain (3.6).

Setting $g(y)$ as the theorem 1,we obtain $J > 0$,if $J = \infty$,then we have (3.6),if $0 < J < \infty$,by (3.5)

$$\int_{-\infty}^{\infty} |y|^{q[1-b(\mu-\lambda)]-1} g^q(y) dy = J = I^*$$

$$> k \left(\int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x) dx \right)^{1/p} \left(\int_{-\infty}^{\infty} |x|^{q[1-b(\mu-\lambda)]-1} g^q(x) dx \right)^{1/q}$$

and we have (3.6),and inequalities (3.5)and (3.6) are equivalent.

Setting

$$[g(x)]_n = \begin{cases} \frac{1}{n} & \text{if } g(x) < \frac{1}{n}, \\ g(x) & \text{if } \frac{1}{n} \leq g(x) \leq n, \\ n & \text{if } g(x) > n \end{cases}$$

$E_n = \left[-n, -\frac{1}{n} \right] \cup \left[\frac{1}{n}, n \right]$ then $\exists n_0 \in \mathbb{N}$, such that $n > n_0$; we have

$$\int_{E_n} |y|^{q[1-b(\mu-\lambda)]-1} g^q(y) dy > 0,$$

And

$$[f(x)]_n = |x|^{qa(\mu-\lambda)-1} \left(\int_{E_n} [g(x)]_n \frac{(\min\{|x|,|y|\})^\lambda}{(\max\{|x|,|y|\})^\mu} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| dy \right)^{q-1}$$

$$[L(x)]_n = \int_{E_n} |x|^{qa(\mu-\lambda)-1} \left(\int_{E_n} [g(x)]_n \frac{(\min\{|x|,|y|\})^\lambda}{(\max\{|x|,|y|\})^\mu} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| dy \right)^q dx$$

Then $\exists n_0 \in \mathbb{N}$, such that $n > n_0$; we have

$$\int_{E_n} |y|^{q[1-b(\mu-\lambda)]-1} g^q(y) dy < \infty,$$

In particular, from (4.1) we get the following particular cases:

1) If $\lambda + \mu = 4$; then $k = 2 \left[\psi(2) - \psi\left(\frac{2}{3}\right) \right] + \frac{2}{3} \left[\psi\left(\frac{1}{3}\right) - \psi(1) \right] = 2 + \frac{3 \ln 3}{2} - \frac{\sqrt{3}\pi}{2}$, we have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)g(y) \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{4-\lambda}} \left| \ln \frac{x^2 + xy + y^2}{x^2 + y^2} \right| dx dy$$

$$< \left(2 + \frac{3 \ln 3}{2} - \frac{\sqrt{3}\pi}{2} \right) \left(\int_{-\infty}^{\infty} |x|^{p(\lambda-1)-1} f^p(x) dx \right)^{1/p} \left(\int_{-\infty}^{\infty} |x|^{q(\lambda-1)-1} g^q(x) dx \right)^{1/q} \quad (4.2)$$

2) If $\lambda + \mu = 3$; then $k = \frac{8}{3} \left[\psi\left(\frac{3}{2}\right) - \psi\left(\frac{1}{2}\right) \right] + \frac{4}{3} \left[\psi\left(\frac{1}{4}\right) - \psi\left(\frac{3}{4}\right) \right] = \frac{4}{3}(4 - \pi)$, we have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)g(y) \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{4-\lambda}} \left| \ln \frac{x^2 + xy + y^2}{x^2 + y^2} \right| dx dy$$

$$< \left(2 + \frac{3 \ln 3}{2} - \frac{\sqrt{3}\pi}{2} \right) \left(\int_{-\infty}^{\infty} |x|^{p(\lambda-1)-1} f^p(x) dx \right)^{1/p} \left(\int_{-\infty}^{\infty} |x|^{q(\lambda-1)-1} g^q(x) dx \right)^{1/q} \quad (4.3)$$

3) If $\lambda + \mu = 2$; then $k = 4 \left[\psi(1) - \psi\left(\frac{1}{3}\right) \right] + 2 \left[\psi\left(\frac{1}{6}\right) - \psi\left(\frac{1}{2}\right) \right] = 6 \ln 3 + \frac{2\pi}{\sqrt{3}} - \sqrt{2}\pi$, we have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)g(y) \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{2-\lambda}} \left| \ln \frac{x^2 + xy + y^2}{x^2 + y^2} \right| dx dy$$

$$< \left(6 \ln 3 + \frac{2\pi}{\sqrt{3}} - \sqrt{2}\pi \right) \left(\int_{-\infty}^{\infty} |x|^{p\lambda-1} f^p(x) dx \right)^{1/p} \left(\int_{-\infty}^{\infty} |x|^{q\lambda-1} g^q(x) dx \right)^{1/q} \quad (4.4)$$

B) Let $\lambda = \mu$ in (3.1), then we have a Integral Inequality with the homogeneous kernel of 0 degree form as follows:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)g(y) \left(\frac{\min\{|x|,|y|\}}{\max\{|x|,|y|\}} \right)^{\lambda} \left| \ln \frac{x^2 + xy + y^2}{x^2 + y^2} \right| dx dy$$

$$< k(\lambda) \left(\int_{-\infty}^{\infty} |x|^{p-1} f^p(x) dx \right)^{1/p} \left(\int_{-\infty}^{\infty} |x|^{q-1} g^q(x) dx \right)^{1/q} \quad (4.5)$$

$$k(\lambda) = \frac{4}{\lambda} \left[\psi(\lambda) - \psi\left(\frac{\lambda}{3}\right) \right] + \frac{2}{\lambda} \left[\psi\left(\frac{\lambda}{6}\right) - \psi\left(\frac{\lambda}{2}\right) \right]$$

There

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