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A New Hilbert-type Integral Inequality with the Homogeneous Kernel of Real Degree Form and the Integral in Whole Plane

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Abstract: In this paper,we build a new Hilbert's inequality with the homogeneous kernel of real order and the integral in whole plane. The equivalent inequality is considered. The best constant factor is calculated using Ψ f unction.

1 INTRODUCTION

If $f(x), g(x) \geq 0$, such that $0<\int_0^\infty f^2(x)\mathrm{d}x<\infty \quad 0<\int_0^\infty g^2(x)\mathrm{d}x<\infty$; then [1]

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y<\pi\left(\int_0^\infty f^2(x)\mathrm{d}x\right)^{1/2}\left(\int_0^\infty g^q(x)\mathrm{d}x\right)^{1/2}. \tag{1.1}$$

where the constant factor π is the best possible. Inequality (1.1) is well-known as Hilbert's integral inequality,which has been extended by Hardy-Riesz as [2]:

If $p>1, 1/p+1/q=1$ $f(x), g(x)\geq 0$; such that $0<\int_0^\infty f^p(x)\mathrm{d}x<\infty$, and $0<\int_0^\infty g^q(x)\mathrm{d}x<\infty$;

then we have the following Hardy-Hilbert's integral inequality:

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y<\frac{\pi}{\sin(\pi/p)}\left(\int_0^\infty f^p(x)\mathrm{d}x\right)^{1/p}\left(\int_0^\infty g^q(x)\mathrm{d}x\right)^{1/q}; \tag{1.2}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ also is the best possible.

Hilbert's inequality attracts some attention in recent years.Actually, inequalities (1.1)and(1.2) have many generalizations and variations. (1.1) has been strengthened by Yang and others (including double series inequalities). [3,4,6-21].

In 2008, Zitian Xie and Zheng Zeng gave a new Hilbert-type Inequality [4] as follows

If $a>0, b>0, c>0$; $p>1, 1/p+1/q=1$ $f(x), g(x)\geq 0$; such that

$0<\int_0^\infty f^{-1-p/2}(x)\mathrm{d}x<\infty$ and $0<\int_0^\infty g^{-1-q/2}(x)\mathrm{d}x<\infty$ then

$$<K\left(\int_0^\infty f^{-1-p/2}(x)\mathrm{d}x\right)^{1/p}\left(\int_0^\infty g^{-1-q/2}(x)\mathrm{d}x\right)^{1/q}; \tag{1.3}$$

where the constant factor $K=\frac{\pi}{(a+b)(a+c)(c+b)}$ is the best possible.

In 2010,Jianhua Xhong and Bicheng Yang gave a new Hilbert-type Inequality [5] as follows :

Assume that

$\lambda, p>0 (p\neq 1), r>1,\ 1/p+1/q=1, 1/r+1/s=1, \phi(x)=x^{p(1-\frac{\lambda}{r})-1}, \varphi(x)=x^{q(1-\frac{\lambda}{s})-1}, x\in(0,\infty)$,

$K=\Gamma(\beta+1)\sum_{k=0}^{\infty}(-1)^k\binom{\alpha-\lambda}{k}\left[\frac{1}{(k+\lambda/r)^{\beta+1}}+\frac{1}{(k+\lambda/s)^{\beta+1}}\right]$,and $f, g\geq 0$,

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$0<\|f\|_{p,\phi}:=\left\{\int_0^\infty x^{p(1-\lambda/r)-1}f^p(x)\mathrm{d}x\right\}^{1/p}<\infty,\ 0<\|g\|_{q,\varphi}<\infty$ then

(1) for $p>1$ we have the following equivalent inequalities:

$$\int_0^\infty\int_0^\infty\frac{|\ln(x/y)|^\beta f(x)g(y)}{|x-y|(\max\{|x|,|y|\})^\alpha}\mathrm{d}x\mathrm{d}y<K\|f\|_{p,\phi}\|g\|_{q,\varphi}$$

(2)For $0<p<1$ the reverse of (1.5) with the best constant factor K .

$$\int_0^\infty\int_0^\infty\frac{|\ln(x/y)|^\beta f(x)g(y)}{|x-y|(\max\{|x|,|y|\})^\alpha}\mathrm{d}x\mathrm{d}y>K\|f\|_{p,\phi}\|g\|_{q,\varphi}$$

The main purpose of this paper is to build a new Hilbert-type inequality with the homogeneous kernel of real order and the integral in whole plane, by estimating the weight function using Ψ function.The equivalent inequality is considered

We knew that (in this paper,γ is the Euler's constant.)

$$\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}=-\gamma+\sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+z}\right),\psi(1)=-\gamma,\psi\left(\frac{1}{2}\right)=-\gamma-2\ln 2.$$

Recent XIE Zitin and ZHOU Qinghua prove that the expression of the Ψ -function admits a finite expression in elementary function for rational number z,and prove that[6]

$$\psi\left(\frac{a}{b}\right)=\Gamma'\left(\frac{a}{b}\right)/\Gamma\left(\frac{a}{b}\right)=-\ln b-\gamma-\ln 2-\frac{\pi}{2}\cot\frac{a\pi}{b}+\sum_{k=1}^{b-1}\cos\frac{2ka\pi}{b}\ln\sin\frac{k\pi}{b}$$

and have

$$\psi\left(\frac{1}{3}\right)=-\gamma-\ln 3-\ln 2-\frac{\pi}{2}\cot\frac{\pi}{3}+\cos\frac{2\pi}{3}\ln\sin\frac{\pi}{3}+\cos\frac{4\pi}{3}\ln\sin\frac{2\pi}{3}$$

$$=-\gamma-\frac{3}{2}\ln 3-\frac{\pi}{2\sqrt{3}};$$

$$\psi\left(\frac{2}{3}\right)=-\gamma-\frac{3}{2}\ln 3+\frac{\pi}{2\sqrt{3}};$$

$$\psi\left(\frac{1}{4}\right)=-\gamma-3\ln 2-\frac{\pi}{2};$$

$$\psi\left(\frac{3}{4}\right)=-\gamma-3\ln 2+\frac{\pi}{2};$$

$$\psi\left(\frac{1}{6}\right)=-\gamma-2\ln 2-\frac{\sqrt{2}\pi}{2};$$

$$\psi\left(\frac{5}{6}\right)=-\gamma-2\ln 2+\frac{\sqrt{2}\pi}{2};$$

$$\psi\left(\frac{1}{5}\right)=-\gamma-\frac{5\ln 5}{4}-\frac{\sqrt{5}}{2}\ln(\sqrt{5}-1)-\frac{\sqrt{5}}{2}\ln 2-\frac{\pi}{40}(5+3\sqrt{5})\sqrt{10-2\sqrt{5}};$$

$$\psi\left(\frac{4}{5}\right)=-\gamma-\frac{5\ln 5}{4}-\frac{\sqrt{5}}{2}\ln(\sqrt{5}-1)-\frac{\sqrt{5}}{2}\ln 2+\frac{\pi}{40}(5+3\sqrt{5})\sqrt{10-2\sqrt{5}}$$

In the following, we always suppose that:

$1/p+1/q=1, p>1, \min\{a\lambda+b\mu, a\mu+b\lambda\}>-1, a\mu+b\lambda\neq 0, a\lambda+b\mu\neq 0, \mu>0, \lambda>0.$

$a+b=1.$

2 SOME LEMMAS

We start by introducing some Lemmas.

Lemma 2.1 If $s>0, r\neq 0, r>-s$, then

$$1)\ \int_0^1 x^{r-1}\ln(1-x^s)dx = -\frac{1}{r}[\gamma+\psi(\frac{r+s}{s})]$$

$$2)\ \int_0^1 x^{r-1}\ln(1+x^s)dx = \frac{1}{r}\ln 2 - \frac{1}{2r}[\psi(\frac{r+2s}{2s})-\psi(\frac{r+s}{2s})] \tag{2.1}$$

Proof. we obtain,'

$$1)\ -\int_0^1 x^{r-1}\ln(1-x^s)dx = \int_0^1 x^{r-1}\sum_{l=1}^{\infty}\frac{x^{ls}}{l}dx$$

$$=\sum_{l=1}^{\infty}\int_0^1\frac{x^{ls+r-1}}{l}dx = \sum_{l=1}^{\infty}\frac{1}{l(r+ls)}$$

$$=\frac{1}{r}\sum_{l=1}^{\infty}(\frac{1}{l}-\frac{1}{l+r/s})$$

$$=\frac{1}{r}[\gamma+\psi(\frac{r+s}{s})]$$

$$2)\int_0^1 x^{r-1}\ln(1+x^s)dx = \int_0^1 x^{r-1}\sum_{l=1}^{\infty}(-1)^{l-1}\frac{x^{ls}}{l}dx$$

$$=\sum_{l=1}^{\infty}\int_0^1(-1)^{l-1}\frac{x^{ls+r-1}}{l}dx = \sum_{l=1}^{\infty}\frac{(-1)^{l-1}}{l(r+ls)}$$

$$=\lim_{N\to\infty}\left\{\frac{1}{r}\sum_{l=1}^{2N+1}(-1)^{l-1}\frac{1}{l}-\frac{1}{2r}\left[\sum_{n=0}^{N}\left(\frac{1}{n+1}-\frac{1}{n+\frac{r+2s}{2s}}\right)-\sum_{n=0}^{N}\left(\frac{1}{n+1}-\frac{1}{n+\frac{r+s}{2s}}\right)\right]\right\}$$

$$=\frac{1}{r}\ln 2-\frac{1}{2r}[\psi(\frac{r+2s}{2s})-\psi(\frac{r+s}{2s})]$$

The lemma is proved.

In particular,if $r>-1$, $r\neq 0$, then

$$\int_0^1 u^{r-1}\ln\frac{u^2+u+1}{u^2-u+1}du$$

$$=\int_0^1 u^{r-1}\ln(1-u^3)du-\int_0^1 u^{r-1}\ln(1-u)du-\left[\int_0^1 u^{r-1}\ln(1+u^3)du-\int_0^1 u^{r-1}\ln(1+u)du\right]$$

$$=\frac{1}{r}\left[\psi(r+1)-\psi(\frac{r+3}{3})\right]+\frac{1}{2r}\left[\psi(\frac{r+6}{6})-\psi(\frac{r+3}{6})-\psi(\frac{r+2}{2})+\psi(\frac{r+1}{2})\right]$$

$$(\text{using } \psi(x+1)=\psi(x)+\frac{1}{x})$$

$$=\frac{1}{r}\left[\psi(r)-\psi(\frac{r}{3})\right]+\frac{1}{2r}\left[\psi(\frac{r}{6})-\psi(\frac{r+3}{6})-\psi(\frac{r}{2})+\psi(\frac{r+1}{2})\right]$$

$$=\frac{1}{r}\left[\psi(r)-\psi(\frac{r}{3})\right]+\frac{1}{2r}\left[2\psi(\frac{r}{6})-\psi(\frac{r}{6})-\psi(\frac{r+3}{6})-2\psi(\frac{r}{2})+\psi(\frac{r}{2})+\psi(\frac{r+1}{2})\right]$$

$$(\text{using } \psi(x+\frac{1}{2})+\psi(x)=2\psi(2x)-\psi(\frac{r+1}{2})$$

$$=\frac{2}{r}\left[\psi(r)-\psi(\frac{r}{3})\right]+\frac{1}{r}\left[(\frac{r}{6})-\psi(\frac{r}{2})\right] \tag{2.2}$$

Lemma 2.2 Define the weight functions as follow:

$$w(x):=\int_{-\infty}^{\infty}\frac{|x|^{a(\mu-\lambda)}}{|y|^{1-b(\mu-\lambda)}}\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}\right|dy,$$

$$\tilde{w}(y):=\int_{-\infty}^{\infty}\frac{|y|^{b(\mu-\lambda)}}{|x|^{1-a(\mu-\lambda)}}\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}\right|dx,$$

Then

$$\begin{aligned}
&w(x)=\tilde{w}(y)\\
&=\frac{2}{a\lambda+b\mu}\left[\psi(a\lambda+b\mu)-\psi(\frac{a\lambda+b\mu}{3})\right]+\frac{1}{a\lambda+b\mu}\left[\psi(\frac{a\lambda+b\mu}{6})-\psi(\frac{a\lambda+b\mu}{2})\right]\\
&+\frac{2}{a\mu+b\lambda}\left[\psi(a\mu+b\lambda)-\psi(\frac{a\mu+b\lambda}{3})\right]+\frac{1}{a\lambda+b\mu}\left[\psi(\frac{a\mu+b\lambda}{6})-\psi(\frac{a\mu+b\lambda}{2})\right].\\
&:=k
\end{aligned}\tag{2.3}$$

Proof We only prove that $w(x)=k$ for $x\in(-\infty,0)$

Using lemma 2.1,setting y = ux ,and y=-ux

$$\begin{aligned}
w(x):=&\int_{-\infty}^{0}\frac{(-x)^{a(\mu-\lambda)}}{(-y)^{1-b(\mu-\lambda)}}\frac{(\min\{(-x),(-y)\})^{\lambda}}{(\max\{(-x),(-y)\})^{\mu}}\ln\frac{x^2+xy+y^2}{x^2+y^2}dy\\
&+\int_{0}^{\infty}\frac{(-x)^{a(\mu-\lambda)}}{y^{1-b(\mu-\lambda)}}\frac{(\min\{(-x),y\})^{\lambda}}{(\max\{(-x),y\})^{\mu}}\ln\frac{x^2+y^2}{x^2+xy+y^2}dy\\
:=&w_1+w_2
\end{aligned}$$

Then

$$\begin{aligned}
w_1&=\int_0^1 u^{-1+a\lambda+b\mu}\ln\frac{u^2+u+1}{u^2+1}du+\int_1^{\infty}u^{-1-a\lambda-b\mu}\ln\frac{u^2+u+1}{u^2+1}du\\
&=\int_0^1 u^{-1+a\lambda+b\mu}\ln\frac{u^2+u+1}{u^2+1}du+\int_0^1 u^{-1+a\mu+b\lambda}\ln\frac{u^2+u+1}{u^2+1}du\\
w_2&=\int_0^1 u^{-1+a\lambda+b\mu}\ln\frac{u^2+1}{u^2-u+1}du+\int_1^{\infty}u^{-1-a\lambda-b\mu}\ln\frac{u^2+1}{u^2-u+1}du\\
&=\int_0^1 u^{-1+a\lambda+b\mu}\ln\frac{u^2+1}{u^2-u+1}du+\int_0^1 u^{-1+a\mu+b\lambda}\ln\frac{u^2+1}{u^2-u+1}du
\end{aligned}$$

And

$$\begin{aligned}
&w=w_1+w_2=\int_0^1 u^{-1+a\lambda+b\mu}\ln\frac{u^2+u+1}{u^2-u+1}du+\int_0^1 u^{-1+a\mu+b\lambda}\ln\frac{u^2+u+1}{u^2-u+1}du\\
&=\frac{2}{a\lambda+b\mu}\left[\psi(a\lambda+b\mu)-\psi(\frac{a\lambda+b\mu}{3})\right]+\frac{1}{a\lambda+b\mu}\left[\psi(\frac{a\lambda+b\mu}{6})-\psi(\frac{a\lambda+b\mu}{2})\right]\\
&+\frac{2}{a\mu+b\lambda}\left[\psi(a\mu+b\lambda)-\psi(\frac{a\mu+b\lambda}{3})\right]+\frac{1}{a\lambda+b\mu}\left[\psi(\frac{a\mu+b\lambda}{6})-\psi(\frac{a\mu+b\lambda}{2})\right].\\
&=k
\end{aligned}$$

Similarly,setting $x=y/u$,and $x=-y/u$

$$\begin{aligned}
\tilde{w}(y)=&\int_{-\infty}^{0}\frac{(-y)^{b(\mu-\lambda)}}{(-x)^{1-a(\mu-\lambda)}}\frac{(\min\{(-x),(-y)\})^{\lambda}}{(\max\{(-x),(-y)\})^{\mu}}\ln\frac{x^2+xy+y^2}{x^2+y^2}dx\\
&+\int_{0}^{\infty}\frac{y^{b(\mu-\lambda)}}{(-x)^{1-a(\mu-\lambda)}}\frac{(\min\{(-x),y\})^{\lambda}}{(\max\{(-x),y\})^{\mu}}\ln\frac{x^2+y^2}{x^2+xy+y^2}dx
\end{aligned}$$

$$= \int_{-\infty}^{0} \frac{(-y)^{b(\mu-\lambda)}}{(-y/u)^{1-a(\mu-\lambda)}} \frac{(\min\{(-y/u),(-y)\})^{\lambda}}{(\max\{(-y/u),(-y)\})^{\mu}} \ln \frac{(y/u)^2+(y/u)y+y^2}{(y/u)^2+y^2} d(y/u)$$

$$+\int_{0}^{\infty} \frac{y^{b(\mu-\lambda)}}{(y/u)^{1-a(\mu-\lambda)}} \frac{(\min\{(y/u),y\})^{\lambda}}{(\max\{(y/u),y\})^{\mu}} \ln \frac{(-y/u)^2+y^2}{(-y/u)^2+(-y/u)y+y^2} d(-y/u)$$

$$= \int_{0}^{\infty} u^{-1+b(\mu-\lambda)} \frac{(\min\{1,u\})^{\lambda}}{(\max\{1,u\})^{\mu}} \ln \frac{u^2+u+1}{u^2+1} du + \int_{0}^{\infty} u^{-1+b(\mu-\lambda)} \frac{(\min\{1,u\})^{\lambda}}{(\max\{1,u\})^{\mu}} \ln \frac{u^2+1}{u^2-u+1} du$$

$$= w_1 + w_2 = k \tag{2.4}$$

and the lemma is proved.

Lemma 2.3 For ε > 0; and $\min\{a\mu+b\lambda-2\varepsilon/q, a\lambda+b\mu-2\varepsilon/q\}>-1$, define both functions, $\tilde{f},\tilde{g}$ as follow:

$$\tilde{f}(x)=\begin{cases} x^{a(\mu-\lambda)-1-2\varepsilon/p}, & \text{if} \quad x\in(1,\infty),\\ 0, & \text{if} \quad x\in[-1,1],\\ (-x)^{a(\mu-\lambda)-1-2\varepsilon/p}, & \text{if} \quad x\in(-\infty,-1),\end{cases}$$

$$\tilde{g}(x)=\begin{cases} x^{b(\mu-\lambda)-1-2\varepsilon/q}, & \text{if} \quad x\in(1,\infty),\\ 0, & \text{if} \quad x\in[-1,1],\\ (-x)^{b(\mu-\lambda)-1-2\varepsilon/q}, & \text{if} \quad x\in(-\infty,-1),\end{cases}$$

Then

$$I(\varepsilon):=\varepsilon\left\{\int_{-\infty}^{\infty}|x|^{p[1-a(\mu-\lambda)]-1}\tilde{f}^{p}(x)dx\right\}^{1/p}\left\{\int_{-\infty}^{\infty}|x|^{q[1-b(\mu-\lambda)]-1}\tilde{g}^{q}(x)dx\right\}^{1/q}=1; \tag{2.5}$$

$$\tilde{I}(\varepsilon):=\varepsilon\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{f}(x)\tilde{g}(y)\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}\right|dxdy \to k(\varepsilon\to 0^{+}) \tag{2.6}$$

Proof Easily

$$I(\varepsilon)=\varepsilon\left\{2\int_{1}^{\infty}x^{-1}x^{-2\varepsilon}dx\right\}^{1/p}\left\{\int_{1}^{\infty}2\int_{1}^{\infty}x^{-1}x^{-2\varepsilon}dxdx\right\}^{1/q}=1$$

Let y-Y, using $\tilde{f}(-x)=\tilde{f}(x)$ $\tilde{g}(-x)=\tilde{g}(x)$ and

$$\tilde{f}(-x)\int_{-\infty}^{\infty}\tilde{g}(y)\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}}\left|\ln\frac{x^2-xy+y^2}{x^2+y^2}\right|dy$$

$$=\tilde{f}(x)\int_{-\infty}^{\infty}\tilde{g}(Y)\frac{(\min\{|x|,|Y|\})^{\lambda}}{(\max\{|x|,|Y|\})^{\mu}}\left|\ln\frac{x^2+xY+Y^2}{x^2+Y^2}\right|dY$$

we have that $\tilde{f}(x)\int_{-\infty}^{\infty}\tilde{g}(y)\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}\right|dy$ is an even function on x, ,then

$$\tilde{I}(\varepsilon):=2\varepsilon\int_{-\infty}^{\infty}\tilde{f}(x)\left(\int_{-\infty}^{\infty}\tilde{g}(y)\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}\right|dy\right)dx$$

$$=2\varepsilon\left[\int_{1}^{\infty}x^{a(\mu-\lambda)-1-2\varepsilon/p}\left(\int_{-\infty}^{-1}(-y)^{b(\mu-\lambda)-1-2\varepsilon/q}\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}\right|dy\right)dx\right.$$

$$\left.+\int_{1}^{\infty}x^{a(\mu-\lambda)-1-2\varepsilon/p}\left(\int_{1}^{\infty}y^{b(\mu-\lambda)-1-2\varepsilon/q}\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}\right|dy\right)dx\right]$$

$$:=I_1+I_2$$

Setting y = tx then

$$I_1 = 2\varepsilon\int_1^{\infty} x^{a(\mu-\lambda)-1-2\varepsilon/p}\left(\int_1^{\infty} y^{b(\mu-\lambda)-1-2\varepsilon/q}\frac{(\min\{x,y\})^{\lambda}}{(\max\{x,y\})^{\mu}}\ln\frac{x^2+y^2}{x^2-xy+y^2}dy\right)dx$$

$$= 2\varepsilon\int_1^{\infty} x^{-1-2\varepsilon}\left(\int_{1/x}^{\infty} t^{b(\mu-\lambda)-1-2\varepsilon/q}\frac{(\min\{1,t\})^{\lambda}}{(\max\{1,t\})^{\mu}}\ln\frac{t^2+1^2}{t^2-t+1^2}dt\right)dx$$

$$= 2\varepsilon\int_1^{\infty} x^{-1-2\varepsilon}\left(\int_{1}^{\infty} t^{b(\mu-\lambda)-1-2\varepsilon/q}\frac{(\min\{1,t\})^{\lambda}}{(\max\{1,t\})^{\mu}}\ln\frac{t^2+1^2}{t^2-t+1^2}dt\right)dx$$

$$+2\varepsilon\int_1^{\infty} x^{-1-2\varepsilon}\left(\int_{1/x}^{1} t^{b(\mu-\lambda)-1-2\varepsilon/q}\frac{(\min\{1,t\})^{\lambda}}{(\max\{1,t\})^{\mu}}\ln\frac{t^2+1^2}{t^2-t+1^2}dt\right)dx$$

$$= \int_1^{\infty} t^{b(\mu-\lambda)-1-2\varepsilon/q}\frac{(\min\{1,t\})^{\lambda}}{(\max\{1,t\})^{\mu}}\ln\frac{t^2+1^2}{t^2-t+1^2}dt$$

$$+2\varepsilon\int_0^{1} t^{b(\mu-\lambda)-1-2\varepsilon/q}\frac{(\min\{1,t\})^{\lambda}}{(\max\{1,t\})^{\mu}}\ln\frac{t^2+1^2}{t^2-t+1^2}\left(\int_{1/t}^{\infty} x^{-1-2\varepsilon}dx\right)dt$$

$$= \int_1^{\infty} t^{b(\mu-\lambda)-1-2\varepsilon/q}\frac{(\min\{1,t\})^{\lambda}}{(\max\{1,t\})^{\mu}}\ln\frac{t^2+1^2}{t^2-t+1^2}dt$$

$$+\int_0^{1} t^{b(\mu-\lambda)-1-2\varepsilon/p}\frac{(\min\{1,t\})^{\lambda}}{(\max\{1,t\})^{\mu}}\ln\frac{t^2+1^2}{t^2-t+1^2}dt$$

$$= \int_0^{\infty} t^{b(\mu-\lambda)-1-2\varepsilon/q}\frac{(\min\{1,t\})^{\lambda}}{(\max\{1,t\})^{\mu}}\ln\frac{t^2+1^2}{t^2-t+1^2}dt$$

$$+\int_0^{1} (t^{2\varepsilon/p}-t^{-2\varepsilon/q})t^{b(\mu-\lambda)-1}\frac{(\min\{1,t\})^{\lambda}}{(\max\{1,t\})^{\mu}}\ln\frac{t^2+1^2}{t^2-t+1^2}dt$$

$$= \int_0^{\infty} t^{b(\mu-\lambda)-1-2\varepsilon/q}\frac{(\min\{1,t\})^{\lambda}}{(\max\{1,t\})^{\mu}}\ln\frac{t^2+1^2}{t^2-t+1^2}dt+\eta_1(\varepsilon),(\lim_{\varepsilon\to 0^+}\eta_1(\varepsilon)=0)$$

Similarly

$$I_2 = 2\varepsilon\int_1^{\infty} x^{a(\mu-\lambda)-1-2\varepsilon/p}\left(\int_1^{\infty} y^{b(\mu-\lambda)-1-2\varepsilon/q}\frac{(\min\{x,y\})^{\lambda}}{(\max\{x,y\})^{\mu}}\ln\frac{x^2+xy+y^2}{x^2+y^2}dy\right)dx$$

$$= \int_0^{\infty} t^{b(\mu-\lambda)-1-2\varepsilon/q}\frac{(\min\{1,t\})^{\lambda}}{(\max\{1,t\})^{\mu}}\ln\frac{t^2+t+1^2}{t^2+1^2}dt+\eta_2(\varepsilon),(\lim_{\varepsilon\to 0^+}\eta_2(\varepsilon)=0)$$

by lemma 2.2,we have

$$\tilde{I}(\varepsilon) = I_1 + I_2$$

$$= \frac{2}{a\lambda+b\mu-2\varepsilon/q}\left[\psi(a\lambda+b\mu-2\varepsilon/q)-\psi(\frac{a\lambda+b\mu-2\varepsilon/q}{3})\right]$$

$$+\frac{1}{a\lambda+b\mu-2\varepsilon/q}\left[\psi(\frac{a\lambda+b\mu-2\varepsilon/q}{6})-\psi(\frac{a\lambda+b\mu-2\varepsilon/q}{2})\right]$$

$$+\frac{2}{a\mu+b\lambda-2\varepsilon/q}\left[\psi(a\mu+b\lambda-2\varepsilon/q)-\psi(\frac{a\mu+b\lambda-2\varepsilon/q}{3})\right]$$

$$+\frac{1}{a\mu+b\lambda-2\varepsilon/q}\left[\psi(\frac{a\mu+b\lambda-2\varepsilon/q}{6})-\psi(\frac{a\mu+b\lambda-2\varepsilon/q}{2})\right]+\eta_1(\varepsilon)+\eta_2(\varepsilon)$$

we know that $\psi(x)$ is a continuous function,then $\lim_{\varepsilon\to 0^+}\tilde{I}(\varepsilon)=k$

The lemma is proved.

Lemma 2.4 If f(x) is a nonnegative measurable function, and $0 < \int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x)dx < \infty$

Then

$$J := \int_{-\infty}^{\infty} |y|^{pb(\mu-\lambda)-1} \left(\int_{-\infty}^{\infty} f(x) \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| dx \right)^p dy$$

$$\leq k^p \int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x)dx \tag{2.7}$$

Proof By lemma 2.2,we find

$$\left(\int_{-\infty}^{\infty} f(x) \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| dx \right)^p$$

$$= \left(\int_{-\infty}^{\infty} \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| \left(\frac{|x|^{[1-a(\mu-\lambda)]/q}}{|y|^{[1-b(\mu-\lambda)]/p}} f(x) \right) \left(\frac{|y|^{[1-b(\mu-\lambda)]/p}}{|x|^{[1-a(\mu-\lambda)]/q}} \right) dx \right)^p$$

$$\leq \int_{-\infty}^{\infty} \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| \frac{|x|^{[1-a(\mu-\lambda)](p-1)}}{|y|^{1-b(\mu-\lambda)}} f^p(x)dx$$

$$\times \left(\int_{-\infty}^{\infty} \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| \frac{|y|^{[1-b(\mu-\lambda)](q-1)}}{|x|^{1-a(\mu-\lambda)}} dx \right)^{p-1}$$

$$= k^{p-1} |y|^{1-pb(\mu-\lambda)} \int_{-\infty}^{\infty} \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| \frac{|x|^{[1-a(\mu-\lambda)](p-1)}}{|y|^{1-b(\mu-\lambda)}} f^p(x)dx$$

$$J \leq k^{p-1} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| \frac{|x|^{[1-a(\mu-\lambda)](p-1)}}{|y|^{1-b(\mu-\lambda)}} f^p(x)dx \right] dy$$

$$= k^{p-1} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| \frac{|x|^{[1-a(\mu-\lambda)](p-1)}}{|y|^{1-b(\mu-\lambda)}} dy \right] f^p(x)dx$$

$$= k^p \int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x)dx$$

3 MAIN RESULTS

Theorem 3.1 If p > 1; both functions, f(x) and g(x) ,are nonnegative measurable functions, and satisfy and $0 < \int_{-\infty}^{\infty} |y|^{q[1-b(\mu-\lambda)]-1} g^q(x)dx < \infty$ then $0 < \int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x)dx < \infty$

$$I^* := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)g(y) \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| dxdy$$

$$< k \left(\int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x)dx \right)^{1/p} \left(\int_{-\infty}^{\infty} |x|^{q[1-b(\mu-\lambda)]-1} g^q(x)dx \right)^{1/q} \tag{3.1}$$

And

$$J = \int_{-\infty}^{\infty} |y|^{pb(\mu-\lambda)-1} \left(\int_{-\infty}^{\infty} f(x) \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| dx \right)^p dy$$

$$< k^p \int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x)dx \tag{3.2}$$

Inequalities (3.1)and (3.2) are equivalent,and where the constant factors k and kp arethe best possible.

Proof If there exist a $y \in (-\infty, 0) \cup (0, \infty)$,such that (2.7) takes the form of equality, then there exists constants M and N, such that they are not all zero, and

$$M \frac{|x|^{[1-a(\mu-\lambda)](p-1)}}{|y|^{1-b(\mu-\lambda)}} f^p(x) = N \frac{|y|^{[1-b(\mu-\lambda)](q-1)}}{|x|^{1-a(\mu-\lambda)}} \text{ a.e. In } (-\infty, \infty)$$

Hence, there exists a constant C, such that

$$M |x|^{p[1-a(\mu-\lambda)]} f^p(x) = N |y|^{q[1-b(\mu-\lambda)]} = C \text{ a.e. In} (-\infty, \infty)$$

It means that M = 0. In fact, if $M \neq 0$, then

$$|x|^{p[1-a(\mu-\lambda)]-1} f^p(x) = \frac{C}{M|x|} \text{ a.e. In} (-\infty, \infty)$$

which contradicts the fact that $0 < \int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x)dx < \infty$ In the same way, we claim that N = 0: This is too a contradiction and hence by (2.7), we have (3.2). By Holder's inequality with weight and (3.2), we have,

$$I^* := \int_{-\infty}^{\infty} \left[|y|^{b(\mu-\lambda)-\frac{1}{p}} \int_{-\infty}^{\infty} f(x) \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| dx \right] \left[|y|^{-b(\mu-\lambda)+\frac{1}{p}} g(y) \right] dy$$

$$\leq J^{1/p} \left(\int_{-\infty}^{\infty} |y|^{q[1-b(\mu-\lambda)]-1} g^q(y)dy \right)^{1/q} \tag{3.3}$$

Using (3.2),we have (3.1).

Setting

$$g(y) = |y|^{pb(\mu-\lambda)-1} \left(\int_{-\infty}^{\infty} f(x) \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| dx \right)^{p-1}$$

Then

$J = \int_{-\infty}^{\infty} |y|^{q[1-b(\mu-\lambda)]-1} g^q(y)dy$ by (2.7)we have $J < \infty$ .if $J = 0$ then (3.2) is proved;

If $0 < J < \infty$ we obtain

$$0 < \int_{-\infty}^{\infty} |y|^{q[1-b(\mu-\lambda)]-1} g^q(y)dy = J = I^*$$

$$< k \left(\int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x)dx \right)^{1/p} \left(\int_{-\infty}^{\infty} |x|^{q[1-b(\mu-\lambda)]-1} g^q(x)dx \right)^{1/q}$$

$$\left(\int_{-\infty}^{\infty} |x|^{q[1-b(\mu-\lambda)]-1} g^q(x)dx \right)^{1/p} = J^{1/p} < k \left(\int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x)dx \right)^{1/p}$$

Inequalities (3.1)and (3.2) are equivalent.

If the constant factor k in (3.1) is not the best possible, then there exists a positive h (with h < k),such that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)g(y) \frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}} \left| \ln \frac{x^2+xy+y^2}{x^2+y^2} \right| dxdy$$

$$< h \left(\int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} f^p(x)dx \right)^{1/p} \left(\int_{-\infty}^{\infty} |x|^{q[1-b(\mu-\lambda)]-1} g^q(x)dx \right)^{1/q}$$

For $\varepsilon > 0$ by (2.5),using lemma 2.3 ,we have

$$k + o(1) < \varepsilon h \left(\int_{-\infty}^{\infty} |x|^{p[1-a(\mu-\lambda)]-1} \tilde{f}^p(x)dx \right)^{1/p} \left(\int_{-\infty}^{\infty} |x|^{q[1-b(\mu-\lambda)]-1} \tilde{g}^q(x)dx \right)^{1/q} = h \tag{3.4}$$

Hence we find, k +o(1) < h: For $\varepsilon \to 0^+$ it follows that $k \leq h$ which contradicts the fact that h < k .

Hence the constant k in (3.1) is the best possible.

Thus we complete the prove of the theorem.

Theorem 3.2 If 1 > p > 0; both functions, f(x) and g(x) ,are nonnegative measurable functions, and satisfy

$0<\int_{-\infty}^{\infty}|x|^{p[1-a(\mu-\lambda)]-1} f^p(x)dx<\infty$ and $0<\int_{-\infty}^{\infty}|x|^{q[1-b(\mu-\lambda)]-1} g^q(x)dx<\infty$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(y)\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}\right|dxdy$$
$$>k\left(\int_{-\infty}^{\infty}|x|^{p[1-a(\mu-\lambda)]-1} f^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{q[1-b(\mu-\lambda)]-1} g^q(x)dx\right)^{1/q} \tag{3.5}$$

$$J>k^p\int_{-\infty}^{\infty}|x|^{p[1-a(\mu-\lambda)]-1} f^p(x)dx \tag{3.6}$$

And

$$L:=\int_{-\infty}^{\infty}|x|^{qa(\mu-\lambda)-1}\left(\int_{-\infty}^{\infty}\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}g(y)\right|dy\right)^q dx$$
$$<k^q\int_{-\infty}^{\infty}|y|^{q[1-b(\mu-\lambda)]-1} g^q(y)dy \tag{3.7}$$

Inequalities (3.5),(3.6)and (3.7) are equivalent,and where the constant factors k, k^p and k^q are the best possible

Proof By the reverse Holder's inequality and the same way, we can obtain the reverseforms of (2.7)and (3.3).And then we deduce the (3.5),by the some way,we obtain (3.6).

Setting g(y) as the theorem 1,we obtain $J>0$,if $J=\infty$,then we have (3.6),if $0<J<\infty$,by (3.5)

$$\int_{-\infty}^{\infty}|y|^{q[1-b(\mu-\lambda)]-1} g^q(y)dy=J=I^*$$
$$>k\left(\int_{-\infty}^{\infty}|x|^{p[1-a(\mu-\lambda)]-1} f^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{q[1-b(\mu-\lambda)]-1} g^q(x)dx\right)^{1/q}$$

and we have (3.6),and inequalities (3.5)and (3.6) are equivalent.

Setting

$$[g(x)]_n=\begin{cases}\frac{1}{n} & \text{if} \quad g(x)<\frac{1}{n},\\ g(x) & \text{if} \quad \frac{1}{n}\le g(x)\le n,\\ n & \text{if} \quad g(x)>n\end{cases}$$

$E_n=\left[-n,-\frac{1}{n}\right]\cup\left[\frac{1}{n},n\right]$ then $\exists n_0\in\mathrm{N}$, such that $n>n_0$; we have

$$\int_{E_n}|y|^{q[1-b(\mu-\lambda)]-1} g^q(y)dy>0,$$

And

$$[f(x)]_n=|x|^{qa(\mu-\lambda)-1}\left(\int_{E_n}[g(x)]_n\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}\right|dy\right)^{q-1}$$

$$[L(x)]_n=\int_{E_n}|x|^{qa(\mu-\lambda)-1}\left(\int_{E_n}[g(x)]_n\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{\mu}}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}\right|dy\right)^q dx$$

Then $\exists n_0\in\mathrm{N}$, such that $n>n_0$; we have

$$\int_{E_n}|y|^{q[1-b(\mu-\lambda)]-1} g^q(y)dy<\infty,$$

In particular, from (4.1) we get the following particular cases:

1)If $\lambda+\mu=4$; then $k=2\left[\psi(2)-\psi(\frac{2}{3})\right]+\frac{2}{3}\left[\psi(\frac{1}{3})-\psi(1)\right]=2+\frac{3\ln 3}{2}-\frac{\sqrt{3}\pi}{2}$, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(y)\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{4-\lambda}}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}\right|dxdy$$
$$<\left(2+\frac{3\ln 3}{2}-\frac{\sqrt{3}\pi}{2}\right)\left(\int_{-\infty}^{\infty}|x|^{p(\lambda-1)-1} f^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{q(\lambda-1)-1} g^q(x)dx\right)^{1/q} \tag{4.2}$$

2) If $\lambda+\mu=3$; then $k=\frac{8}{3}\left[\psi(\frac{3}{2})-\psi(\frac{1}{2})\right]+\frac{4}{3}\left[\psi(\frac{1}{4})-\psi(\frac{3}{4})\right]=\frac{4}{3}(4-\pi)$, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(y)\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{4-\lambda}}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}\right|dxdy$$
$$<\left(2+\frac{3\ln 3}{2}-\frac{\sqrt{3}\pi}{2}\right)\left(\int_{-\infty}^{\infty}|x|^{p(\lambda-1)-1} f^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{q(\lambda-1)-1} g^q(x)dx\right)^{1/q} \tag{4.3}$$

3) If $\lambda+\mu=2$; then $k=4\left[\psi(1)-\psi(\frac{1}{3})\right]+2\left[\psi(\frac{1}{6})-\psi(\frac{1}{2})\right]=6\ln 3+\frac{2\pi}{\sqrt{3}}-\sqrt{2}\pi$, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(y)\frac{(\min\{|x|,|y|\})^{\lambda}}{(\max\{|x|,|y|\})^{2-\lambda}}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}\right|dxdy$$
$$<\left(6\ln 3+\frac{2\pi}{\sqrt{3}}-\sqrt{2}\pi\right)\left(\int_{-\infty}^{\infty}|x|^{p\lambda-1} f^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{q\lambda-1} g^q(x)dx\right)^{1/q} \tag{4.4}$$

B) Let $\lambda=\mu$ in (3.1),then we have a 1ntegral 1nequality with the homogeneous kernel of 0 degree form as follows:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)g(y)\left(\frac{\min\{|x|,|y|\}}{\max\{|x|,|y|\}}\right)^{\lambda}\left|\ln\frac{x^2+xy+y^2}{x^2+y^2}\right|dxdy$$
$$<k(\lambda)\left(\int_{-\infty}^{\infty}|x|^{p-1} f^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{q-1} g^q(x)dx\right)^{1/q} \tag{4.5}$$

There $k(\lambda)=\frac{4}{\lambda}\left[\psi(\lambda)-\psi(\frac{\lambda}{3})\right]+\frac{2}{\lambda}\left[\psi(\frac{\lambda}{6})-\psi(\frac{\lambda}{2})\right]$

References

[1] Hardy G. H., Littlewood J. E. and Polya G., Inequalities,London, Cambridge University Press, Cambridge, 1952.

[2] Hardy G. H., Note on a theorem of Hilbert concerning series of positive terms, Proceedings London Math. Soc., 1925,23(2):45-46.

[3] Zitian Xie and Zheng Zeng, A Hilbert-type integral inequality whose kernel is a homogeneous form of degree-331

[4] Zitian Xie, Zheng Zeng, A Hilbert-type integral inequality with a non- homogeneous form and a best constant factor, Advances and Applications in Mathematical Sciens 2010,3(1),61-71

[5] Jianhua Xhong and Bicheng Yang, On a relation to four basic Hilbert-type integral inequalities, Applied Mathematical Sciences,Vol. 4, 2010,no,19,923-930.

[6] XIE Zitian and ZHOU Qinghua,A note on a Property of psi -function, Wuhan University Journal of Natural Sciences, Wuhan University Journal of Natural Science, 2011,16(2),139-142.

[7] Bicheng Yang, On a Hilber-type integral inequality with a combination kerne and applications, Kyungpook Mathematical Journal, 2010(50),281-288

[8] Zitian Xie , Zheng Zeng, The Hilbert-type integral inequality with the system Kernel of $-\lambda$ degree homogeneous form, Kyungpook Mathematical Journal, 2010(50),297-306

[9] Xie Zitian, A New Hilbert-type integral inequality with the homogeneous kernel of real number-degre, Journal of Jishou University (Natural Science Edition), 2011,32(4)26-30

[10] Zitian Xie and Bicheng Yang,A new Hilbert-type integral inequality with some parameters and its reverse,Kyungpook Mathematical Journal,2008,(48):93-100

[11] Zitian Xie,A new Hilbert-type inequality with the kernel of -30 -homogeneous, Journal of Jilin University (Science edition),2007,45(3):369-373.

[12] Zitian Xie and Juming Murong,A rewerse Hilbert-type inequality with some parameters,Journal of Jilin University (Science edition), 2008,46 (4):665-669.

[13] Zitian Xie,A new reverse Hilbert-type inequality with a best constant facter, J.Math.Appl, 2008,(343):1154-1160.

[14] Zheng Zeng and Zi-tian Xie, A Hilbert's inequality with a best constant factor, Journal of Inequalities and Applications, 2009, Article ID 820176, 8 pages doi:10.1155/2009/820176

[15] Xie Zitian ,Zeng Zheng,On generality of Hilbert's inequality with best constant factor, Natural Science Journal of Xiangtan University, 2010, 32(3) : 1-4

[16] Zheng Zeng and Zitian Xie, On a new Hilbert-type integral inequality with the integral in whole plane, Journal of Inequalities and Applications vol. 2010, Article ID 256796, 8 pages, 2010. doi:10.1155/2010/256796

[17] Zitian Xie,Bicheng Yang, Zheng Zeng, A New Hilbert-type integral inequality with the homogeneous kernel of real number-degree ,Journal of Jilin University(Science Edition), 2010,48(6) : 941-945.

[18] Zitian Xie and Fang min Zhou, A generalization of a Hilbert-type inequality with the best constant factor, Journal of Sichuan Normal University (Natural Science) 2009, 32(5): 626-629

[19] Xie Zitian,Zeng Zheng, A Hilbert-type integral inequality with the integral in whole plane and the kernel of -4 degree homogeneous form, i-manager's Journal on Mathematics, Vol. 1 No. 3, July-september 2012,6-12

[20] Zitian Xie,Zheng Zeng,Qinghua Zhou, A new Hilbert-type integral inequality with the homogeneous kernel of real number-degree and its equivalent inequality forms,Journal of Jilin University(Science Edition), 2012,50(4), 693-697.

[21] Xie Zitian,Zeng Zheng, A new half-discrete Hilbert-type inequality with the homogeneous kernel of degree -4μ, Journal of Jishou University(Natural Science Edition), 2012,33 (2), 15-19