

## An embedding algorithm for a special case of extended grids

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**Abstract:** A *book* consists of a line in the 3-dimensional space, called the spine, and a number of *pages*, each a half-plane with the spine as boundary. A *book embedding*  $(\pi, \rho)$  of a graph consists of a linear ordering of  $\pi$ , of vertices, called the *spine ordering*, along the spine of a book and an assignment  $\rho$ , of edges to pages so that edges assigned to the same page can be drawn on that page without crossing. That is, we cannot find vertices  $u, v, x, y$  with  $\pi(u) < \pi(x) < \pi(v) < \pi(y)$ , yet the edges  $uv$  and  $xy$  are assigned to the same page, that is  $\rho(uv) = \rho(xy)$ . The *book thickness* or *page number* of a graph  $G$  is the minimum number of pages in required to embed  $G$  in a book. In this paper we consider the extended grid and prove that the  $1 \times n$  extended grid can be embedded in two pages. We also give a linear time algorithm to embed the  $1 \times n$  extended grid in two pages.

**1. Introduction:** The growth of the subject ‘graph theory’ has been very rapid in recent years, particularly since the domain of its application is varied. Graph algorithms play a very important role in design of various computer networks.

Among the problems one comes across in graph theory, is the embedding of graphs. A particular way of embedding graphs is in the pages of a book. The book embedding of graphs was first introduced by Bernhart and Kainen [1] and since then, many researchers have actively studied it. Determining the book thickness for general graphs is *NP*-hard. But obtaining the book thickness for particular graphs have been found to be possible. The book embeddings have been studied for many classes of graphs. To name a few, we have: Complete Graphs [1, 2], Complete Bipartite Graphs [10], Trees, Grids and X-trees [3], hypercubes [3, 9], incomplete hypercubes [8], iterated line digraphs [6], de Bruijn graphs, Kautz graphs, shuffle-exchange graphs [7], for each of which embedding in books have been studied.

The book embedding problem has many different applications, which include sorting with parallel stacks, single-row routing of printed circuit boards, and the design of fault-tolerant processor arrays [4, 12].

In this paper, we discuss the book embedding of extended grids

## 2. Preliminaries

The *Extended Grid* [11] is an extension of the Mesh network very popular for its layout design and ease of implementation. It is a large graph with big edge to vertex ratio. The Extended Grid graphs are sub graphs of the tessellation of the  $K_4$  graphs.

The 2-dimensional grid graph (also called *Mesh*)  $M(r, s)$  is a graph whose vertex set is the set of ordered pairs of nonnegative integers  $\{(i, j): 0 \leq i < r, 0 \leq j < s\}$ , in which there is an edge between vertex  $(i, j)$  and  $(k, l)$  if either  $|i - k| = 1$  and  $j = l$ , or  $i = k$  and  $|j - l| = 1$ . See Figure 1.

The *Extended Grid*  $EM(r, s)$  is the graph whose vertex set is the set of ordered pairs of nonnegative integers  $\{(i, j): 0 \leq i < r, 0 \leq j < s\}$ , in which there is an edge between vertex  $(i, j)$  and  $(k, l)$  if and only if  $|i - k| \leq 1$  and  $|j - l| \leq 1$ . Thus the extended grid is obtained from a 2-dimensional grid by adding diagonal edges to the nodes. See Figure 2.

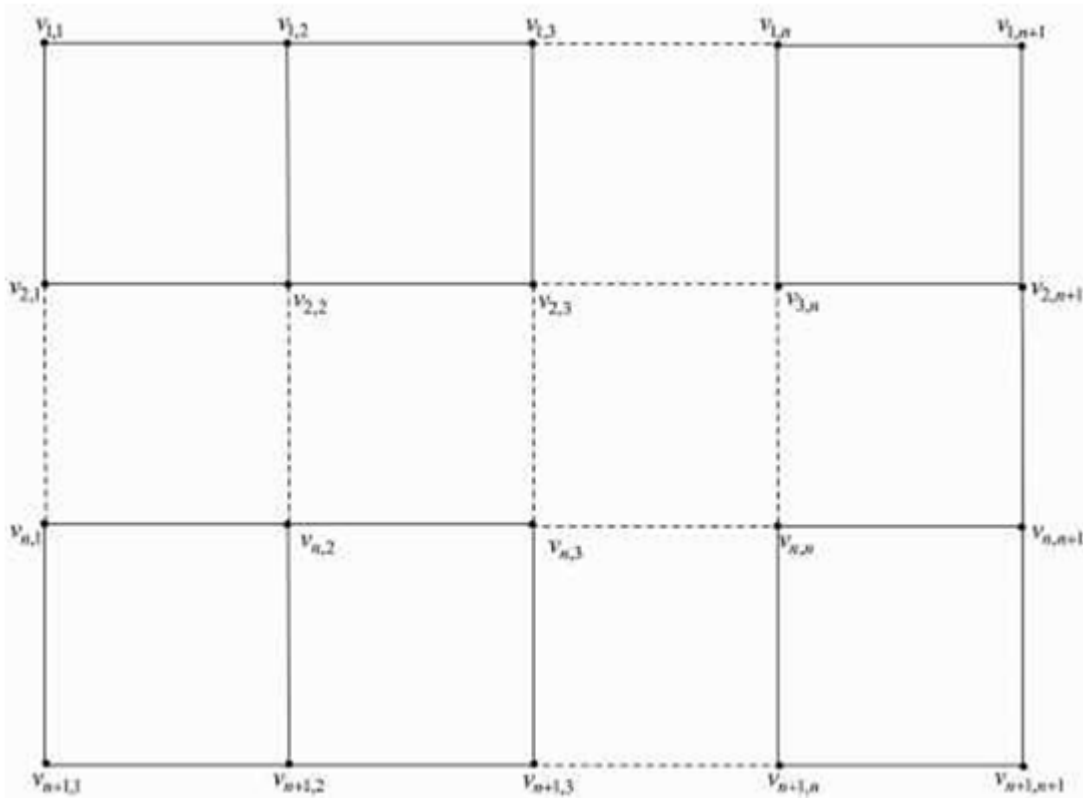


Figure 1. A grid graph

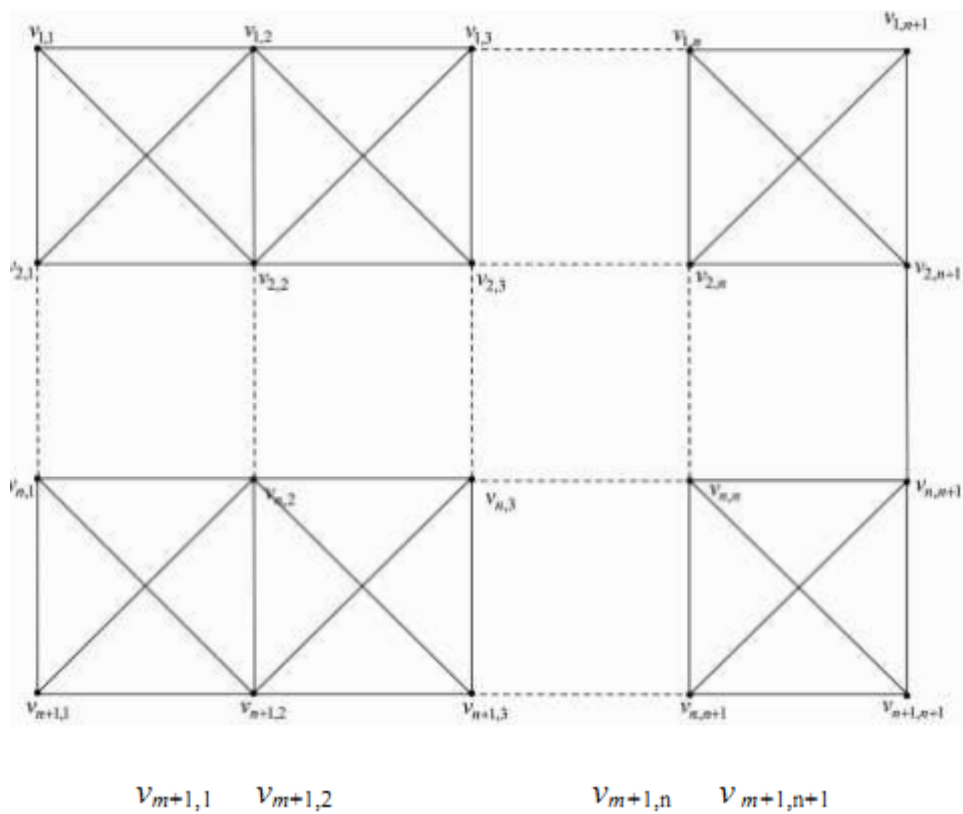


Figure 2 - An extended Grid

The graph  $EM(m, n)$  consists of  $m$  rows of ' $n$ '  $K_4$  graphs and  $n$  columns of ' $m$ '  $K_4$  graphs. In this paper, we shall denote the  $K_4$  in the  $i^{th}$  row,  $j^{th}$  column by  $G_{ij}$ .

**3. Embedding  $EM(1, n)$  in two pages**

We shall now prove two preliminary lemmas in respect of the extended grids.

**Lemma 1:** Extended grids  $EM(m, n)$  cannot be embedded in a single page.

**Proof:** It is enough to prove the lemma for the special case  $m = 1, n = 1$ .

Let  $EM(1, 1)$  have vertices  $u_1, u_2, u_3, u_4$  as shown in Figure 3. Let these vertices be embedded in any manner on  $P_4$ , which after renaming be  $v_1, v_2, v_3, v_4$ . Then  $v_1, v_2, v_3, v_4$  is just a permutation of  $u_1, u_2, u_3, u_4$ .

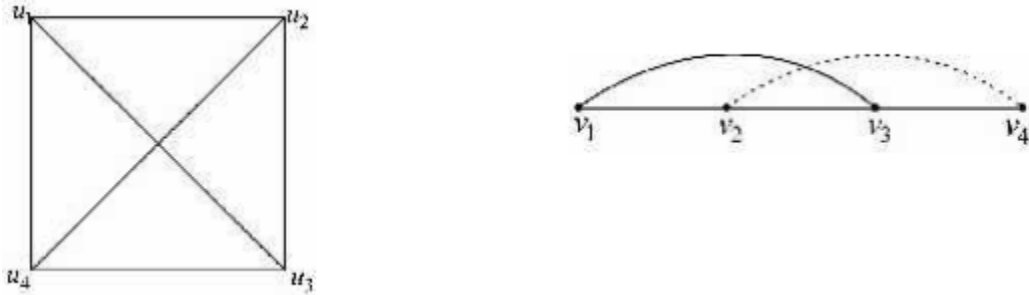


Figure 3

We see that  $EM(1, 1)$  is just  $K_4$  and hence 3-regular. Hence  $deg(u_i) = 3$  for each  $i = 1, 2, 3, 4$ . Consequently  $deg(v_i) = 3$  for each  $i = 1, 2, 3, 4$ . Therefore, each  $v_i$  is adjacent to each of the  $v_j, i \neq j$ . In particular, there has to be an edge between  $v_1$  and  $v_4$ . But then, there cannot be an edge between  $v_2$  and  $v_4$  in the same page. See Figure 3. Thus, a single page is not sufficient to embed  $EM(1, 1)$ . ■

Now we shall prove the main result of this paper.

**Theorem 1:**  $EM(1, n)$  can be embedded in two pages.

**Proof:** The graph  $EM(1, n)$  is a subhamiltonian graph [1] since it is planar (see fig. 5) and has the Hamilton cycle  $v_1, 1, v_1, 2, \dots, v_1, n+1, v_2, n+1, \dots, v_2, 1$ . Hence it requires exactly two pages for embedding the edges of  $EM(1, n)$  in a book [1]. Hence the theorem. ■

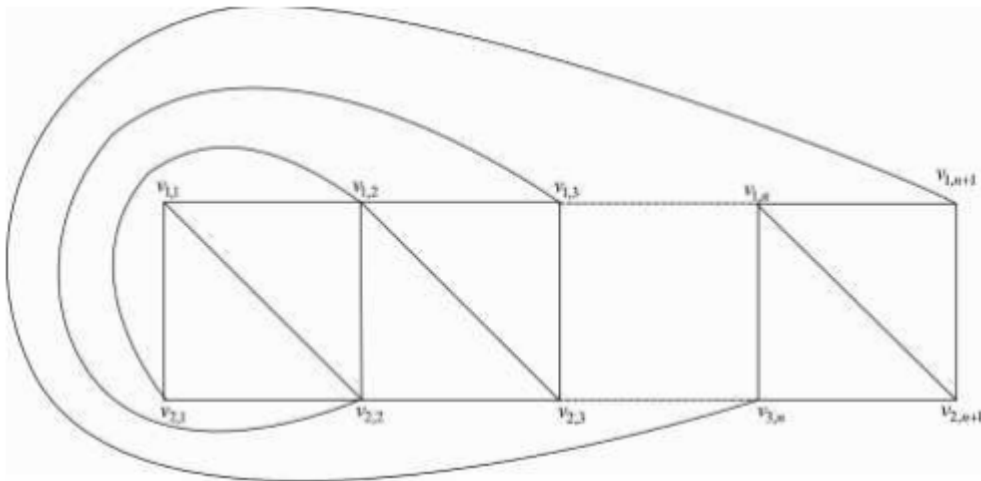


Fig. 5

We shall now present a linear time algorithm for embedding  $EM(1, n)$  in two pages and give a proof of correctness.

Note: If  $S = \{a_1, \dots, a_k\}$  is a finite sequence of symbols then  $S$  reversed is the sequence  $\{a_k, \dots, a_1\}$  and is denoted by  $S^R$ .

#### 4. Embedding Algorithm

ALGORITHM *BKEMBED*(1,  $n$ )

Step 1: Label the vertices of  $P_{2n+2}$ , the spine of the book embedding with the sequence  $S^0 S^R$  where  $S^0$  is the sequence  $\{v_{1,1}, \dots, v_{1,n+1}\}$  and  $S^1$  is the sequence

$\{v_{2,1}, \dots, v_{2,n+1}\}$

Step 2: Join  $v_{1,1}$  to  $v_{1,2}$  in page 1

Step 3: for  $i = 1$  to  $n$

join  $v_{1,i+1}$  to  $v_{2,i+1}$  in page  $i \% 2 + 1$

join  $v_{1,i}$  to  $v_{1,i+1}$  in page 2

join  $v_{2,i+1}$  to  $v_{2,i+1}$  in page 2

join  $v_{1,i}$  to  $v_{2,i+1}$  in page 1

join  $v_{1,i+1}$  to  $v_{2,i}$  in page 2

Note: Let  $1 \leq i \leq n$ . The above algorithm works in the following way

(i) The printing cycle  $S^0 S^R$  is obtained by nesting  $(v_{1,i}, v_{2,i})$  within  $(v_{1,i-1}, v_{2,i-1})$ , for every

$i = 2, \dots, n+1$ , which produces the Hamilton cycle of  $EM(1, n)$ .

(ii) The algorithm embeds the horizontal edges  $(v_{1,i}, v_{1,i+1})$  and  $(v_{2,i}, v_{2,i+1})$  in page 2.

(iii) The vertical edges  $(v_{1,i}, v_{2,i})$  are embedded alternately in page 1 and page 2 starting with embedding  $(v_{1,1}, v_{2,1})$  in page 1.

(iv) The forward diagonals  $(v_{1,i}, v_{2,i+1})$  are embedded in page 1 and the backward diagonals  $(v_{1,i+1}, v_{2,i})$  are embedded in page 2.

#### 5. Proof of Correctness of the Embedding algorithm

**Theorem2:** Algorithm *BKEMBED*(1,  $n$ ) embeds all the edges of  $EM(1, n)$  in two pages for any  $n$ .

**Proof:** We give the proof of correctness using loop-invariants [5]. In step 1, the algorithm labels the spine  $P_{2n+2}$  with the labels  $v_{1,1}, \dots, v_{1,n+1}, v_{2,n+1}, \dots, v_{2,1}$ , in that order. See Figure 6. We shall now prove using loop-invariants that step 3 correctly completes the embedding of  $EM(1, n)$  in two pages in a book.

(i) The vertical edge  $(v_{1,2}, v_{2,2})$  is embedded in page 2.

(ii) The horizontal edge  $(v_{1,1}, v_{1,2})$  is embedded in page 2. (iii) The horizontal edge  $(v_{2,1}, v_{2,2})$  is embedded in page 2. (iv) The forward diagonal  $(v_{1,1}, v_{2,2})$  is embedded in page 1.

(v) The backward diagonal  $(v_{1,2}, v_{2,1})$  is embedded in page 2.

Thus all the six edges of  $G_{1,1}$  are embedded in two pages without crossing

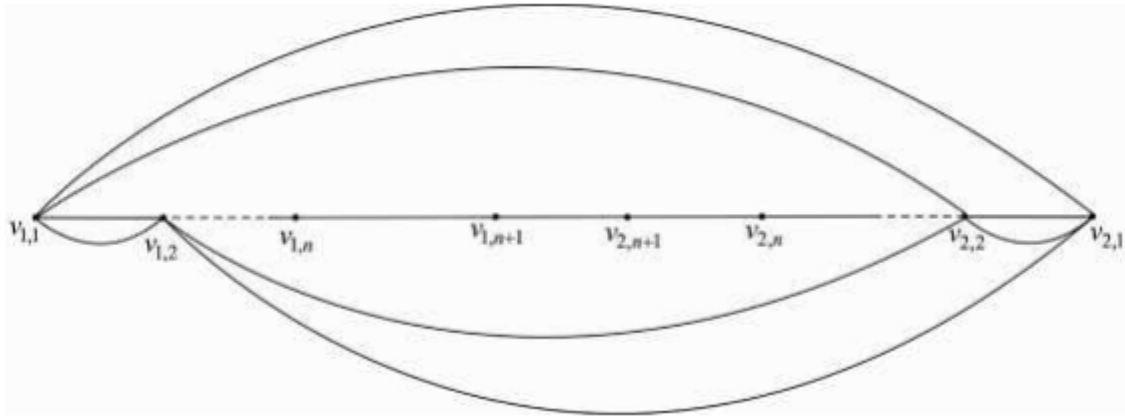


Figure 6

**Maintenance:** Assume that at the beginning of the iteration corresponding to  $i = k$ , the first  $k-1$  grids starting from  $G_{1,1}$  up to  $G_{1,k-1}$  have been embedded in two pages. Let us prove that at the end of this iteration, all the grids starting from  $G_{1,1}$  up to  $G_{1,k}$  are embedded in two pages on the spine  $P_{2n+2}$ . When the iteration reaches  $i = k$ , we observe that the  $2k$  edges with both ends in  $\{v_{1,1}, v_{1,2}, \dots, v_{1,k}, v_{2,k}, \dots, v_{2,2}, v_{2,1}\}$  have been embedded in two pages. But then the edges incident with  $v_{1,k+1}, v_{2,k+1}$  have not been embedded. Since the pair of vertices  $(v_{1,k+1}, v_{2,k+1})$  have been nested between  $(v_{1,k}, v_{2,k})$ , embedding the edges incident with the vertices  $v_{1,k+1}$  and  $v_{2,k+1}$  would embed the grid  $G_{1,k+1}$  completely in two pages of a book. Thus at the end of the iteration corresponding to  $i = k$ , all the grids  $G_{1,1}, \dots, G_{1,k+1}$  would have been embedded in two pages of a book.

**Termination:** When the value of  $i$  reaches the final value  $n$ , all the grids from  $G_{1,1}$  up to  $G_{1,n}$  would have been embedded in two pages of a book. Hence by loop-invariants, Algorithm  $BKEMBED(1,n)$  embeds  $EM(1,n)$  in two pages. ■

## 6. Time Complexity

Now we shall find the time complexity of  $BKEMBED(1, n)$ . We shall directly compute the time complexity calculating the effort in each step of the algorithm.

**Theorem 3:** Algorithm  $BKEMBED(1, n)$  works in  $O(n)$  time.

Proof: Looking at the algorithm, Step 1 works in  $O(n)$  time. Step 2 works in  $O(1)$  time. Step 3 contains a simple loop, which works for  $n$  iterations, within which constant-time work is done. Hence Step 3 works in  $O(n)$  time. In all, Algorithm  $BKEMBED(1,n)$  works in  $O(n)$  time. ■

## 7. Conclusion

The Extended Grid  $EM(1, n)$  is planar while  $EM(m, n)$  is non-planar. Hence  $EM(m, n)$  will require at least 3 pages to embed them. At the first instance it looks as if the number of pages will increase as the number of rows increase. However as the number of rows and columns increase, the number of vertices also increase. Hence these graphs may be embedded in considerably less number of pages. But to prove such a possibility would be a challenge.

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