Twin prime numbers and Diophantine equations

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Abstract: For two millennia, the prime numbers have continued to fascinate mathematicians. Indeed, a conjecture which dates back to this period states that the number of twin primes is infinite. In 1949 Clement showed a theorem on twin primes. Starting from Wilson's theorem, Clement's theorem and the corollary of Clement's theorem [1], I came to find Diophantine equations whose solution could lead to the proof of the infinitude of twin primes.

Introduction:

Or (p, q) a pair of integers such that p and q are both prime and p <q. We say that (p, q) form a pair of twin primes if q = p + 2.
The couple (2, 3) is the only pair of consecutive primes.
Omitting the pair (2, 3), 2 is the smallest possible distance between two primes, twin primes are two and two consecutive odd numbers.
Any pair of twin primes (with the exception of the couple (3, 5)) is of the form (6n - 1,6n + 1) for some integer n. Indeed, any set of three consecutive natural numbers has at least a multiple of 2 (possibly two) and one multiple of 3, these two are confused between multiple two twin primes [4].
It is possible to show that, for any integer, the pair (m, m + 2) consists of twin primes if and only if:

$$2[(M-1)!+1]+m=0 \bmod m(m+2)$$

This characterization of factorial and modular twin primes was discovered by P. A. Clement in 1949 [2].
The series of reciprocals of twin primes converges to Brun's constant, unlike the series of reciprocals of prime numbers. This property was demonstrated by Viggo Brun in 1919 [3].

The twin prime conjecture states that there are infinitely many twin primes. In other words, there are infinitely many primes p such that (p +2) is also prime.
In 1940, Paul Erdős proved the existence of a constant c <1 and infinitely many primes p such that:
p '- p <cln (p) where p' denotes the number immediately following the first p.
This result was improved several times, in 1986, Helmut Maier showed a constant c <0.25 could be reached.
In 2003, Daniel Goldston and Cem Yildirim have shown that, for all c> 0, there are infinitely many primes p such that p '- p <c ln (p).
In 1966, Chen Jingrun demonstrated the existence of infinitely many primes p such that p + 2 is the product of at most two prime factors (such a number, product of at most two prime factors, 2 is said -almost-first).
His approach was that of the theory of the screen, he used to treat similarly the twin prime conjecture and Goldbach's Conjecture.
As for me I establish relationships between the twin primes and special Diophantine equations.

Theorem 1: Theorem Wilson

Statement: An integer p strictly greater than 1, is a prime number if and only if divides (p - 1)! + 1, that is to say if and only if: (p-1)!+1=0 (mod p)

Theorem 2: Theorem Clement

For any integer, the pair (m, m + 2) consists of twin primes if $4[(m-1)!+1]+m=0 \mod m(m+2)$

Theorem 3: Corollary of Clement's theorem [1]

For any integer, the pair (m, m + 2) consists of twin primes if: $m(m+2)$ divides $((m-2)(m-1)! - 2)$.

Theorem 4: Bachet-Bézout Theorem

Given two integers p and q nonzero, if r is the GCD of p and q then there exist two integers x and y such that: $r = x*p + y*q$

In particular, two integers p and q are coprime if and only if there exist two integers x and y such that $x*p + y*q = 1$

Conjecture about Diophantine equations and twin primes:

Is the set of integers n> 4 such that n and (n +2) are twin primes. Let a, b, c and d non-zero integers. The following three Diophantine equations admit infinitely many solutions.

$$f(n)= a*d*M +3*a*d*n^2*n + (2*a*d+2*a-b*d)*n^2 + (3*a-2*b*d-2*a*b-b)*n +4*a*b-2*a=0$$

$$g(n)=a*c*M + a*(3*c-1)*n^2*n+ (2*a*c-b*c-5*a)*n^2 +(b-2*b*c-4*a)*n–8*a*b+4*a=0$$

$$h(n)=c*n^2 - (4*d+1)*n – 2*(4*d+1+2*c)=0$$

M means n exponent 4 ;

Attempt of proof:

By Wilson's theorem n is a prime number if it divides $((n-1)! + 1)$. And (n +2) is a prime prime number if it divides $((n +1)! + 1)$. Consider all values of n such that n and (n +2) are twin primes. So there are two nonzero integers a and b such that:

$$a = ((n-1)! + 1) / n \text{ and } b = ((n +1)! + 1) / (n +2)$$

$$b = ((n^2 + n)*(n-1)! + 1) / (n +2)$$

So, $n = ((n-1)! + 1) / a$; $(n +2) = ((n^2 + n)*(n-1)! + 1) / b$
$(n+2)-n = ((n^2+ n)*(n-1)! +1)/b - ((n-1)! +1)/a = ((a*n^2+a*n-b) (n- 1)! + (a+b)) /a*b = 2$
After calculations we arrive at:

$$(n-1)!= (2*a*b –(a-b))/(a*n^2+a*n-b)$$

By the theorem of Clement of 1949 and we know that n (n +2) are twin primes if:

$$n*(n +2) \text{ divides } (4*((n-1)! + 1) + n).$$

So there exists an integer c such that c = $(4*((n-1)! + 1) + n) / (n*(n +2))$ $(n-1)! = (c*n*(n +2) - n -4) / 4$
By the corollary of the theorem of Clement and n*(n +2) are twin primes if:
$$n*(n +2) \text{ divides } ((n-2)*(n-1)! - 2).$$

So there exists an integer d such that $d = ((n-2)*(n-1)! - 2) / (n*(n+2))$ $(n-1)! = (d*n*(n+2) + 2) / (n-2)$

Hence,

$$(n-1)! = (2*a*b - (a-b))/(a*n^2 + a*n - b) = (c*n*(n+2) - n - 4)/4 = (d*n*(n+2) + 2)/(n-2)$$

If we associate two to two, the three expressions we obtain the following three Diophantine equations:

$$f(n) = a*d*M + 3*a*d*n^2*n + (2*a*d + 2*a - b*d)*n^2 + (3*a - 2*b*d - 2*a*b - b)*n + 4*a*b - 2*a = 0$$

$$g(n) = a*c*M + a*(3*c-1)*n^2*n + (2*a*c - b*c - 5*a)*n^2 + (b - 2*b*c - 4*a)*n - 8*a*b + 4*a = 0$$

$$h(n) = c*n^2 - (4*d+1)*n - 2*(4*d+1+2*c) = 0$$

M means n exponent 4.

Easiest to solve is $h(n) = c*n^2 - (4*d+1)*n - 2(4*d+1+2*c) = 0$

I call it **"The Diophantine equation of twin primes"**

c and d are nonzero integers

n > 4 such that n and (n+2) are twin primes.

If we can demonstrate the existence of infinitely many solutions then we can conclude of the infinite of twin primes.

Solve the equation $h(n) = c*n^2 - (4*d+1)*n - 2*(4*d+1+2*c) = 0$

$Delta = 16*d^2 + 32*c*d + 16*c^2 + 8*d + 8*c + 1 = (4*d + 4*c + 1)^2$

The only solution is $n = (4*d + 2*c + 1)/c$, the other solution is necessarily negative.

$$n = (4*d+1)/c + 2$$

$$(n+2) = (4*d+1)/c + 4$$

Recall that the twin primes are of the form (6*m-1) and (6*m+1)

Let n = (6*m-1); So $(n-2) = (4*d+1) / c = (6*m-3)$

$m = (4*d + 3*c + 1)/(6*c)$

Let k be a nonzero as entire: $k = (4*d+1)/c$

$$c*k = 4*d+1;$$
$$c*k - 4*d = 1$$

According to Bachet-Bézout theorem, c and d are coprime.

Example: The first solution is for n=5; (n+2)=7; c=3 and d=2 ; in same time we remark that a=5 and b=103

The second solution is for n=11;(n+2)=13;c=101505 and d=228386

Generalization

I think that characterization of Clement of 1949 is just a special case of a more comprehensive characterization of twin primes.

Now consider the set of integers n > 4. I propose the following characterization simpler and more general about twin primes:

Consider the nonzero integers d' and c' and m such that:

$$(6*m - 1) = n = (4d'+1) / c' + 2 \text{ and } (6*m + 1) = (n + 2) = (4d'+1) / c' + 4$$

I conjecture that for each twin primes we can find two primes c' and d'. In this case m = $(4*d' + 3*c' + 1)/(6*c')$

Examples:

d'=5 and c'=7 ; n=5 and (n+2)=7

d'=29 and c'=13; n=11 and (n+2)=13

d'=11 et c'=3; n=17 and (n+2)=19

d'=47 and c'=7 we have n=29 and (n+2)=31

d'=29 and c'=3 we have n=41 and (n+2)=43 Etc…

References

[1] Ibrahima GUEYE, A note on the twin primes, South Asian Journal of Mathematics, Volume 2 (2012) Issue 2, p. 159-161

[2] PA Clement, Congruences for sets of premiums, American Mathematical Monthly 56 (1949), p. 23-25

[3] Viggo Brun, Series 1/5 + 1/7 + 1/11 + 1/13 + 1/17 + 1/19 + 1/29 + 1/31 + 1/41 + 1/43 + 1/59 + 1/61 + ... where denominators are "twin primes" is convergent or over, Bulletin of Mathematical Sciences 43 (1919), p. 100-104 and 124-128.

[4] http://fr.wikipedia.org/wiki/Nombres_premiers_jumeaux