

Twin prime numbers and Diophantine equations

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Abstract: For two millennia, the prime numbers have continued to fascinate mathematicians. Indeed, a conjecture which dates back to this period states that the number of twin primes is infinite. In 1949 Clement showed a theorem on twin primes. Starting from Wilson's theorem, Clement's theorem and the corollary of Clement's theorem [1], I came to find Diophantine equations whose solution could lead to the proof of the infinitude of twin primes.

Introduction:

Or (p, q) a pair of integers such that p and q are both prime and $p < q$. We say that (p, q) form a pair of twin primes if $q = p + 2$.

The couple $(2, 3)$ is the only pair of consecutive primes.

Omitting the pair $(2, 3)$, 2 is the smallest possible distance between two primes, twin primes are two and two consecutive odd numbers.

Any pair of twin primes (with the exception of the couple $(3, 5)$) is of the form $(6n - 1, 6n + 1)$ for some integer n . Indeed, any set of three consecutive natural numbers has at least a multiple of 2 (possibly two) and one multiple of 3, these two are confused between multiple two twin primes [4].

It is possible to show that, for any integer, the pair $(m, m + 2)$ consists of twin primes if and only if:

$$2[(M-1)!+1]+m=0 \pmod{m(m+2)}$$

This characterization of factorial and modular twin primes was discovered by P. A. Clement in 1949 [2].

The series of reciprocals of twin primes converges to Brun's constant, unlike the series of reciprocals of prime numbers. This property was demonstrated by Viggo Brun in 1919 [3].

The twin prime conjecture states that there are infinitely many twin primes. In other words, there are infinitely many primes p such that $(p + 2)$ is also prime.

In 1940, Paul Erdős proved the existence of a constant $c < 1$ and infinitely many primes p such that:

$p' - p < c \ln(p)$ where p' denotes the number immediately following the first p .

This result was improved several times, in 1986, Helmut Maier showed a constant $c < 0.25$ could be reached.

In 2003, Daniel Goldston and Cem Yildirim have shown that, for all $c > 0$, there are infinitely many primes p such that $p' - p < c \ln(p)$.

In 1966, Chen Jingrun demonstrated the existence of infinitely many primes p such that $p + 2$ is the product of at most two prime factors (such a number, product of at most two prime factors, 2 is said -almost-first).

His approach was that of the theory of the screen, he used to treat similarly the twin prime conjecture and Goldbach's Conjecture.

As for me I establish relationships between the twin primes and special Diophantine equations.

Theorem 1: Theorem Wilson

Statement: An integer p strictly greater than 1, is a prime number if and only if divides $(p - 1)! + 1$, that is to say if and only if: $(p-1)!+1=0 \pmod{p}$

Theorem 2: Theorem Clement

For any integer, the pair $(m, m + 2)$ consists of twin primes if $4[(m-1)!+1]+m=0 \pmod{m(m+2)}$

Theorem 3: Corollary of Clement’s theorem [1]

For any integer, the pair $(m, m + 2)$ consists of twin primes if: $m(m+2)$ divides $((m-2)(m-1)! - 2)$.

Theorem 4: Bachet-Bézout Theorem

Given two integers p and q nonzero, if r is the GCD of p and q then there exist two integers x and y such that: $r = x*p + y*q$

In particular, two integers p and q are coprime if and only if there exist two integers x and y such that $x*p + y*q = 1$

Conjecture about Diophantine equations and twin primes:

Is the set of integers $n > 4$ such that n and $(n + 2)$ are twin primes. Let a, b, c and d non-zero integers. The following three Diophantine equations admit infinitely many solutions.

$$f(n) = a*d*M + 3*a*d*n^2*n + (2*a*d + 2*a-b*d)*n^2 + (3*a-2*b*d-2*a*b-b)*n + 4*a*b-2*a=0$$

$$g(n) = a*c*M + a*(3*c-1)*n^2*n + (2*a*c-b*c-5*a)*n^2 + (b-2*b*c-4*a)*n - 8*a*b + 4*a=0$$

$$h(n) = c*n^2 - (4*d+1)*n - 2*(4*d+1+2*c)=0$$

M means n exponent 4 ;

Attempt of proof:

By Wilson's theorem n is a prime number if it divides $((n-1)! + 1)$. And $(n + 2)$ is a prime number if it divides $((n+1)! + 1)$. Consider all values of n such that n and $(n + 2)$ are twin primes. So there are two nonzero integers a and b such that:

$$a = ((n-1)! + 1) / n \text{ and } b = ((n + 1)! + 1) / (n + 2)$$

$$b = ((n^2 + n)*(n-1)! + 1) / (n + 2)$$

So, $n = ((n-1)! + 1) / a$; $(n + 2) = ((n^2 + n)*(n-1)! + 1) / b$
 $(n+2)-n = ((n^2 + n)*(n-1)! + 1)/b - ((n-1)! + 1)/a = ((a*n^2 + a*n - b) (n-1)! + (a+b)) / a*b = 2$
 After calculations we arrive at:

$$(n-1)! = (2*a*b - (a-b)) / (a*n^2 + a*n - b)$$

By the theorem of Clement of 1949 and we know that n and $(n + 2)$ are twin primes if:

$$n*(n + 2) \text{ divides } (4*((n-1)! + 1) + n).$$

So there exists an integer c such that $c = (4*((n-1)! + 1) + n) / (n*(n + 2))$ $(n-1)! = (c*n*(n + 2) - n - 4) / 4$

By the corollary of the theorem of Clement and $n*(n + 2)$ are twin primes if:

$$n*(n + 2) \text{ divides } ((n-2)*(n-1)! - 2).$$

So there exists an integer d such that $d = ((n-2)*(n-1)! - 2) / (n*(n+2)) (n-1)! = (d*n*(n+2) + 2) / (n-2)$

Hence,

$$(n-1)! = (2*a*b - (a-b)) / (a*n^2 + a*n - b) = (c*n*(n+2) - n - 4) / 4 = (d*n*(n+2) + 2) / (n-2)$$

If we associate two to two, the three expressions we obtain the following three Diophantine equations:

$$f(n) = a*d*M + 3*a*d*n^2 + (2*a*d + 2*a*b*d)*n^2 + (3*a - 2*b*d - 2*a*b - b)*n + 4*a*b - 2*a = 0$$

$$g(n) = a*c*M + a*(3*c - 1)*n^2 + (2*a*c - b*c - 5*a)*n^2 + (b - 2*b*c - 4*a)*n - 8*a*b + 4*a = 0$$

$$h(n) = c*n^2 - (4*d + 1)*n - 2*(4*d + 1 + 2*c) = 0$$

M means n exponent 4.

Easiest to solve is $h(n) = c*n^2 - (4*d + 1)*n - 2*(4*d + 1 + 2*c) = 0$

I call it "**The Diophantine equation of twin primes**"

c and d are nonzero integers

$n > 4$ such that n and $(n+2)$ are twin primes.

If we can demonstrate the existence of infinitely many solutions then we can conclude of the infinite of twin primes.

Solve the equation $h(n) = c*n^2 - (4*d + 1)*n - 2*(4*d + 1 + 2*c) = 0$

$$\Delta = 16*d^2 + 32*c*d + 16*c^2 + 8*d + 8*c + 1 = (4*d + 4*c + 1)^2$$

The only solution is $n = (4*d + 2*c + 1)/c$, the other solution is necessarily negative.

$$n = (4*d + 1)/c + 2$$

$$(n+2) = (4*d + 1)/c + 4$$

Recall that the twin primes are of the form $(6*m-1)$ and $(6*m+1)$

Let $n = (6*m-1)$; So $(n-2) = (4*d + 1) / c = (6*m-3)$

$$m = (4*d + 3*c + 1) / (6*c)$$

Let k be a nonzero as entire: $k = (4*d + 1)/c$

$$c*k = 4*d + 1;$$

$$c*k - 4*d = 1$$

According to Bachet-Bézout theorem, c and d are coprime.

Example: The first solution is for $n=5$; $(n+2)=7$; $c=3$ and $d=2$; in same time we remark that $a=5$ and $b=103$

The second solution is for $n=11$; $(n+2)=13$; $c=101505$ and $d=228386$

Generalization

I think that characterization of Clement of 1949 is just a special case of a more comprehensive characterization of twin primes.

Now consider the set of integers $n > 4$. I propose the following characterization simpler and more general about twin primes:

Consider the nonzero integers d' and c' and m such that:

$$(6*m - 1) = n = (4d'+1) / c' + 2 \text{ and } (6*m + 1) = (n + 2) = (4d'+1) / c' + 4$$

I conjecture that for each twin primes we can find two primes c' and d' . In this case $m = (4*d' + 3*c' + 1) / (6*c')$

Examples:

$$d'=5 \text{ and } c'=7 ; n=5 \text{ and } (n+2)=7$$

$$d'=29 \text{ and } c'=13 ; n=11 \text{ and } (n+2)=13$$

$$d'=11 \text{ et } c'=3 ; n=17 \text{ and } (n+2)=19$$

$$d'=47 \text{ and } c'=7 \text{ we have } n=29 \text{ and } (n+2)=31$$

$$d'=29 \text{ and } c'=3 \text{ we have } n=41 \text{ and } (n+2)=43 \text{ Etc...}$$

References

- [1] Ibrahima GUEYE, A note on the twin primes, South Asian Journal of Mathematics, Volume 2 (2012) Issue 2, p. 159-161
- [2] PA Clement, Congruences for sets of premiums, American Mathematical Monthly 56 (1949), p. 23-25
- [3] Viggo Brun, Series $1/5 + 1/7 + 1/11 + 1/13 + 1/17 + 1/19 + 1/29 + 1/31 + 1/41 + 1/43 + 1/59 + 1/61 + \dots$ where denominators are "twin primes" is convergent or over, Bulletin of Mathematical Sciences 43 (1919), p. 100-104 and 124-128.
- [4] http://fr.wikipedia.org/wiki/Nombres_premiers_jumeaux