ON DEGREE OF APPROXIMATION BY PRODUCT MEANS OF CONJUGATE SERIES OF A FOURIER SERIES

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Abstract: In this paper a theorem on degree of approximation of a function $f \in Lip(\alpha , r)$ by product summability $(E, q)(\overline{N} , p_n)$ of conjugate series of Fourier series associated with f has been established.

1. Introduction

Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of positive real numbers such that

$$P_n = \sum_{\upsilon=0}^{n} p_\upsilon \to \infty, \quad \text{as} \quad n \to \infty , \ (P_{-i} = p_{-i} = 0 , i \geq 0). \qquad (1.1)$$

The sequence –to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{\upsilon=0}^{n} p_\upsilon s_\upsilon \qquad (1.2)$$

defines the sequence $\{ t_n \}$ of the $(\overline{N} , p_n)$ -mean of the sequence $\{s_n\}$ generated by the sequence of coefficient $\{p_n\}$. If

$$t_n \to s \quad \text{,as} \ n \to \infty \qquad (1.3)$$

then the series Σ an is said to be $(\overline{N, p_n})$ summable to s.
The conditions for regularity of $(\overline{N} , p_n)$- summability are easily seen to be[1]

$$\begin{cases} (i) \ P_n \to \infty, as \ n \to \infty. \\ (ii) \ \sum_{i=0}^{n} p_i \leq C|P_n| , as \ n \to \infty. \end{cases} \qquad (1.4)$$

The sequence –to-sequence transformation, [1]

$$T_n = \frac{1}{(1+q)^n} \sum_{\upsilon=0}^{n} \binom{n}{\upsilon} q^{n-\upsilon} s_\upsilon \qquad (1.5)$$

defines the sequence $\{T_n \}$ of the (E , q) mean of the sequence $\{s_n\}$.
If

$$T_n \to s \ , \text{as} \ n \to \infty , \qquad (1.6)$$

then the series $\sum a_n$ is said to be (E , q) summable to s .

Clearly (E, q) method is regular. Further, the (E, q) transform of the $(\underline{N}, p_n)$ transform of $\{s_n\}$ is defined by

$$\tau_n = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} t_k$$

$$= \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\upsilon=0}^{k} p_\upsilon s_\upsilon \right\} \tag{1.7}$$

If

$$\tau_n \to s \quad , \text{ as } \quad n \to \infty, \tag{1.8}$$

then $\sum a_n$ is said to be *(E, q)(N , p_n)*-summable to s .

Let $f(t)$ be a periodic function with period 2π and L-integrable over $(-\pi,\pi)$. The Fourier series associated with f at any point x is defined by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right) \equiv \sum_{n=0}^{\infty} A_n(x) \tag{1.9}$$

and the conjugate series of the Fourier series (1.9) is

$$\sum_{n=1}^{\infty} \left(b_n \cos nx - a_n \sin nx \right) \equiv \sum_{n=1}^{\infty} B_n(x) \tag{1.10}$$

Let $\overline{s_n}(f; x)$ be the n-th partial sum of (1.10).The L∞ -norm of a function $f : R \to R$ is defined by

$$\|f\|_{\infty} = \sup\left\{ |f(x)| : x \in R \right\} \tag{1.11}$$

and the L_v -norm is defined by

$$\|f\|_{\upsilon} = \left(\int_{0}^{2\pi} |f(x)|^{\upsilon} \right)^{\frac{1}{\upsilon}} \quad , \upsilon \geq 1 \tag{1.12}$$

The degree of approximation of a function $f : R \to R$ by a trigonometric polynomial $P_n(x)$ of degree n under norm $\| \|_{\infty}$ is defined by [5].

$$\|P_n - f\|_{\infty} = \sup\left\{ |p_n(x) - f(x)| : x \in R \right\} \tag{1.13}$$

and the degree of approximation $E_n(f)$ of a function $f \in L_v$ is given by

$$E_n(f) = \min_{P_n} \|P_n - f\|_{\upsilon} \tag{1.14}$$

A function f is said to satisfy Lipschitz condition (here after we write $f \in Lip a$) if

$$\left| f(x+t) - f(x) \right| = O\left(|t|^{\alpha} \right), 0 < \alpha \leq 1 \tag{1.15}$$

and $f(x) \in Lip (a, r)$, for $0 \leq x \leq 2\pi$, if

$$\left(\int_{0}^{2\pi} |f(x+t) - f(x)|^r \, dx \right)^{\frac{1}{r}} = O\left(|t|^{\alpha} \right), 0 < \alpha \leq 1, r \geq 1, t > 0 \tag{1.16}$$

We use the following notation throughout this paper :

$$\psi(t) = \frac{1}{2}\{f(x+t) - f(x-t)\},\qquad(1.17)$$

and

$$\overline{K_n}(t) = \frac{1}{\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\frac{\cos\frac{t}{2} - \cos\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\qquad(1.18)$$

Further, the method $(E, q)(N, p_n)$ is assumed to be regular .

2. Known Theorem

Dealing with The degree of approximation by the product $(E, q)(C, 1)$ -mean of Fourier series, Nigam et al [3] proved the following theorem:

Theorem 2.1. If a function f is 2π - periodic and belonging to class _Lipa_ , then its degree of

approximation by $(E, q)(C, 1)$ summability mean on its Fourier series $\sum\limits_{n=0}^{\infty}A_n(t)$ is given by

$\left\|E_n^q C_n^1 - f\right\|_\infty = O\left(\dfrac{1}{(n+1)^\alpha}\right)$,$0 < \alpha < 1$ where $E_n^q C_n^1$ represents the (E, q) transform of $(C, 1)$

transform of $S_n(f; x)$.

Recently, Misra et al [2] proved the following theorem using $(E, q)(N, p_n)$ mean of conjugate series of the Fourier series :

Theorem 2.2. If f be 2π - Periodic function of class _Lipa_ , then degree of approximation by the product $(E, q)(\overline{N}, p_n)$ summability means of the conjugate series (1.10) of Fourier series (1.9) is

given $\left\|\tau_n - f\right\|_\infty = O\left(\dfrac{1}{(n+1)^\alpha}\right)$,$0 < \alpha < 1$ where τ_n is as defined in 1.7) .

3. Main theorem

In this paper, we have proved a theorem on degree of approximation by the product mean $(E, q)(\overline{N}, p)$ of conjugate series of Fourier series of a function of class $Lip (a, r)$. We prove:

Theorem.3.1. If f be 2π - Periodic function of class Lipa , then degree of approximation by the product $(E, q)(\overline{N}, p_n)$ summability means of the conjugate series of Fourier series (1.10) is given

by $\left\|\tau_n - f\right\|_\infty = O\left(\dfrac{1}{(n+1)^{\alpha-1/r}}\right)$,$0 < \alpha < 1$ where τ is as defined in (1.7) .

4. Required Lemmas

We require the following Lemmas to prove the theorem.

Lemma 4.1

$$\left|\overline{K_n}(t)\right| = O(n)\quad,0 \le t \le \frac{1}{n+1}$$

Proof.

For $0 \le t \le \dfrac{1}{n+1}$ we have $\sin nt \le n \sin t$ then

$$\left|\overline{K_n}(t)\right| = \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\frac{\cos\frac{t}{2}-\cos\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right|$$

$$\le \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\frac{\cos\frac{t}{2}-\cos\upsilon t.\cos\frac{t}{2}+\sin\upsilon t.\sin\frac{t}{2}}{\sin\frac{t}{2}}\right\}\right|$$

$$\le \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\left(\frac{\cos\frac{t}{2}\left(2\sin^2\upsilon\frac{t}{2}\right)}{\sin\frac{t}{2}}+\sin\upsilon t\right)\right\}\right|$$

$$\le \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\left(O\left(2\sin\upsilon\frac{t}{2}\sin\upsilon\frac{t}{2}\right)+\upsilon\sin t\right)\right\}\right|$$

$$\le \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\left(O(\upsilon)+O(\upsilon)\right)\right\}\right|$$

$$\le \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{O(k|)}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\right|$$

$$= O(n)$$

This proves the lemma.

Lemma 4.2

$$\left|\overline{K_n}(t)\right| = O\left(\frac{1}{t}\right), \text{ for } \frac{1}{n+1} \le t \le \pi$$

Proof.

For $\dfrac{1}{n+1} \le t \le \pi$, by Jordan's lemma, we have $\sin\left(\dfrac{t}{2}\right) \ge \dfrac{t}{\pi}$

Then

$$\left|\overline{K_n}(t)\right| = \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\frac{\cos\frac{t}{2}-\cos\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right|$$

$$= \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon\frac{\cos\frac{t}{2}-\cos\upsilon\frac{t}{2}.\cos\frac{t}{2}+\sin\upsilon\frac{t}{2}.\sin\frac{t}{2}}{\sin\frac{t}{2}}\right\}\right|$$

$$\leq \frac{1}{\pi(1+q)^n} \left| \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} \frac{\pi}{2t} P_v \left(\cos\frac{t}{2} \left(2\sin^2 v\frac{t}{2} \right) + \sin v\frac{t}{2} . \sin\frac{t}{2} \right) \right\} \right|$$

$$\leq \frac{\pi}{2\pi(1+q)^n t} \left| \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} P_v \right\} \right| = \frac{1}{2(1+q)^n t} \left| \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} P_v \right\} \right|$$

$$= \frac{1}{2(1+q)^n t} \left| \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \right|$$

$$= O\left(\frac{1}{t} \right).$$

This proves the lemma.

5. Proof of theorem 3.1

Using Riemann –Lebesgue theorem, we have for the n-th partial sum $s_n\left(\overline{f};x\right)$ of the conjugate Fourier series (1.10) of $f(x)$,following Titchmarch [4]

$$\overline{s_n}(f;x) - \overline{f}(x) = \frac{2}{\pi} \int_0^\pi \psi(t) \, \overline{K_n} \, dt$$

the (N, pn) transform of $\overline{s_n}(f;x)$ using (1.2) is given by

$$t_n - \overline{f}(x) = \frac{2}{\pi P_n} \int_0^\pi \psi(t) \sum_{k=0}^{n} P_k \frac{\cos\frac{t}{2} - \sin\left(n + \frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)} dt$$

denoting the $(E, q)(N, pn)$ transform of $\overline{s_n}(f;x)$ by τ_n , we have

$$\|\tau_n - \overline{f}\| = \frac{2}{\pi(1+q)^n} \int_0^\pi \psi(t) \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{v=0}^{k} P_v \frac{\cos\frac{t}{2} - \sin\left(v + \frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)} \right\} dt$$

$$= \int_0^\pi \psi(t) \, \overline{K_n}(t) dt$$

$$= \left\{ \int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^\pi \right\} \psi(t) \, \overline{K_n}(t) \, dt$$

$$= I_1 + I_2 \text{ , say} \tag{5.1}$$

Now

$$|I_1| = \frac{2}{\pi(1+q)^n} \left| \int_0^{1/n+1} \psi(t) \sum_{k=0}^n \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\upsilon=0}^k P_\upsilon \frac{\cos\frac{t}{2} - \cos\left(\upsilon + \frac{1}{2}\right)t}{2\sin\frac{t}{2}} \right\} dt \right|$$

$$\leq \left| \int_0^{\frac{1}{n+1}} \psi(t)\, \overline{K_n}(t)\, dt \right|$$

$$= \left(\int_0^{\frac{1}{n+1}} (\psi(t))^r\, dt \right)^{\frac{1}{r}} \left(\int_0^{\frac{1}{n+1}} \left(\overline{K_n}(t)\right)^s dt \right)^{\frac{1}{s}} \quad \text{, using Holder's inequality}$$

$$\leq O\left(\frac{1}{(n+1)^\alpha}\right) \left(\int_0^{\frac{1}{n+1}} n^s\, dt \right)^{\frac{1}{s}} \qquad \text{using Lemma 4.1}$$

$$= O\left(\frac{1}{(n+1)^\alpha}\right) \left(\frac{n^s}{s+1} \right)^{\frac{1}{s}}$$

$$= O\left(\frac{n}{(n+1)^{\alpha+\frac{1}{s}}}\right) .$$

$$= O\left(\frac{1}{(n+1)^{\alpha+\frac{1}{s}-1}}\right) = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right) \tag{5.2}$$

Next

$$|I_2| \leq \left(\int_{\frac{1}{n+1}}^{\pi} (\psi(t))^r\, dt \right)^{\frac{1}{r}} \left(\int_{\frac{1}{n+1}}^{\pi} \left(\overline{K_n}(t)\right)^s dt \right)^{\frac{1}{s}} \qquad \text{using Holder's inequality}$$

$$\leq O\left(\frac{1}{(n+1)^\alpha}\right) \left(\int_{\frac{1}{n+1}}^{\pi} \left(\frac{1}{t}\right)^s dt \right)^{\frac{1}{s}} \qquad \text{using Lemma 4.2}$$

$$= O\left(\frac{1}{(n+1)^\alpha}\right) \left(\frac{1}{n+1} \right)^{\frac{1-s}{s}}$$

$$= O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right) \tag{5.3}$$

Then from (5.2) and (5.3) , we have

$$\left|\tau_n - f(x)\right| = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right) , \text{ for } 0 < \alpha < 1, r \geq 1.$$

Hence,

$$\left\|\tau_n - f(x)\right\|_\infty = \sup_{-\pi<x<\pi}\left|\tau_n - f(x)\right| = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right), 0 < \alpha < 1, r \geq 1$$

This completes the proof of the theorem.

References

[1]   Hardy, G.H: Divergent series, First edition, Oxford University press 70(19).

[2]   Misra, U.K., Misra, M., Padhy, B.P. and Buxi, S.K: On degree of approximation by product means of conjugate series of Fourier series", International Jour. of Math. Sciences, and Engineering Applications, ISSN 0973 – 9424, Vol.6 No.122 (Jan. 2012), pp 363 – 370.

[3]   Nigam, H.K and Ajay Sharma: On degree of Approximation by product means, Ultra Scientist of Physical Sciences, Vol.22 (3) M, 889-894, (2010).

[4]   Titchmarch, E.C: The theory of functions, oxford university press, p.p402-403(1939).

[5]   Zggmund , A : Trigonometric Series , second Edition ,Vol.I , Cambridge University press , Cambridge , (1959).