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ON DEGREE OF APPROXIMATION BY PRODUCT MEANS OF CONJUGATE SERIES OF A FOURIER SERIES

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Abstract: In this paper a theorem on degree of approximation of a function $f \in Lip(\alpha\ , r)$ by product summability $(E, q\ )(\overline{N}\ , p_n)$ of conjugate series of Fourier series associated with f has been established.

1. Introduction

Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of positive real numbers such that

$$P_n = \sum_{\upsilon=0}^{n} p_\upsilon \to \infty, \qquad \text{as} \qquad n \to \infty\ ,\ (P_{-i} = p_{-i} = 0\ \ , i \geq 0). \tag{1.1}$$

The sequence –to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{\upsilon=0}^{n} p_\upsilon\, s_\upsilon \tag{1.2}$$

defines the sequence $\{\ t_n\ \}$ of the $(\overline{N}\ , p_n)$ -mean of the sequence $\{s_n\}$ generated by the sequence of coefficient $\{p_n\}$. If

$$t_n \to s \quad ,\text{as} \quad n \to \infty \tag{1.3}$$

then the series $\Sigma\ a_n$ is said to be $(\overline{N\ , p_n}\ )$ summable to s.
The conditions for regularity of $(\overline{N}\ , p_n)$- summability are easily seen to be[1]

$$\begin{cases} (i)\ P_n \to \infty, as\ n \to \infty. \\ (ii)\ \sum_{i=0}^{n} p_i \leq C|P_n|\ , as\ n \to \infty. \end{cases} \tag{1.4}$$

The sequence –to-sequence transformation, [1]

$$T_n = \frac{1}{(1+q)^n} \sum_{\upsilon=0}^{n} \binom{n}{\upsilon} q^{n-\upsilon}\, s_\upsilon \tag{1.5}$$

defines the sequence $\{T_n\ \}$ of the $(E\ , q\ )$ mean of the sequence $\{s_n\}$.
If

$$T_n \to s\ \ , \text{as} \quad n \to \infty\ , \tag{1.6}$$

then the series $\sum a_n$ is said to be $(E\ , q\ )$ summable to s .

Clearly (E, q) method is regular. Further, the (E, q) transform of the $(\underline{N}, p_n)$ transform of $\{s_n\}$ is defined by

$$\tau_n = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\, t_k$$

$$= \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k} p_\upsilon s_\upsilon\right\} \tag{1.7}$$

If

$$\tau_n \to s \text{ , as } n \to \infty, \tag{1.8}$$

then $\sum a_n$ is said to be $(E, q)(N, p_n)$-summable to s.

Let $f(t)$ be a periodic function with period 2π and L-integrable over $(-\pi,\pi)$. The Fourier series associated with f at any point x is defined by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right) \equiv \sum_{n=0}^{\infty} A_n(x) \tag{1.9}$$

and the conjugate series of the Fourier series (1.9) is

$$\sum_{n=1}^{\infty}\left(b_n \cos nx - a_n \sin nx\right) \equiv \sum_{n=1}^{\infty} B_n(x) \tag{1.10}$$

Let $\overline{s}_n(f; x)$ be the n-th partial sum of (1.10). The $L\infty$ -norm of a function $f: R \to R$ is defined by

$$\|f\|_\infty = \sup\left\{|f(x)| : x \in R\right\} \tag{1.11}$$

and the L_v -norm is defined by

$$\|f\|_\upsilon = \left(\int_0^{2\pi} |f(x)|^\upsilon\right)^{\frac{1}{\upsilon}} \quad , \upsilon \ge 1 \tag{1.12}$$

The degree of approximation of a function $f: R \to R$ by a trigonometric polynomial $P_n(x)$ of degree n under norm $\|\ \|_\infty$ is defined by [5].

$$\|P_n - f\|_\infty = \sup\left\{|p_n(x) - f(x)| : x \in R\right\} \tag{1.13}$$

and the degree of approximation $E_n(f)$ of a function $f \in L_v$ is given by

$$E_n(f) = \min_{P_n} \|P_n - f\|_\upsilon \tag{1.14}$$

A function f is said to satisfy Lipschitz condition (here after we write $f \in Lip\alpha$) if

$$|f(x+t) - f(x)| = O\left(|t|^\alpha\right) , 0 < \alpha \le 1 \tag{1.15}$$

and $f(x) \in Lip(a, r)$, for $0 \le x \le 2\pi$, if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^r dx\right)^{\frac{1}{r}} = O\left(|t|^\alpha\right) , 0 < \alpha \le 1,\ r \ge 1,\ t > 0 \tag{1.16}$$

We use the following notation throughout this paper :

$$\psi(t) = \frac{1}{2}\{f(x+t) - f(x-t)\}, \tag{1.17}$$

and

$$\overline{K_n}(t) = \frac{1}{\pi(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}p_\upsilon \frac{\cos\frac{t}{2} - \cos\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\} \tag{1.18}$$

Further, the method *(E, q)(*$\overline{N}$*, p_n)* is assumed to be regular .

2. Known Theorem

Dealing with The degree of approximation by the product (E, q) (C ,1) -mean of Fourier series, Nigam et al [3] proved the following theorem:

Theorem 2.1. If a function f is 2π - periodic and belonging to class $Lip\alpha$, then its degree of approximation by *(E, q) (C ,1)* summability mean on its Fourier series $\sum_{n=0}^{\infty} A_n(t)$ is given by $\|E_n^q C_n^1 - f\|_\infty = O\left(\frac{1}{(n+1)^\alpha}\right), 0<\alpha<1$ where $E_n^q C_n^1$ represents the *(E , q)*transform of (C ,1) transform of $s_n\,(f\,;x)$.

Recently, Misra et al [2] proved the following theorem using *(E, q)(N , p_n)* mean of conjugate series of the Fourier series :

Theorem 2.2. If f be 2π - Periodic function of class $Lip\alpha$, then degree of approximation by the product *(E, q)(*$\overline{N}$*, p_n)* summability means of the conjugate series (1.10) of Fourier series (1.9) is given $\|\tau_n - f\|_\infty = O\left(\frac{1}{(n+1)^\alpha}\right), 0<\alpha<1$ where τ_n is as defined in 1.7) .

3. Main theorem

In this paper, we have proved a theorem on degree of approximation by the product mean *(E, q)(*$\overline{N}$*, p)* of conjugate series of Fourier series of a function of class *Lip (a , r)* . We prove:

Theorem.3.1. If f be 2π - Periodic function of class Lipa , then degree of approximation by the product *(E, q)(*$\overline{N}$*, p_n)* summability means of the conjugate series of Fourier series (1.10) is given by $\|\tau_n - f\|_\infty = O\left(\frac{1}{(n+1)^{\alpha-1/r}}\right), 0<\alpha<1$ where τ is as defined in (1.7) .

4. Required Lemmas

We require the following Lemmas to prove the theorem.

Lemma 4.1

$$\left|\overline{K_n}(t)\right| = O(n) \qquad ,0 \le t \le \frac{1}{n+1}$$

Proof.

For $0 \le t \le \frac{1}{n+1}$ we have $\sin nt \le n\sin t$ then

$$\left|\overline{K_n}(t)\right| = \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k} p_\upsilon \frac{\cos\frac{t}{2}-\cos\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right|$$

$$\le \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k} p_\upsilon \frac{\cos\frac{t}{2}-\cos\upsilon t.\cos\frac{t}{2}+\sin\upsilon t.\sin\frac{t}{2}}{\sin\frac{t}{2}}\right\}\right|$$

$$\le \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k} p_\upsilon \left(\frac{\cos\frac{t}{2}\left(2\sin^2\upsilon\frac{t}{2}\right)}{\sin\frac{t}{2}}+\sin\upsilon t\right)\right\}\right|$$

$$\le \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k} p_\upsilon \left(O\left(2\sin\upsilon\frac{t}{2}\,\sin\upsilon\frac{t}{2}\right)+\upsilon\sin t\right)\right\}\right|$$

$$\le \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k} p_\upsilon (O(\upsilon)+O(\upsilon))\right\}\right|$$

$$\le \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\frac{O(k|)}{P_k}\sum_{\upsilon=0}^{k} p_\upsilon\right|$$

$$= O(n)$$

This proves the lemma.

Lemma 4.2

$$\left|\overline{K_n}(t)\right| = O\left(\frac{1}{t}\right),\ for\ \ \frac{1}{n+1} \le t \le \pi$$

Proof.

For $\frac{1}{n+1} \le t \le \pi$, by Jordan's lemma, we have $\sin\left(\frac{t}{2}\right) \ge \frac{t}{\pi}$

Then

$$\left|\overline{K_n}(t)\right| = \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k} p_\upsilon \frac{\cos\frac{t}{2}-\cos\left(\upsilon+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right\}\right|$$

$$= \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k} p_\upsilon \frac{\cos\frac{t}{2}-\cos\upsilon\frac{t}{2}.\cos\frac{t}{2}+\sin\upsilon\frac{t}{2}.\sin\frac{t}{2}}{\sin\frac{t}{2}}\right\}\right|$$

$$\leq \frac{1}{\pi(1+q)^n}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k}\frac{\pi}{2t} p_\upsilon\left(\cos\frac{t}{2}\left(2\sin^2 \upsilon\frac{t}{2}\right)+\sin\upsilon\frac{t}{2}.\sin\frac{t}{2}\right)\right\}\right|$$

$$\leq \frac{\pi}{2\pi(1+q)^n t}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k} p_\upsilon\right\}\right| = \frac{1}{2(1+q)^n t}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k} p_\upsilon\right\}\right|$$

$$= \frac{1}{2(1+q)^n t}\left|\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\right|$$

$$= O\left(\frac{1}{t}\right).$$

This proves the lemma.

5. Proof of theorem 3.1

Using Riemann –Lebesgue theorem, we have for the n-th partial sum $s_n(\overline{f}; x)$ of the conjugate Fourier series (1.10) of *f (x)* ,following Titchmarch [4]

$$\overline{s_n}(f;x) - f(x) = \frac{2}{\pi}\int_0^{\pi} \psi(t)\, \overline{K_n}\, dt$$

the (N, pn) transform of $\overline{s_n}(f; x)$ using (1.2) is given by

$$t_n - f(x) = \frac{2}{\pi P_n}\int_0^{\pi} \psi(t) \sum_{k=0}^{n} p_k \frac{\cos\frac{t}{2} - \sin\left(n+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}\, dt$$

denoting the $(E, q)(N, pn)$ transform of $\overline{s_n}(f; x)$ by τ_n , we have

$$\|\tau_n - f\| = \frac{2}{\pi(1+q)^n}\int_0^{\pi} \psi(t) \sum_{k=0}^{n}\binom{n}{k} q^{n-k}\left\{\frac{1}{P_k}\sum_{\upsilon=0}^{k} p_\upsilon \frac{\cos\frac{t}{2} - \sin\left(\upsilon+\frac{1}{2}\right)t}{2\sin\left(\frac{t}{2}\right)}\right\} dt$$

$$= \int_0^{\pi} \psi(t)\, \overline{K_n}(t)\, dt$$

$$= \left\{\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right\} \psi(t)\, \overline{K_n}(t)\, dt$$

$= I_1 + I_2$, say (5.1)

Now

$$|I_1| = \frac{2}{\pi(1+q)^n}\left| \int_0^{1/n+1} \psi(t) \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\upsilon=0}^{k} p_\upsilon \frac{\cos\frac{t}{2} - \cos\left(\upsilon + \frac{1}{2}\right)t}{2\sin\frac{t}{2}} \right\} dt \right|$$

$$\leq \left| \int_0^{\frac{1}{n+1}} \psi(t)\ \overline{K_n}(t)\, dt \right|$$

$$= \left(\int_0^{\frac{1}{n+1}} (\psi(t))^r\, dt \right)^{\frac{1}{r}} \left(\int_0^{\frac{1}{n+1}} (\overline{K}_n(t))^s\, dt \right)^{\frac{1}{s}} \text{, using Holder's inequality}$$

$$\leq O\left(\frac{1}{(n+1)^\alpha}\right) \left(\int_0^{\frac{1}{n+1}} n^s\, dt \right)^{\frac{1}{s}} \qquad \text{using Lemma 4.1}$$

$$= O\left(\frac{1}{(n+1)^\alpha}\right) \left(\frac{n^s}{s+1} \right)^{\frac{1}{s}}$$

$$= O\left(\frac{n}{(n+1)^{\alpha+\frac{1}{s}}}\right).$$

$$= O\left(\frac{1}{(n+1)^{\alpha+\frac{1}{s}-1}}\right) = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right) \tag{5.2}$$

Next

$$|I_2| \leq \left(\int_{\frac{1}{n+1}}^{\pi} (\psi(t))^r\, dt \right)^{\frac{1}{r}} \left(\int_{\frac{1}{n+1}}^{\pi} (\overline{K}_n(t))^s\, dt \right)^{\frac{1}{s}} \qquad \text{using Holder's inequality}$$

$$\leq O\left(\frac{1}{(n+1)^\alpha}\right) \left(\int_{\frac{1}{n+1}}^{\pi} \left(\frac{1}{t}\right)^s dt \right)^{\frac{1}{s}} \qquad \text{using Lemma 4.2}$$

$$= O\left(\frac{1}{(n+1)^\alpha}\right) \left(\frac{1}{n+1} \right)^{\frac{1-s}{s}}$$

$$= O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right) \tag{5.3}$$

Then from (5.2) and (5.3) , we have

$$\left|\tau_n - f(x)\right| = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right) \text{ , for } 0<\alpha<1,\ r\geq 1 .$$

Hence,

$$\left\|\tau_n - f(x)\right\|_\infty = \sup_{-\pi<x<\pi}\left|\tau_n - f(x)\right| = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right) \text{ ,} 0<\alpha<1,\ r\geq 1$$

This completes the proof of the theorem.

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