ON DEGREE OF APPROXIMATION BY PRODUCT MEANS OF CONJUGATE SERIES OF A FOURIER SERIES

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Keywords: Degree of Approximation $f \in Lip(\alpha, r)$ class of function, $(E, q)$ mean, $(\overline{N}, p_n)$ product mean, Fourier series, Conjugate of the Fourier series, Lebesgue integral

Abstract: In this paper a theorem on degree of approximation of a function $f \in Lip(\alpha, r)$ by product summability $(E, q)(\overline{N}, p_n)$ of conjugate series of Fourier series associated with $f$ has been established.

1. Introduction

Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of positive real numbers such that

$$P_n = \sum_{v=0}^{n} p_v \rightarrow \infty, \quad \text{as} \quad n \rightarrow \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 0).$$

(1.1)

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

(1.2)

defines the sequence $\{t_n\}$ of the $(\overline{N}, p_n)$-mean of the sequence $\{s_n\}$ generated by the sequence of coefficient $\{p_n\}$. If

$$t_n \rightarrow s \quad \text{as} \quad n \rightarrow \infty$$

(1.3)

then the series $\sum a_n$ is said to be $(\overline{N}, p_n)$ summable to $s$.

The conditions for regularity of $(\overline{N}, p_n)$-summability are easily seen to be [1]

$$\begin{align*}
(i) & \quad P_n \rightarrow \infty, \quad \text{as} \quad n \rightarrow \infty. \\
(ii) & \quad \sum_{i=0}^{n} p_i \leq C |P_n|, \quad \text{as} \quad n \rightarrow \infty.
\end{align*}$$

(1.4)

The sequence-to-sequence transformation, [1]

$$T_n = \frac{1}{(1+q)^n} \sum_{v=0}^{n} \left(\begin{array}{c} n \\ v \end{array}\right) q^{n-v} s_v$$

(1.5)

defines the sequence $\{T_n\}$ of the $(E, q)$ mean of the sequence $\{s_n\}$. If

$$T_n \rightarrow s \quad \text{as} \quad n \rightarrow \infty,$$

(1.6)

then the series $\sum a_n$ is said to be $(E, q)$ summable to $s$. 

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Clearly \((E, q)\) method is regular. Further, the \((E, q)\) transform of the \((N, p_n)\) transform of \(\{s_n\}\) is defined by

\[
\tau_n = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} t_k
\]

\[
= \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{-k} \left\{ \frac{1}{p_k} \sum_{\nu=0}^{k} p_{\nu} s_{\nu} \right\}
\]

If

\[
\tau_n \to s, \quad \text{as} \quad n \to \infty,
\]

then \(\sum a_n\) is said to be \((E, q)(N, p_n)\)-summable to \(s\).

Let \(f(t)\) be a periodic function with period \(2\pi\) and \(L\)-integrable over \((-\pi, \pi)\). The Fourier series associated with \(f\) at any point \(x\) is defined by

\[
f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right) \equiv \sum_{n=0}^{\infty} A_n(x)
\]

and the conjugate series of the Fourier series (1.9) is

\[
\sum_{n=1}^{\infty} \left( b_n \cos nx - a_n \sin nx \right) = \sum_{n=1}^{\infty} B_n(x)
\]

Let \(s_n(f; x)\) be the \(n\)-th partial sum of (1.10). The \(L^\infty\) norm of a function \(f : R \to R\) is defined by

\[
\|f\|_{\infty} = \sup \{|f(x)| : x \in R\}
\]

and the \(L_v\) norm is defined by

\[
\|f\|_{L_v} = \left( \frac{2\pi}{0} \int |f(x)|^v dx \right)^{1/v}, \quad v \geq 1
\]

The degree of approximation of a function \(f : R \to R\) by a trigonometric polynomial \(P_n(x)\) of degree \(n\) under norm \(\|\|\) is defined by [5].

\[
\|P_n - f\|_{\infty} = \sup \{|P_n(x) - f(x)| : x \in R\}
\]

and the degree of approximation \(E_n(f)\) of a function \(f \in L_v\) is given by

\[
E_n(f) = \min \|P_n - f\|_{L_v}
\]

A function \(f\) is said to satisfy Lipschitz condition (hereafter we write \(f \in Lip\)) if

\[
|f(x+t) - f(x)| = O\left(|t|^\alpha\right), 0 < \alpha \leq 1
\]

and \(f(x) \in Lip(a, r)\), for \(0 \leq x \leq 2\pi\), if

\[
\left( \int_0^{2\pi} |f(x+t) - f(x)|^r dx \right)^{1/r} = O\left(|t|^\alpha\right), 0 < \alpha \leq 1, r \geq 1, t > 0
\]
We use the following notation throughout this paper:

\[ \psi(t) = \frac{1}{2} \{ f(x+t) - f(x-t) \}, \quad (1.17) \]

and

\[ K_n(t) = \frac{1}{n} \sum_{k=0}^{n-1} \left( \frac{n}{k} \right) \left( \frac{q}{n} \right)^{n-k} \left( \frac{1}{p_n} \sum_{v=0}^{n-1} \frac{\cos \left( \frac{t}{2} \right) - \cos \left( \frac{v+1}{2} \right) t}{\sin \left( \frac{t}{2} \right)} \right), \quad (1.18) \]

Further, the method \((E, q ) (N , p_n)\) is assumed to be regular.

2. Known Theorem

Dealing with The degree of approximation by the product \((E, q ) (C ,1)\) -mean of Fourier series, Nigam et al [3] proved the following theorem:

**Theorem 2.1.** If a function \(f\) is \(2\pi\) - periodic and belonging to class \(Lipa\), then its degree of approximation by \((E, q ) (C ,1)\) summability mean on its Fourier series is given by

\[ \| E_n^q C_n^1 f \|_\infty = O \left( \frac{1}{(n+1)^\alpha} \right), \quad 0 < \alpha < 1 \]

where \(E_n^q C_n^1\) represents the \((E, q )\) transform of \((C ,1)\) transform of \(s_n(f; x)\).

Recently, Misra et al [2] proved the following theorem using \((E, q ) (N , p_n)\) mean of conjugate series of the Fourier series:

**Theorem 2.2.** If \(f\) be \(2\pi\) - Periodic function of class \(Lipa\), then degree of approximation by the product \((E, q ) (N , p_n)\) summability means of the conjugate series \((1.10)\) of Fourier series \((1.9)\) is given by

\[ \| \tau_n - f \|_\infty = O \left( \frac{1}{(n+1)^\alpha} \right), \quad 0 < \alpha < 1 \]

where \(\tau_n\) is as defined in \((1.7)\).

3. Main theorem

In this paper, we have proved a theorem on degree of approximation by the product mean \((E, q ) (N , p )\) of conjugate series of Fourier series of a function of \(Lip (a , r )\). We prove:

**Theorem 3.1.** If \(f\) be \(2\pi\) - Periodic function of class \(Lipa\), then degree of approximation by the product \((E, q ) (N , p_n)\) summability means of the conjugate series of Fourier series \((1.10)\) is given by

\[ \| \tau_n - f \|_\infty = O \left( \frac{1}{(n+1)^\alpha} \right), \quad 0 < \alpha < 1 \]

where \(\tau\) is as defined in \((1.7)\).

4. Required Lemmas

We require the following Lemmas to prove the theorem.

**Lemma 4.1**

\[ \left| K_n(t) \right| = O(n) \quad 0 \leq t \leq \frac{1}{n+1} \]
Proof.

For $0 \leq t \leq \frac{1}{n+1}$ we have $\sin nt \leq nsin t$ then

\[
|K_n(t)| = \frac{1}{\pi(1+q)} \sum_{k=0}^{n} \binom{n}{k} q^{-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^{k} P_{\nu} \frac{\cos \frac{t}{2} - \cos \left( \nu + \frac{1}{2} \right) t}{\sin \frac{t}{2}} \right\}
\]

\[
\leq \frac{1}{\pi(1+q)} \sum_{k=0}^{n} \binom{n}{k} q^{-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^{k} P_{\nu} \frac{\cos \frac{t}{2} - \cos \nu t \cos \frac{t}{2} + \sin \nu t \sin \frac{t}{2}}{\sin \frac{t}{2}} \right\}
\]

\[
\leq \frac{1}{\pi(1+q)} \sum_{k=0}^{n} \binom{n}{k} q^{-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^{k} P_{\nu} \left( \frac{2 \sin \nu \frac{t}{2}}{\sin \frac{t}{2}} \right) \right\}
\]

\[
\leq \frac{1}{\pi(1+q)} \sum_{k=0}^{n} \binom{n}{k} q^{-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^{k} P_{\nu} \left( O(\nu) + O(\nu) \right) \right\}
\]

\[
\leq \frac{1}{\pi(1+q)} \sum_{k=0}^{n} \binom{n}{k} q^{-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^{k} P_{\nu} O(k) \right\}
\]

\[
= O(n)
\]

This proves the lemma.

Lemma 4.2

\[
|K_n(t)| = O\left(\frac{1}{t}\right), \text{ for } \frac{1}{n+1} \leq t \leq \pi
\]

Proof.

For $\frac{1}{n+1} \leq t \leq \pi$, by Jordan’s lemma, we have $\sin \left( \frac{t}{2} \right) \geq \frac{t}{\pi}$

Then

\[
|K_n(t)| = \frac{1}{\pi(1+q)} \sum_{k=0}^{n} \binom{n}{k} q^{-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^{k} P_{\nu} \frac{\cos \frac{t}{2} - \cos \left( \nu + \frac{1}{2} \right) t}{\sin \frac{t}{2}} \right\}
\]

\[
= \frac{1}{\pi(1+q)} \sum_{k=0}^{n} \binom{n}{k} q^{-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^{k} P_{\nu} \left( \frac{\cos \frac{t}{2} - \cos\nu \frac{t}{2} \cos \frac{t}{2} + \sin\nu \frac{t}{2} \sin \frac{t}{2}}{\sin \frac{t}{2}} \right) \right\}
\]
This proves the lemma.

5. Proof of theorem 3.1

Using Riemann–Lebesgue theorem, we have for the n-th partial sum $s_n (f; x)$ of the conjugate Fourier series (1.10) of $f(x)$, following Titchmarsh [4]

$$\overline{s_n (f; x)} - f(x) = 2 \pi \int_0^\pi \psi(t) \overline{K_n} \ dt$$

the $(N, p_n)$ transform of $\overline{s_n (f; x)}$ using (1.2) is given by

$$t_n - f(x) = \frac{2}{\pi P_n} \int_0^\pi \psi(t) \sum_{k=0}^n p_k \left[ \cos \frac{t}{2} - \sin \frac{n+1}{2} \right] \ dt$$

denoting the $(E, q)(N, p_n)$ transform of $\overline{s_n (f; x)}$ by $\tau_n$, we have

$$\|\tau_n - f\| = \frac{2}{\pi (1 + q)} \int_0^\pi \psi(t) \sum_{k=0}^n \left( \frac{n}{k} \right) q^{n-k} \left[ \frac{1}{P_k} \sum_{\nu=0}^k p_\nu \frac{\cos \frac{t}{2} - \sin \frac{\nu+1}{2} \right] \ dt$$

$$= \int_0^\pi \psi(t) \overline{K_n}(t) \ dt$$

$$= \int_0^{n+1} + \int_1^{\pi} \psi(t) \overline{K_n}(t) \ dt$$

$$= I_1 + I_2, \text{ say}$$

(5.1)
Now

\[
|I_1| = \frac{2}{\pi (1 + q)^{\frac{n+1}{r}}} \left| \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left( \frac{1}{p_k} \sum_{\nu=0}^{k} p_{\nu} \frac{\cos \frac{t}{2} - \cos \left( \nu + \frac{1}{2} \right)}{2 \sin \frac{t}{2}} \right) \right| dt
\]

\[
\leq \left| \int_0^{\pi} \psi(t) \overline{K_n(t)} dt \right|
\]

\[
= \left( \int_0^{\frac{1}{n+1}} (\psi(t))^s dt \right)^{\frac{1}{r}} \left( \int_0^{\frac{1}{n+1}} (\overline{K_n(t)})^{-\frac{s}{r}} dt \right)^{-\frac{1}{s}}, \text{ using Holder’s inequality}
\]

\[
\leq O\left( \frac{1}{(n+1)^{\alpha-1}} \right) \left( \int_0^{\frac{1}{n+1}} n^2 dt \right)^{\frac{1}{s}} \quad \text{using Lemma 4.1}
\]

\[
= O\left( \frac{n}{(n+1)^{\alpha-1}} \right) = O\left( \frac{1}{(n+1)^{\alpha-1}} \right)
\]

(5.2)

Next

\[
|I_2| \leq \left( \int_{\frac{\pi}{n+1}}^{\pi} (\psi(t))^s dt \right)^{\frac{1}{r}} \left( \int_{\frac{\pi}{n+1}}^{\pi} (\overline{K_n(t)})^{-\frac{s}{r}} dt \right)^{\frac{1}{s}} \quad \text{using Holder’s inequality}
\]

\[
\leq O\left( \frac{1}{(n+1)^{\alpha}} \right) \left( \int_{\frac{1}{n+1}}^{\frac{1}{n+1}} \left( \frac{1}{t} \right)^s dt \right)^{\frac{1}{s}} \quad \text{using Lemma 4.2}
\]

\[
= O\left( \frac{1}{(n+1)^{\alpha}} \right) \left( \frac{1}{n+1} \right)^{\frac{1-s}{s}} \quad \text{using Lemma 4.2}
\]

\[
= O\left( \frac{1}{(n+1)^{\alpha-1}} \right)
\]

(5.3)
Then from (5.2) and (5.3), we have

\[ |r_n - f(x)| = O\left( \frac{1}{(n+1)^{\frac{\alpha}{r}}} \right), \text{ for } 0 < \alpha < 1, \ r \geq 1. \]

Hence,

\[ \|r_n - f(x)\|_{x} = \sup_{-\pi < x < \pi} |r_n - f(x)| = O\left( \frac{1}{(n+1)^{\frac{\alpha}{r}}} \right), 0 < \alpha < 1, \ r \geq 1 \]

This completes the proof of the theorem.

References


