ON DEGREE OF APPROXIMATION BY PRODUCT MEANS OF CONJUGATE SERIES OF A FOURIER SERIES

S. K. Paikray¹, R. K. Jati², N. C. Sahoo³, U.K. Misra⁴

¹P.G. Department of Mathematics, Ravenshaw University, Cuttack-753003, Odisha, India
²Department of Mathematics, DRIEMS, Tangi, Cuttack, Odisha, India
³Department of Mathematics, S.B. Women’s College (Auto), Cuttack, Odisha, India
⁴Science and Technology, Pallur Hills, Golanathara-761008, Odisha, India

Keywords: Degree of Approximation \( f \in \text{Lip}(\alpha, r) \) class of function, \((E, q)\) mean, \((\hat{N}, p)\) mean, \((E, q)(\hat{N}, p)\) product mean, Fourier series, Conjugate of the Fourier series, Lebesgue integral

Abstract: In this paper a theorem on degree of approximation of a function \( f \in \text{Lip}(\alpha, r) \) by product summability \((E, q)(\hat{N}, p)\) of conjugate series of Fourier series associated with \( f \) has been established.

1. Introduction

Let \( \sum a_n \) be a given infinite series with the sequence of partial sums \( \{ s_n \} \). Let \( \{ p_n \} \) be a sequence of positive real numbers such that

\[
P_n = \sum_{v=0}^{n} p_v \rightarrow \infty, \quad \text{as} \quad n \rightarrow \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 0).
\]

(1.1)

The sequence \( t_n \)-to-sequence transformation

\[
t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v
\]

(1.2)

defines the sequence \( \{ t_n \} \) of the \((\hat{N}, p)\)-mean of the sequence \( \{ s_n \} \) generated by the sequence of coefficient \( \{ p_n \} \). If

\[
t_n \rightarrow s, \quad \text{as} \quad n \rightarrow \infty
\]

(1.3)

then the series \( \Sigma a_n \) is said to be \((\hat{N}, p)\) summable to \( s \).

The conditions for regularity of \((\hat{N}, p)\)-summability are easily seen to be \([1]\)

\[
(i) \quad P_n \rightarrow \infty, \quad \text{as} \quad n \rightarrow \infty.
\]

\[
(ii) \quad \sum_{i=0}^{n} |P_i| \leq CLP_n, \quad \text{as} \quad n \rightarrow \infty.
\]

(1.4)

The sequence \( t_n \)-to-sequence transformation, \([1]\)

\[
T_n = \frac{1}{(1+q)^n} \sum_{v=0}^{n} \left( \frac{n}{v} \right) q^{n-v} s_v
\]

(1.5)

defines the sequence \( \{ T_n \} \) of the \((E, q)\)-mean of the sequence \( \{ s_n \} \).

If

\[
T_n \rightarrow s, \quad \text{as} \quad n \rightarrow \infty,
\]

(1.6)

then the series \( \Sigma a_n \) is said to be \((E, q)\) summable to \( s \).
Clearly \((E, q)\) method is regular. Further, the \((E, q)\) transform of the \((N, p_n)\) transform of \(\{s_n\}\) is defined by

\[
\tau_n = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} t_k
\]

Further, the \((E, q)\) transform of the \((N, p_n)\) transform of \(\{s_n\}\) is defined by

\[
(1.7)
\]

If

\[
\tau_n \rightarrow s, \quad n \rightarrow \infty,
\]

then \(\sum a_n\) is said to be \((E, q)(N, p_n)\)-summable to \(s\).

Let \(f(t)\) be a periodic function with period \(2\pi\) and \(L\)-integrable over \((-\pi, \pi)\). The Fourier series associated with \(f\) at any point \(x\) is defined by

\[
f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right) \equiv \sum_{n=0}^{\infty} A_n(x)
\]

and the conjugate series of the Fourier series (1.9) is

\[
\sum_{n=1}^{\infty} \left( b_n \cos nx - a_n \sin nx \right) = \sum_{n=1}^{\infty} B_n(x)
\]

Let \(s_n(f; x)\) be the \(n\)-th partial sum of (1.10). The \(L^\infty\)-norm of a function \(f: R \rightarrow R\) is defined by

\[
\|f\|_{\infty} = \sup \{f(x) : x \in R\}
\]

and the \(L^\nu\)-norm is defined by

\[
\|f\|_{\nu} = \left( \int_0^{2\pi} |f(x)|^\nu \, dx \right)^{\frac{1}{\nu}}, \quad \nu \geq 1
\]

The degree of approximation of a function \(f: R \rightarrow R\) by a trigonometric polynomial \(P_n(x)\) of degree \(n\) under norm \(\|\|\) is defined by [5].

\[
\|P_n - f\|_{\infty} = \sup \{|P_n(x) - f(x)| : x \in R\}
\]

and the degree of approximation \(E_n(f)\) of a function \(f \in L^\nu\) is given by

\[
E_n(f) = \min_{P_n} \|P_n - f\|_{\nu}
\]

A function \(f\) is said to satisfy Lipschitz condition (here after we write \(f \in Lipa\)) if

\[
|f(x + t) - f(x)| = O\left( t^\alpha \right), \quad 0 < \alpha \leq 1
\]

and \(f(x) \in Lip (a, r)\), for \(0 \leq x \leq 2\pi\), if

\[
\left( \int_0^{2\pi} |f(x + t) - f(x)|^r \, dx \right)^{\frac{1}{r}} = O\left( t^\alpha \right), \quad 0 < \alpha \leq 1, \, r \geq 1, \, t > 0
\]
We use the following notation throughout this paper:

\[
\psi(t) = \frac{1}{2} \{ f(x + t) - f(x - t) \},
\]

(1.17)

and

\[
\overline{K_n}(t) = \frac{1}{\pi(1 + q)^\nu} \sum_{k=0}^{n} \frac{1}{q^n} \sum_{v=0}^{k} \frac{\cos t}{2 + \cos \left( \frac{t + 1}{2} \right)} \frac{\sin \frac{t}{2}}{\sin \frac{t}{2}}
\]

(1.18)

Further, the method \((E, q)(N, p_n)\) is assumed to be regular.

2. Known Theorem

Dealing with The degree of approximation by the product \((E, q)(C, 1)\) -mean of Fourier series, Nigam et al [3] proved the following theorem:

**Theorem 2.1.** If a function \(f\) is \(2\pi\)-periodic and belonging to class \(Lip \alpha\), then its degree of approximation by \((E, q)(C, 1)\) summability mean on its Fourier series \(\sum A_n(t)\) is given by

\[
\left\| E_n^q C_n^1 - f \right\|_\infty = O\left( \frac{1}{(n+1)^\alpha} \right), \quad 0 < \alpha < 1
\]

where \(E_n^q C_n^1\) represents the \((E, q)\) transform of \((C, 1)\) transform of \(s_n(f; x)\).

Recently, Misra et al [2] proved the following theorem using \((E, q)(N, p_n)\) mean of conjugate series of the Fourier series:

**Theorem 2.2.** If \(f\) be \(2\pi\)-Periodic function of class \(Lip \alpha\), then degree of approximation by the product \((E, q)(N, p_n)\) summability means of the conjugate series (1.10) of Fourier series (1.9) is given

\[
\left\| \tau_n - f \right\|_\infty = O\left( \frac{1}{(n+1)^\alpha} \right), \quad 0 < \alpha < 1
\]

where \(\tau_n\) is as defined in (1.7).

3. Main Theorem

In this paper, we have proved a theorem on degree of approximation by the product mean \((E, q)(N, p)\) of conjugate series of Fourier series of a function of class \(Lip(\alpha, \beta)\). We prove:

**Theorem 3.1.** If \(f\) be \(2\pi\)-Periodic function of class \(Lip \alpha\), then degree of approximation by the product \((E, q)(N, p_n)\) summability means of the conjugate series of Fourier series (1.10) is given

\[
\left\| \tau_n - f \right\|_\infty = O\left( \frac{1}{(n+1)^{\alpha-1/\beta}} \right), \quad 0 < \alpha < 1
\]

by

where \(\tau\) is as defined in (1.7).

4. Required Lemmas

We require the following Lemmas to prove the theorem.

**Lemma 4.1**

\[
\left| K_n(t) \right| = O(n), \quad 0 \leq t \leq \frac{1}{n + 1}
\]
Proof.

For \( 0 \leq t \leq \frac{1}{n+1} \) we have \( \sin nt \leq n \sin t \) then

\[
|K_n(t)| = \frac{1}{\pi (1+q)^n} \left| \sum_{k=0}^{n} \binom{n}{k} q^{-k} \left( \frac{1}{P_k} \sum_{\upsilon=0}^{k} p_{\upsilon} \frac{\cos \frac{t}{2} - \cos \left( \upsilon + \frac{1}{2} \right) t}{\sin \frac{t}{2}} \right) \right|
\]

\[
\leq \frac{1}{\pi (1+q)^n} \left| \sum_{k=0}^{n} \binom{n}{k} q^{-k} \left( \frac{1}{P_k} \sum_{\upsilon=0}^{k} p_{\upsilon} \frac{\cos \frac{t}{2} - \cos \upsilon t \cos \frac{t}{2} + \sin \upsilon t \sin \frac{t}{2}}{\sin \frac{t}{2}} \right) \right|
\]

\[
\leq \frac{1}{\pi (1+q)^n} \left| \sum_{k=0}^{n} \binom{n}{k} q^{-k} \left( \frac{1}{P_k} \sum_{\upsilon=0}^{k} p_{\upsilon} \frac{\cos \frac{t}{2} \left( 2 \sin \frac{\upsilon}{2} \frac{t}{2} \right) + \sin \upsilon t}{\sin \frac{t}{2}} \right) \right|
\]

\[
\leq \frac{1}{\pi (1+q)^n} \left| \sum_{k=0}^{n} \binom{n}{k} q^{-k} \left( \frac{1}{P_k} \sum_{\upsilon=0}^{k} p_{\upsilon} \left( O\left( \frac{2 \sin \frac{\upsilon}{2} \frac{t}{2} \right) \right) + O(\upsilon) \right) \right) \right|
\]

\[
\leq \frac{1}{\pi (1+q)^n} \left| \sum_{k=0}^{n} \binom{n}{k} q^{-k} \frac{O(k)}{P_k} \sum_{\upsilon=0}^{k} P_{\upsilon} \right|
\]

\[
= O(n)
\]

This proves the lemma.

**Lemma 4.2**

\[
|K_n(t)| = O\left( \frac{1}{t} \right), \text{ for } \frac{1}{n+1} \leq t \leq \pi
\]

**Proof.**

For \( \frac{1}{n+1} \leq t \leq \pi \), by Jordan’s lemma, we have \( \sin \left( \frac{t}{2} \right) \geq \frac{t}{\pi} \)

Then

\[
|K_n(t)| = \frac{1}{\pi (1+q)^n} \left| \sum_{k=0}^{n} \binom{n}{k} q^{-k} \left( \frac{1}{P_k} \sum_{\upsilon=0}^{k} p_{\upsilon} \frac{\cos \frac{t}{2} - \cos \left( \upsilon + \frac{1}{2} \right) t}{\sin \frac{t}{2}} \right) \right|
\]

\[
= \frac{1}{\pi (1+q)^n} \left| \sum_{k=0}^{n} \binom{n}{k} q^{-k} \left( \frac{1}{P_k} \sum_{\upsilon=0}^{k} p_{\upsilon} \frac{\cos \frac{t}{2} - \cos \upsilon t \cos \frac{t}{2} + \sin \upsilon t \sin \frac{t}{2}}{\sin \frac{t}{2}} \right) \right|
\]

\[
= \frac{1}{\pi (1+q)^n} \left| \sum_{k=0}^{n} \binom{n}{k} q^{-k} \left( \frac{1}{P_k} \sum_{\upsilon=0}^{k} p_{\upsilon} \frac{\cos \frac{t}{2} \left( 2 \sin \frac{\upsilon}{2} \frac{t}{2} \right) + \sin \upsilon t}{\sin \frac{t}{2}} \right) \right|
\]

\[
\leq \frac{1}{\pi (1+q)^n} \left| \sum_{k=0}^{n} \binom{n}{k} q^{-k} \left( \frac{1}{P_k} \sum_{\upsilon=0}^{k} p_{\upsilon} \left( O\left( \frac{2 \sin \frac{\upsilon}{2} \frac{t}{2} \right) \right) + O(\upsilon) \right) \right) \right|
\]

\[
\leq \frac{1}{\pi (1+q)^n} \left| \sum_{k=0}^{n} \binom{n}{k} q^{-k} \frac{O(k)}{P_k} \sum_{\upsilon=0}^{k} P_{\upsilon} \right|
\]

\[
= O(n)
\]
This proves the lemma.

5. Proof of theorem 3.1

Using Riemann–Lebesgue theorem, we have for the n-th partial sum \( s_n(f; x) \) of the conjugate Fourier series (1.10) of \( f(x) \), following Titchmarsh [4]

\[
\overline{s_n(f; x)} = f(x) = \frac{2}{\pi} \int_0^\pi \psi(t) \overline{K_n} \, dt
\]

the \((N, p_n)\) transform of \( \overline{s_n(f; x)} \) using (1.2) is given by

\[
t_n - f(x) = \frac{2}{\pi P_n} \int_0^\pi \psi(t) \sum_{k=0}^n p_k \frac{\cos t - \sin \left( n + \frac{1}{2} \right) t}{2 \sin \left( \frac{t}{2} \right)} \, dt
\]

denoting the \((E, q)(N, p_n)\) transform of \( \overline{s_n(f; x)} \) by \( \tau_n \), we have

\[
\|\tau_n - f\| = \frac{2}{\pi (1 + q) P_n} \int_0^\pi \psi(t) \sum_{k=0}^n \left( \begin{array}{c} n \\ k \end{array} \right) q^{n-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^k p_\nu \frac{\cos \frac{t}{2} - \sin \left( \frac{n + \frac{1}{2}}{2} \right) t}{2 \sin \left( \frac{t}{2} \right)} \right\} dt
\]

\[
= \int_0^\pi \psi(t) \overline{K_n(t)} \, dt
\]

\[
= \left\{ \frac{1}{n+1} \int_0^\pi \psi(t) \overline{K_n(t)} \, dt \right\}
\]

\[
= I_1 + I_2, \text{ say}
\]
Now

$$|I_1| = \frac{2}{\pi (1 + q)^r} \left| \int_0^{\frac{\pi}{n+1}} \psi(t) \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^{k} P_{\nu} \cos \left( \frac{\nu - 1}{2} \right) \frac{\cos \left( \frac{\nu + 1}{2} \right)}{2 \sin \frac{\nu}{2}} \right\} \right| \right|_{0}^{\pi}$$

$$\leq \int_0^{\frac{\pi}{n+1}} \psi(t) \frac{1}{K_n(t)} dt$$

$$= \left( \frac{1}{n+1} \right)^\frac{1}{r} \left( \frac{1}{n+1} \right)^\frac{1}{s} \left( \int_0^{\pi} K_n(t) dt \right)^\frac{1}{s}, \text{ using Holder’s inequality}$$

$$\leq O\left( \frac{1}{(n+1)^a} \right) \left( \int_0^{\pi} n^2 \right)^\frac{1}{s}, \text{ using Lemma 4.1}$$

$$= O\left( \frac{1}{(n+1)^a} \right) \left( \frac{n^2}{s+1} \right)^\frac{1}{s}$$

$$= O\left( \frac{n}{(n+1)^{\alpha + \frac{1}{\alpha}}} \right) = O\left( \frac{1}{(n+1)^{\alpha + 1}} \right)$$

(5.2)

Next

$$|I_2| \leq \left( \int_0^{\pi} \varphi(t) dt \right)^\frac{1}{r} \left( \int_0^{\pi} K_n(t) dt \right)^\frac{1}{s}, \text{ using Holder’s inequality}$$

$$\leq O\left( \frac{1}{(n+1)^a} \right) \left( \int_0^{\pi} \frac{1}{t} dt \right)^\frac{1}{s}, \text{ using Lemma 4.2}$$

$$= O\left( \frac{1}{(n+1)^a} \right) \left( \frac{1}{n+1} \right)^\frac{1-s}{s}$$

$$= O\left( \frac{1}{(n+1)^{\alpha + 1}} \right)$$

(5.3)
Then from (5.2) and (5.3), we have

\[ |r_n - f(x)| = O \left( \frac{1}{(n+1)^\frac{1}{\alpha} + r} \right), \quad \text{for} \quad 0 < \alpha < 1, \quad r \geq 1. \]

Hence,

\[ \|r_n - f(x)\|_x = \sup_{-\pi < x < \pi} |r_n - f(x)| = O \left( \frac{1}{(n+1)^\frac{1}{\alpha} + r} \right), \quad 0 < \alpha < 1, \quad r \geq 1. \]

This completes the proof of the theorem.

References


