ON DEGREE OF APPROXIMATION BY PRODUCT MEANS OF
CONJUGATE SERIES OF A FOURIER SERIES

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Abstract: In this paper a theorem on degree of approximation of a function f ∈ Lip(α, r) by product summability (E, q)(N, pₙ) of conjugate series of Fourier series associated with f has been established.

1. Introduction

Let \( \sum aₙ \) be a given infinite series with the sequence of partial sums \( \{ sₙ \} \). Let \( \{ pₙ \} \) be a sequence of positive real numbers such that

\[
Pₙ = \sum_{\nu=0}^{n} pₙ \rightarrow \infty, \quad \text{as} \quad n \rightarrow \infty, \quad (P₋ᵢ = p₋ᵢ = 0, i \geq 0).
\]

(1.1)

The sequence –to–sequence transformation

\[
tₙ = \frac{1}{Pₙ} \sum_{\nu=0}^{n} pₙ sₙ \quad (1.2)
\]

defines the sequence \( \{ tₙ \} \) of the \( (\bar{N}, pₙ) \)-mean of the sequence \( \{ sₙ \} \) generated by the sequence of coefficient \( \{ pₙ \} \). If

\[
tₙ \rightarrow s \quad \text{as} \quad n \rightarrow \infty
\]

(1.3)

then the series \( \Sigma aₙ \) is said to be \( (\bar{N}, pₙ) \) summable to \( s \).

The conditions for regularity of \( (\bar{N}, pₙ) \)-summability are easily seen to be[1]

\[
(i) \quad Pₙ \rightarrow \infty, \text{as} \quad n \rightarrow \infty.
\]

\[
(ii) \quad \sum_{\nu=0}^{n} pₙ \leq C \left| Pₙ \right|, \text{as} \quad n \rightarrow \infty.
\]

(1.4)

The sequence –to–sequence transformation, [1]

\[
Tₙ = \frac{1}{(1 + q)^{n}} \sum_{\nu=0}^{n} \left( \begin{array}{c} n \\ \nu \end{array} \right) q^{n-\nu} sₙ
\]

(1.5)

defines the sequence \( \{ Tₙ \} \) of the \( (E, q) \)-mean of the sequence \( \{ sₙ \} \). If

\[
Tₙ \rightarrow s \quad \text{as} \quad n \rightarrow \infty,
\]

then the series \( \Sigma aₙ \) is said to be \( (E, q) \) summable to \( s \).
Clearly \((E, q)\) method is regular. Further, the \((E, q)\) transform of the \((N, p_n)\) transform of \(\{s_n\}\) is defined by

\[
\tau_n = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} t_k
\]

\[
= \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{p_k} \sum_{\ell=0}^{k} p_{\ell} s_{\ell} \right\}
\]

(1.7)

If

\[
\tau_n \to s, \quad \text{as} \quad n \to \infty,
\]

(1.8)

then \(\sum a_n\) is said to be \((E, q)(N, p_n)\)-summable to \(s\).

Let \(f(t)\) be a periodic function with period \(2\pi\) and \(L\)-integrable over \((-\pi, \pi)\). The Fourier series associated with \(f\) at any point \(x\) is defined by

\[
f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x)
\]

(1.9)

and the conjugate series of the Fourier series (1.9) is

\[
\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) = \sum_{n=1}^{\infty} B_n(x)
\]

(1.10)

Let \(s_n(f; x)\) be the \(n\)-th partial sum of (1.10). The \(L^\infty\)-norm of a function \(f: \mathbb{R} \to \mathbb{R}\) is defined by

\[
\|f\|_{\infty} = \sup \left\{ f(x) : x \in \mathbb{R} \right\}
\]

(1.11)

and the \(L^v\)-norm is defined by

\[
\|f\|_v = \left( \int_0^{2\pi} |f(x)|^v \, dx \right)^{1/v}, \quad v \geq 1
\]

(1.12)

The degree of approximation of a function \(f: \mathbb{R} \to \mathbb{R}\) by a trigonometric polynomial \(P_n(x)\) of degree \(n\) under norm \(\|\|\) is defined by [5].

\[
\|P_n - f\|_{\infty} = \sup \left\{ |P_n(x) - f(x)| : x \in \mathbb{R} \right\}
\]

(1.13)

and the degree of approximation \(L_n(f)\) of a function \(f \in L_v\) is given by

\[
E_n(f) = \min \|P_n - f\|_v
\]

(1.14)

A function \(f\) is said to satisfy Lipschitz condition (here after we write \(f \in \text{Lipa}\)) if

\[
|f(x + t) - f(x)| = O\left(|t|^\alpha \right), 0 < \alpha \leq 1
\]

(1.15)

and \(f(x) \in \text{Lip}(a, r), \) for \(0 \leq x \leq 2\pi,\) if

\[
\left( \int_0^{2\pi} |f(x + t) - f(x)|^r \, dx \right)^{1/r} = O\left(|t|^\alpha \right), 0 < \alpha \leq 1, \ r \geq 1, \ t > 0
\]

(1.16)
We use the following notation throughout this paper:

\[ \psi(t) = \frac{1}{2} \{ f(x + t) - f(x - t) \} \tag{1.17} \]

and

\[ \overline{K_n}(t) = \frac{1}{\pi (1 + q)^n} \sum_{k=0}^{n} \left( \frac{1}{k!} \sum_{l=0}^{k} p_l \right) \cos \left( \frac{t}{2} \cos \left( \frac{v + \frac{1}{2}}{2} \right) \right) \sin \left( \frac{t}{2} \right) \tag{1.18} \]

Further, the method \((E, q) (N, p_n)\) is assumed to be regular.

2. Known Theorem

Dealing with the degree of approximation by the product \((E, q) (C, 1)\) - mean of Fourier series, Nigam et al [3] proved the following theorem:

**Theorem 2.1.** If a function \(f\) is \(2\pi\)-periodic and belonging to class \(Lip_a\), then its degree of approximation by \((E, q) (C, 1)\) summability mean on its Fourier series \(\sum_{n=0}^{\infty} A_n(t)\) is given by

\[ \| E_n^q C_n^1 - f \|_\infty = O\left( \frac{1}{(n+1)^\alpha} \right), 0 < \alpha < 1 \]

where \(E_n^q C_n^1\) represents the \((E, q)\) transform of \((C, 1)\) transform of \(s_n(f; x)\).

Recently, Misra et al [2] proved the following theorem using \((E, q)(N, p_n)\) mean of conjugate series of the Fourier series:

**Theorem 2.2.** If \(f\) be \(2\pi\)-Periodic function of class \(Lip_a\), then degree of approximation by the product \((E, q)(N, p_n)\) summability means of the conjugate series (1.10) of Fourier series (1.9) is given

\[ \| \tau_n - f \|_\infty = O\left( \frac{1}{(n+1)^\alpha} \right), 0 < \alpha < 1 \]

where \(\tau_n\) is as defined in (1.7).

3. Main theorem

In this paper, we have proved a theorem on degree of approximation by the product mean \((E, q)(N, p)\) of conjugate series of Fourier series of a function of class \(Lip(a, r)\). We prove:

**Theorem 3.1.** If \(f\) be \(2\pi\)-Periodic function of class \(Lip_a\), then degree of approximation by the product \((E, q)(N, p_n)\) summability means of the conjugate series of Fourier series (1.10) is given

\[ \| \tau_n - f \|_\infty = O\left( \frac{1}{(n+1)^{\alpha-1/r}} \right), 0 < \alpha < 1 \]

by

\[ \| \tau_n - f \|_\infty = O\left( \frac{1}{(n+1)^{\alpha-1/r}} \right), 0 < \alpha < 1 \]

where \(\tau\) is as defined in (1.7).

4. Required Lemmas

We require the following Lemmas to prove the theorem.

**Lemma 4.1**

\[ \left| \overline{K_n}(t) \right| = O(n), 0 \leq t \leq \frac{1}{n+1} \]
Proof.

For \(0 \leq t \leq \frac{1}{n+1}\) we have \(\sin nt \leq n\sin t\) then

\[
\left| K_n(t) \right| = \frac{1}{\pi (1 + q)^n} \left| \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^{k} p_{\nu} \frac{\cos \frac{t}{2} - \frac{\nu}{2} + \frac{1}{2} t}{\sin \frac{t}{2}} \right\} \right|
\]

\[
\leq \frac{1}{\pi (1 + q)^n} \left| \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^{k} p_{\nu} \left( \cos \frac{t}{2} \frac{2 \sin \frac{t}{2} \sin \frac{t}{2}}{\sin \frac{t}{2}} + \sin \nu t \right) \right\} \right|
\]

\[
\leq \frac{1}{\pi (1 + q)^n} \left| \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^{k} p_{\nu} \left( O(2 \sin \frac{t}{2} \sin \frac{t}{2} \sin \nu t) + \nu \sin t \right) \right\} \right|
\]

\[
= O(n)
\]

This proves the lemma.

Lemma 4.2

\[
\left| K_n(t) \right| = O\left(\frac{1}{t}\right), \text{ for } \frac{1}{n+1} \leq t \leq \pi
\]

Proof.

For \(\frac{1}{n+1} \leq t \leq \pi\), by Jordan’s lemma, we have \(\sin\left(\frac{t}{2}\right) \geq \frac{t}{\pi}\)

Then

\[
\left| K_n(t) \right| = \frac{1}{\pi (1 + q)^n} \left| \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^{k} p_{\nu} \frac{\cos \frac{t}{2} - \frac{\nu}{2} + \frac{1}{2} t}{\sin \frac{t}{2}} \right\} \right|
\]

\[
= \frac{1}{\pi (1 + q)^n} \left| \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^{k} p_{\nu} \left( \cos \frac{t}{2} \frac{2 \sin \frac{t}{2} \cos \frac{t}{2} + \sin \nu t \sin \frac{t}{2}}{\sin \frac{t}{2}} \right) \right\} \right|
\]
This proves the lemma.

5. Proof of theorem 3.1

Using Riemann–Lebesgue theorem, we have for the n-th partial sum $s_n(f; x)$ of the conjugate Fourier series (1.10) of $f(x)$, following Titchmarsh [4]

$$\overline{s_n}(f; x) - f(x) = \frac{2}{\pi} \int_0^\pi \psi(t) \overline{K_n} \ dt$$

the $(N, pn)$ transform of $\overline{s_n}(f; x)$ using (1.2) is given by

$$t_n - f(x) = \frac{2}{\pi P_n} \int_0^\pi \psi(t) \sum_{k=0}^n \frac{\cos t}{2} \frac{\sin \left(\frac{n+1}{2} \right) t}{2 \sin \left(\frac{t}{2}\right)} dt$$

denoting the $(E, q)(N, pn)$ transform of $\overline{s_n}(f; x)$ by $\tau_n$, we have

$$\|\tau_n - f\| = \frac{2}{\pi(1+q)^n} \int_0^\pi \psi(t) \sum_{k=0}^n \left(\begin{array}{c} n \\ k \end{array}\right) q^{n-k} \left(\frac{1}{P_k} \sum_{\nu=0}^k P_{\nu} \frac{\cos t}{2} \frac{\sin \left(\frac{\nu+1}{2} \right) t}{2 \sin \left(\frac{t}{2}\right)} \right) dt$$

$$= \int_0^\pi \psi(t) \overline{K_n}(t) dt$$

$$= \int_0^{\pi/2} \psi(t) \overline{K_n}(t) dt + \int_{\pi/2}^\pi \psi(t) \overline{K_n}(t) dt$$

$$= I_1 + I_2$$

(5.1)
Now

\[ |I_1| = \frac{2}{\pi (1 + q)^r} \left| \int_0^{r+1} \psi(t) \sum_{k=0}^{n} \binom{n}{k} q^{n-k} \left( \frac{1}{p_k} \sum_{\nu=0}^{k} p_{\nu} \cos \left( \frac{t}{2} - \cos \left( \frac{\nu + \frac{1}{2}}{2} \right) \right) \right) dt \right| \]

\[ \leq \int_0^{n+1} \psi(t) K_n(t) dt \]

\[ = \left( \int_0^{n+1} (\psi(t))^r dt \right)^{\frac{1}{r}} \left( \int_0^{n+1} (K_n(t))^\frac{1}{r} dt \right)^{\frac{1}{r}}, \text{ using Holder’s inequality} \]

\[ \leq O\left( \frac{1}{(n+1)^2} \right) \left( \int_0^{n+1} n^z dt \right)^{\frac{1}{s}}, \text{ using Lemma 4.1} \]

\[ = O\left( \frac{n}{(n+1)^{s+1}} \right) = O\left( \frac{1}{(n+1)^{s-1}} \right) = O\left( \frac{1}{(n+1)^{\frac{1}{r}}} \right) \]

(5.2)

Next

\[ |I_2| \leq \left( \int_{\frac{1}{n+1}}^{\frac{n+1}{n+1}} (\psi(t))^r dt \right)^{\frac{1}{r}} \left( \int_{\frac{1}{n+1}}^{\frac{n+1}{n+1}} (K_n(t))^\frac{1}{r} dt \right)^{\frac{1}{r}}, \text{ using Holder’s inequality} \]

\[ \leq O\left( \frac{1}{(n+1)^2} \right) \left( \int_{\frac{1}{n+1}}^{\frac{n+1}{n+1}} \left( \frac{1}{t} \right)^s dt \right)^{\frac{1}{s}}, \text{ using Lemma 4.2} \]

\[ = O\left( \frac{1}{(n+1)^{s-1}} \right) \frac{1}{n+1} \]

(5.3)
Then from (5.2) and (5.3), we have

\[ |r_n - f(x)| = O \left( \frac{1}{(n+1)^{\frac{1}{r}} - \frac{1}{r}} \right), \text{ for } 0 < \alpha < 1, r \geq 1. \]

Hence,

\[ \|r_n - f(x)\|_z = \sup_{-\pi < x < \pi} |r_n - f(x)| = O \left( \frac{1}{(n+1)^{\frac{1}{r}}} \right), 0 < \alpha < 1, r \geq 1 \]

This completes the proof of the theorem.

References


