

## ON DEGREE OF APPROXIMATION BY PRODUCT MEANS OF CONJUGATE SERIES OF A FOURIER SERIES

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**Abstract:** In this paper a theorem on degree of approximation of a function  $f \in Lip(\alpha, r)$  by product summability  $(E, q)(\overline{N}, p_n)$  of conjugate series of Fourier series associated with  $f$  has been established.

### 1. Introduction

Let  $\sum a_n$  be a given infinite series with the sequence of partial sums  $\{s_n\}$ . Let  $\{p_n\}$  be a sequence of positive real numbers such that

$$P_n = \sum_{\nu=0}^n p_\nu \rightarrow \infty, \quad \text{as } n \rightarrow \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 0). \quad (1.1)$$

The sequence –to–sequence transformation

$$t_n = \frac{1}{P_n} \sum_{\nu=0}^n p_\nu s_\nu \quad (1.2)$$

defines the sequence  $\{t_n\}$  of the  $(\overline{N}, p_n)$ -mean of the sequence  $\{s_n\}$  generated by the sequence of coefficient  $\{p_n\}$ . If

$$t_n \rightarrow s, \quad \text{as } n \rightarrow \infty \quad (1.3)$$

then the series  $\sum a_n$  is said to be  $(\overline{N}, p_n)$  summable to  $s$ .

The conditions for regularity of  $(\overline{N}, p_n)$ - summability are easily seen to be [1]

$$\begin{cases} (i) P_n \rightarrow \infty, \text{ as } n \rightarrow \infty. \\ (ii) \sum_{i=0}^n p_i \leq C|P_n|, \text{ as } n \rightarrow \infty. \end{cases} \quad (1.4)$$

The sequence –to–sequence transformation, [1]

$$T_n = \frac{1}{(1+q)^n} \sum_{\nu=0}^n \binom{n}{\nu} q^{n-\nu} s_\nu \quad (1.5)$$

defines the sequence  $\{T_n\}$  of the  $(E, q)$  mean of the sequence  $\{s_n\}$ .

If

$$T_n \rightarrow s, \quad \text{as } n \rightarrow \infty, \quad (1.6)$$

then the series  $\sum a_n$  is said to be  $(E, q)$  summable to  $s$ .

Clearly  $(E, q)$  method is regular. Further, the  $(E, q)$  transform of the  $(N, p_n)$  transform of  $\{s_n\}$  is defined by

$$\begin{aligned}\tau_n &= \frac{1}{(1+q)^n} \sum_{k=0}^n \binom{n}{k} q^{n-k} t_k \\ &= \frac{1}{(1+q)^n} \sum_{k=0}^n \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^k P_\nu s_\nu \right\}\end{aligned}\quad (1.7)$$

If

$$\tau_n \rightarrow s, \text{ as } n \rightarrow \infty, \quad (1.8)$$

then  $\sum a_n$  is said to be  $(E, q)(N, p_n)$ -summable to  $s$ .

Let  $f(t)$  be a periodic function with period  $2\pi$  and L-integrable over  $(-\pi, \pi)$ . The Fourier series associated with  $f$  at any point  $x$  is defined by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x) \quad (1.9)$$

and the conjugate series of the Fourier series (1.9) is

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(x) \quad (1.10)$$

Let  $S_n(f; x)$  be the  $n$ -th partial sum of (1.10). The  $L^\infty$ -norm of a function  $f: R \rightarrow R$  is defined by

$$\|f\|_\infty = \sup \{ |f(x)| : x \in R \} \quad (1.11)$$

and the  $L_\nu$ -norm is defined by

$$\|f\|_\nu = \left( \int_0^{2\pi} |f(x)|^\nu dx \right)^{\frac{1}{\nu}}, \quad \nu \geq 1 \quad (1.12)$$

The degree of approximation of a function  $f: R \rightarrow R$  by a trigonometric polynomial  $P_n(x)$  of degree  $n$  under norm  $\|\cdot\|_\infty$  defined by [5].

$$\|P_n - f\|_\infty = \sup \{ |P_n(x) - f(x)| : x \in R \} \quad (1.13)$$

and the degree of approximation  $E_n(J)$  of a function  $J \in L_\nu$  is given by

$$E_n(f) = \min_{P_n} \|P_n - f\|_\nu \quad (1.14)$$

A function  $f$  is said to satisfy Lipschitz condition (here after we write  $f \in Lip_\alpha$ ) if

$$|f(x+t) - f(x)| = O(|t|^\alpha), \quad 0 < \alpha \leq 1 \quad (1.15)$$

and  $f(x) \in Lip_\alpha(a, r)$ , for  $0 \leq x \leq 2\pi$ , if

$$\left( \int_0^{2\pi} |f(x+t) - f(x)|^r dx \right)^{\frac{1}{r}} = O(|t|^\alpha), \quad 0 < \alpha \leq 1, r \geq 1, t > 0 \quad (1.16)$$

We use the following notation throughout this paper :

$$\psi(t) = \frac{1}{2} \{f(x+t) - f(x-t)\}, \tag{1.17}$$

and

$$\overline{K}_n(t) = \frac{1}{\pi(1+q)^n} \sum_{k=0}^n \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^k p_\nu \frac{\cos \frac{t}{2} - \cos \left(\nu + \frac{1}{2}\right) t}{\sin \frac{t}{2}} \right\} \tag{1.18}$$

Further, the method  $(E, q)(N, p_n)$  is assumed to be regular .

**2. Known Theorem**

Dealing with The degree of approximation by the product  $(E, q)(C, 1)$  -mean of Fourier series, Nigam et al [3] proved the following theorem:

**Theorem 2.1.** If a function  $f$  is  $2\pi$  - periodic and belonging to class  $\underline{Lipa}$  , then its degree of approximation by  $(E, q)(C, 1)$  summability mean on its Fourier series  $\sum_{n=0}^{\infty} A_n(t)$  is given by  $\|E_n^q C_n^1 - f\|_{\infty} = O\left(\frac{1}{(n+1)^\alpha}\right), 0 < \alpha < 1$  where  $E_n^q C_n^1$  represents the  $(E, q)$  transform of  $(C, 1)$  transform of  $S_n(f; x)$ .

Recently, Misra et al [2] proved the following theorem using  $(E, q)(N, p_n)$  mean of conjugate series of the Fourier series :

**Theorem 2.2.** If  $f$  be  $2\pi$  - Periodic function of class  $Lipa$  , then degree of approximation by the product  $(E, q)(\overline{N}, p_n)$  summability means of the conjugate series (1.10) of Fourier series (1.9) is

given  $\|\tau_n - f\|_{\infty} = O\left(\frac{1}{(n+1)^\alpha}\right), 0 < \alpha < 1$  where  $\tau_n$  is as defined in 1.7) .

**3. Main theorem**

In this paper, we have proved a theorem on degree of approximation by the product mean  $(E, q)(N, p)$  of conjugate series of Fourier series of a function of class  $Lip(a, r)$  . We prove:

**Theorem.3.1.** If  $f$  be  $2\pi$  - Periodic function of class  $Lipa$  , then degree of approximation by the product  $(E, q)(\overline{N}, p_n)$  summability means of the conjugate series of Fourier series (1.10) is given

by  $\|\tau_n - f\|_{\infty} = O\left(\frac{1}{(n+1)^{\alpha-1/r}}\right), 0 < \alpha < 1$  where  $\tau$  is as defined in (1.7) .

**4. Required Lemmas**

We require the following Lemmas to prove the theorem.

**Lemma 4.1**

$$\left| \overline{K}_n(t) \right| = O(n) \quad , 0 \leq t \leq \frac{1}{n+1}$$

**Proof.**

For  $0 \leq t \leq \frac{1}{n+1}$  we have  $\sin nt \leq n \sin t$  then

$$\begin{aligned}
|\overline{K}_n(t)| &= \frac{1}{\pi(1+q)^n} \left| \sum_{k=0}^n \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^k p_\nu \frac{\cos \frac{t}{2} - \cos \left(\nu + \frac{1}{2}\right) t}{\sin \frac{t}{2}} \right\} \right| \\
&\leq \frac{1}{\pi(1+q)^n} \left| \sum_{k=0}^n \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^k p_\nu \frac{\cos \frac{t}{2} - \cos \nu t \cdot \cos \frac{t}{2} + \sin \nu t \cdot \sin \frac{t}{2}}{\sin \frac{t}{2}} \right\} \right| \\
&\leq \frac{1}{\pi(1+q)^n} \left| \sum_{k=0}^n \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^k p_\nu \left( \frac{\cos \frac{t}{2} (2 \sin^2 \nu \frac{t}{2})}{\sin \frac{t}{2}} + \sin \nu t \right) \right\} \right| \\
&\leq \frac{1}{\pi(1+q)^n} \left| \sum_{k=0}^n \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^k p_\nu \left( O\left(2 \sin \nu \frac{t}{2} \sin \nu \frac{t}{2}\right) + \nu \sin t \right) \right\} \right| \\
&\leq \frac{1}{\pi(1+q)^n} \left| \sum_{k=0}^n \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^k p_\nu (O(\nu) + O(\nu)) \right\} \right| \\
&\leq \frac{1}{\pi(1+q)^n} \left| \sum_{k=0}^n \binom{n}{k} q^{n-k} \frac{O(k!)}{P_k} \sum_{\nu=0}^k p_\nu \right| \\
&= O(n)
\end{aligned}$$

This proves the lemma.

**Lemma 4.2**

$$|\overline{K}_n(t)| = O\left(\frac{1}{t}\right), \text{ for } \frac{1}{n+1} \leq t \leq \pi$$

**Proof.**

For  $\frac{1}{n+1} \leq t \leq \pi$ , by Jordan's lemma, we have  $\sin\left(\frac{t}{2}\right) \geq \frac{t}{\pi}$

Then

$$\begin{aligned}
|\overline{K}_n(t)| &= \frac{1}{\pi(1+q)^n} \left| \sum_{k=0}^n \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^k p_\nu \frac{\cos \frac{t}{2} - \cos \left(\nu + \frac{1}{2}\right) t}{\sin \frac{t}{2}} \right\} \right| \\
&= \frac{1}{\pi(1+q)^n} \left| \sum_{k=0}^n \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^k p_\nu \frac{\cos \frac{t}{2} - \cos \nu \frac{t}{2} \cdot \cos \frac{t}{2} + \sin \nu \frac{t}{2} \cdot \sin \frac{t}{2}}{\sin \frac{t}{2}} \right\} \right|
\end{aligned}$$

$$\begin{aligned}
 &\leq \frac{1}{\pi(1+q)^n} \left| \sum_{k=0}^n \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^k \frac{\pi}{2t} p_\nu \left( \cos \frac{t}{2} \left( 2 \sin^2 \nu \frac{t}{2} \right) + \sin \nu \frac{t}{2} \cdot \sin \frac{t}{2} \right) \right\} \right| \\
 &\leq \frac{\pi}{2\pi(1+q)^n t} \left| \sum_{k=0}^n \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^k p_\nu \right\} \right| = \frac{1}{2(1+q)^n t} \left| \sum_{k=0}^n \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^k p_\nu \right\} \right| \\
 &= \frac{1}{2(1+q)^n t} \left| \sum_{k=0}^n \binom{n}{k} q^{n-k} \right| \\
 &= O\left(\frac{1}{t}\right).
 \end{aligned}$$

This proves the lemma.

**5. Proof of theorem 3.1**

Using Riemann –Lebesgue theorem, we have for the n-th partial sum  $s_n(\bar{f}; x)$  of the conjugate Fourier series (1.10) of  $f(x)$ , following Titchmarsh [4]

$$\overline{s_n}(f; x) - f(x) = \frac{2}{\pi} \int_0^\pi \psi(t) \overline{K_n} dt$$

the  $(N, pn)$  transform of  $s_n(\bar{f}; x)$  using (1.2) is given by

$$\tau_n - f(x) = \frac{2}{\pi P_n} \int_0^\pi \psi(t) \sum_{k=0}^n p_k \frac{\cos \frac{t}{2} - \sin\left(n + \frac{1}{2}\right) t}{2 \sin\left(\frac{t}{2}\right)} dt$$

denoting the  $(E, q)(N, pn)$  transform of  $s_n(\bar{f}; x)$  by  $\tau_n$ , we have

$$\begin{aligned}
 \|\tau_n - f\| &= \frac{2}{\pi(1+q)^n} \int_0^\pi \psi(t) \sum_{k=0}^n \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^k p_\nu \frac{\cos \frac{t}{2} - \sin\left(\nu + \frac{1}{2}\right) t}{2 \sin\left(\frac{t}{2}\right)} \right\} dt \\
 &= \int_0^\pi \psi(t) \overline{K_n}(t) dt \\
 &= \left\{ \int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^\pi \right\} \psi(t) \overline{K_n}(t) dt \\
 &= I_1 + I_2, \text{ say}
 \end{aligned}$$

(5.1)

Now

$$\begin{aligned}
 |I_1| &= \frac{2}{\pi(1+q)^n} \left| \int_0^{1/n+1} \psi(t) \sum_{k=0}^n \binom{n}{k} q^{n-k} \left\{ \frac{1}{P_k} \sum_{\nu=0}^k p_\nu \frac{\cos \frac{t}{2} - \cos \left( \nu + \frac{1}{2} \right) t}{2 \sin \frac{t}{2}} \right\} dt \right| \\
 &\leq \left| \int_0^{1/n+1} \psi(t) \overline{K}_n(t) dt \right| \\
 &= \left( \int_0^{1/n+1} (\psi(t))^r dt \right)^{\frac{1}{r}} \left( \int_0^{1/n+1} (\overline{K}_n(t))^s dt \right)^{\frac{1}{s}}, \text{ using Holder's inequality} \\
 &\leq O\left(\frac{1}{(n+1)^\alpha}\right) \left( \int_0^{1/n+1} n^s dt \right)^{\frac{1}{s}} \quad \text{using Lemma 4.1} \\
 &= O\left(\frac{1}{(n+1)^\alpha}\right) \left(\frac{n^s}{s+1}\right)^{\frac{1}{s}} \\
 &= O\left(\frac{n}{(n+1)^{\alpha+\frac{1}{s}}}\right) \\
 &= O\left(\frac{1}{(n+1)^{\alpha+\frac{1}{s}-1}}\right) = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right) \tag{5.2}
 \end{aligned}$$

Next

$$\begin{aligned}
 |I_2| &\leq \left( \int_{\frac{1}{n+1}}^{\pi} (\psi(t))^r dt \right)^{\frac{1}{r}} \left( \int_{\frac{1}{n+1}}^{\pi} (\overline{K}_n(t))^s dt \right)^{\frac{1}{s}} \text{ using Holder's inequality} \\
 &\leq O\left(\frac{1}{(n+1)^\alpha}\right) \left( \int_{\frac{1}{n+1}}^{\pi} \left(\frac{1}{t}\right)^s dt \right)^{\frac{1}{s}} \text{ using Lemma 4.2} \\
 &= O\left(\frac{1}{(n+1)^\alpha}\right) \left(\frac{1}{n+1}\right)^{\frac{1-s}{s}} \\
 &= O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right) \tag{5.3}
 \end{aligned}$$

Then from (5.2) and (5.3), we have

$$|\tau_n - f(x)| = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right), \text{ for } 0 < \alpha < 1, r \geq 1.$$

Hence,

$$\|\tau_n - f(x)\|_{\infty} = \sup_{-\pi < x < \pi} |\tau_n - f(x)| = O\left(\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right), 0 < \alpha < 1, r \geq 1$$

This completes the proof of the theorem.

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