A Hilbert-Type Integral Inequality with the Homogeneous Kernel of Degree -3 and a Best Constant Factor

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Abstract. By establishing the weight function, we present a new Hilbert-type inequality with the integral in whole plane and with a best constant factor,and its kernel is a homogeneous form of degree-3, and also we put forward its equivalent form.

1 Introduction

If *f (x), g(x)* ≥ 0, such that $0 < \int_0^\infty f^2(x)\,dx < \infty$ and $0 < \int_0^\infty g^2(x)\,dx < \infty$ then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \pi \left\{ \int_0^\infty f(x)dx \int_0^\infty g(x)dx \right\}^{1/2}$$

(1.1)

where the constant factor π is the best possible. Inequality (1.1) is well-known as Hilbert's integral inequality, which has been extended by Hardy-Riesz as [2]:

If $p > 1, 1/p + 1/q = 1, f(x), g(x) \geq 0$, such that $0 < \int_0^\infty f^p(x)\,dx < \infty$ and $0 < \int_0^\infty g^q(x)\,dx < \infty$ then we have the following Hardy-Hilbert's integral inequality:

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)} \left(\int_0^\infty f^p(x)dx \right)^{1/p} \left(\int_0^\infty g^q(x)dx \right)^{1/q}.$$

(1.2)

Where the constant factor $\frac{\pi}{\sin(\pi/p)}$ also is the best possible.

In recent years, by introducing some parameters and estimating the way of weight function, inequalities (1.1) and (1.2) have many generalizations and variants. (1.1) has been strengthened by Yang and others. (including double series inequalities)[3-15].
In this paper, we give a new Hilbert-type inequality with the integral in whole plane.
In the following, we always suppose that: p >1, 1/ p +1/ q =1, p ≠1, a <1. .

2 Some lemmas

Lemma 2.1 Define the weight functions as follow:

$$w(x) := \int_{-\infty}^{\infty} \frac{|x||y|dy}{(|x|+|y|)(y^2+2axy+x^2)}, \quad w(y) := \int_{-\infty}^{\infty} \frac{y^2 dx}{(|x|+|y|)(y^2+2axy+x^2)}.$$

Then $w(x) = w(y) = \frac{\pi}{2\sqrt{1-a^2}}$.

(2.1)

Proof. We only prove that $w(x) = \frac{\pi}{2\sqrt{1-a^2}}$ for x∈(−∞,0) .

$$w(x) := \int_{-\infty}^{0} \frac{|x||y|dy}{(|x|+|y|)(y^2+2axy+x^2)} + \int_{0}^{\infty} \frac{|x||y|dy}{(|x|+|y|)(y^2+2axy+x^2)} = w_1 + w_2 .$$

Setting y = tx , then

$$w_1 = \int_{-\infty}^{0} \frac{(-x)(-y)dy}{(-x-y)(y^2+2axy+x^2)} = \int_{0}^{\infty} \frac{tdt}{(t+1)(t^2+2at+1)} = \frac{1}{2(a-1)} \int_{0}^{\infty} \left(\frac{1}{t+1} - \frac{t+1}{t^2+2at+1} \right) dt$$

$$= \frac{1}{2(a-1)} \left[\ln\frac{t+1}{t^2+2at+1} \Big|_0^\infty - (1-a)\int_0^\infty \frac{dt}{t^2+2at+1} \right] = \frac{1}{2}\int_0^\infty \frac{dt}{t^2+2at+1} = \frac{1}{2\sqrt{1-a^2}} \left(\frac{\pi}{2} - \arctan\frac{a}{\sqrt{1-a^2}} \right).$$

Setting y = −tx , then

$$w_2 = \int_0^\infty \frac{(-x)ydy}{(-x+y)(y^2+2axy+x^2)} = \int_0^\infty \frac{tdt}{(t+1)(t^2-2at+1)} = \frac{1}{2\sqrt{1-a^2}} \left(\frac{\pi}{2} + \arctan\frac{a}{\sqrt{1-a^2}} \right)$$

and $w(x) = \dfrac{\pi}{2\sqrt{1-a^2}}$.

Easily if $x \in (-\infty, 0)$, setting x = y / t ,we have

$$w(y) = \int_{-\infty}^{\infty} \frac{y^2 dx}{(|x|+|y|)(y^2+2axy+x^2)} = \int_{-\infty}^{0} \frac{y^2 dx}{(|x|+|y|)(y^2+2axy+x^2)} + \int_{0}^{\infty} \frac{y^2 dx}{(|x|+|y|)(y^2+2axy+x^2)}$$

:= $w_1 + w_2$. And

$$w_1 = \int_{-\infty}^{0} \frac{y^2 dx}{(-x-y)(y^2+2axy+x^2)} = \int_0^\infty \frac{y^2 dx}{(x+y)(y^2+2axy+x^2)} = \int_0^\infty \frac{y^2 d(\frac{y}{t})}{((\frac{y}{t})+y)(y^2+2a(\frac{y}{t})y+(\frac{y}{t})^2)}$$

$$= \int_0^\infty \frac{tdt}{(t+1)(t^2+2at+1)} = w_1 .$$

$$w(x) = w(y) = \frac{\pi}{2\sqrt{1-a^2}}$$

Similarly, w_2 (x) = w_2 ,and

The lemma is proved.

Lemma 2.2 For $\frac{q}{2} > \varepsilon >$, define both functions, f and g as follow:

$$f(x) = \begin{cases} x^{-2\varepsilon/p}, & \text{if } x \in (1,\infty), \\ 0, & \text{if } x \in [-1,1], \\ (-x)^{-2\varepsilon/p}, & \text{if } x \in (-\infty,1); \end{cases} \quad \text{and } g(x) = \begin{cases} x^{1-2\varepsilon/q}, & \text{if } x \in (1,\infty), \\ 0, & \text{if } x \in [-1,1], \\ (-x)^{1-2\varepsilon/q}, & \text{if } x \in (-\infty,1); \end{cases}$$

Then

$$I(\varepsilon) := \varepsilon \left\{ \int_{-\infty}^{\infty} |x|^{-1} f^p(x)dx \right\}^{1/p} \left\{ \int_{-\infty}^{\infty} |x|^{-q-1} g^q(x)dx \right\}^{1/p} = 1;$$

(2.2)

$$\tilde{I}(\varepsilon) := 2\varepsilon \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(x)g(y)}{(|x|+|y|)(y^2+2axy+x^2)} dxdy = \frac{\pi}{2\sqrt{1-a^2}} + o(1), \text{ (for } \varepsilon \to 0^+)$$

(2.3)

Proof Easily,

$$I(\varepsilon) := 2\varepsilon \left\{ \int_1^\infty x^{-1} x^{-2\varepsilon} dx \right\}^{1/p} \left\{ \int_0^1 x^{-1} x^{-2\varepsilon} dx \right\}^{1/p} = 1;$$

Let $y = -Y$,using $(f-x) = f(x)$, $g(-x) = g(x)$ and

$$f(-x)\int_{-\infty}^{\infty} \frac{g(y)dy}{(|x|+|y|)(y^2 - 2axy + x^2)} = f(x)\int_{-\infty}^{\infty} g(Y)\frac{g(Y)dY}{(|x|+|Y|)(Y^2 + 2axY + x^2)}$$

$$f(x)\int_{-\infty}^{\infty} \frac{g(y)dy}{(|x|+|y|)(y^2 + 2axy + x^2)}$$

we have that   is an even function, then

$$\tilde{I}(\varepsilon) = 2\varepsilon \int_0^{\infty} f(x)\left(\int_{-\infty}^{\infty} \frac{g(y)dy}{(|x|+|y|)(y^2 + 2axy + x^2)} \right) dx$$

$$= 2\varepsilon\left[\int_1^{\infty} x^{-2\varepsilon/p}\left(\int_{-\infty}^{-1} \frac{(-y)^{1-\frac{2\varepsilon}{q}} dy}{(x-y)(y^2 + 2axy + x^2)} \right) dx + \int_1^{\infty} x^{-2\varepsilon/p}\left(\int_1^{\infty} \frac{y^{1-\frac{2\varepsilon}{q}} dy}{(x+y)(y^2 + 2axy + x^2)} \right) dx \right]$$

$$= I_1 + I_2.$$

Setting $y = xu$ then

$$I_1 = 2\varepsilon \int_1^{\infty} x^{-2\varepsilon/p}\left(\int_1^{\infty} \frac{y^{1-\frac{2\varepsilon}{q}}}{(x+y)(y^2 - 2axy + x^2)} dy \right) dx = 2\varepsilon \int_1^{\infty} x^{-1-2\varepsilon}\left(\int_{\frac{1}{x}}^{\infty} \frac{u^{1-\frac{2\varepsilon}{q}}}{(u+1)(u^2 - 2au + 1)} du \right) dx$$

$$= 2\varepsilon\left[\int_1^{\infty} x^{-1-2\varepsilon}\left(\int_1^{\infty} \frac{u^{1-\frac{2\varepsilon}{q}}}{(u+1)(u^2 - 2au + 1)} du \right) dx + \int_1^{\infty} x^{-1-2\varepsilon}\left(\int_{\frac{1}{x}}^{\infty} \frac{u^{1-\frac{2\varepsilon}{q}}}{(u+1)(u^2 - 2au + 1)} du \right) dx \right]$$

$$= \int_1^{\infty} \frac{u^{1-\frac{2\varepsilon}{q}}}{(u+1)(u^2 - 2au + 1)} du + 2\varepsilon \int_0^1 \frac{u^{1-\frac{2\varepsilon}{q}}}{(u+1)(u^2 - 2au + 1)}\left(\int_{1/u}^{\infty} x^{-1-2\varepsilon} dx \right) du$$

$$= \int_1^{\infty} \frac{u^{1-\frac{2\varepsilon}{q}}}{(u+1)(u^2 - 2au + 1)} du + \int_0^1 \frac{u^{1+\frac{2\varepsilon}{p}}}{(u+1)(u^2 - 2au + 1)} du$$

$$= \int_0^{\infty} \frac{u^{1-\frac{2\varepsilon}{q}}}{(u+1)(u^2 - 2au + 1)} du + \int_0^1 (u^{2\varepsilon/p} - u^{-2\varepsilon/q})\frac{u}{(u+1)(u^2 - 2au + 1)} du$$

$$= \int_0^{\infty} \frac{u^{1-\frac{2\varepsilon}{q}}}{(u+1)(u^2 - 2au + 1^2)} du + \eta(\varepsilon)$$

$$= \frac{1}{2\sqrt{1-a^2}}\left(\frac{\pi}{2} + \arctan\frac{a}{\sqrt{1-a^2}} \right) + o(1)\ (\text{for}\ \varepsilon \to 0^+).$$

The reason for the last equation is that, because of $\exists M \in \square_+$, such that

$$\int_0^{\infty} \frac{u^{1-\frac{2\varepsilon}{q}}}{(u+1)(u^2 - 2au + 1)} du < \int_0^1 \frac{1}{(u+1)(u^2 - 2au + 1)} du + \int_0^{\infty} \frac{u}{(u+1)(u^2 - 2au + 1)} du < M$$

and by using the theorem of control convergence, we have the conclusion as follows:

$$\lim_{\varepsilon \to 0^+} \int_1^{\infty} \frac{u^{1-\frac{2\varepsilon}{q}}}{(u+1)(u^2 - 2au + 1)} du = \int_1^{\infty} \frac{u}{(u+1)(u^2 - 2au + 1)} du = \frac{1}{2\sqrt{1-a^2}}\left(\frac{\pi}{2} + \arctan\frac{a}{\sqrt{1-a^2}} \right).$$

and we have
$$I_1 \to \frac{1}{2\sqrt{1-a^2}}\left(\frac{\pi}{2}+\arctan\frac{a}{\sqrt{1-a^2}}\right)\ (\varepsilon \to 0^+)$$

$$I_2 \to \frac{1}{2\sqrt{1-a^2}}\left(\frac{\pi}{2}-\arctan\frac{a}{\sqrt{1-a^2}}\right)\ (\text{for } \varepsilon \to 0^+).$$

Similarly   The lemma is proved.

Lemma 2.3. If $0 < \int_{-\infty}^{\infty}|x|^{-1}f^p(x)dx < \infty$ we have

$$J:=\int_{-\infty}^{\infty}|y|^{2p-1}\left(\int_{-\infty}^{\infty}\frac{f(x)}{(|x|+|y|)(y^2+2axy+x^2)}dx\right)^p dy \le \left(\frac{\pi}{2\sqrt{1-a^2}}\right)^p \int_{-\infty}^{\infty}|x|^{-1}f^p(x)dx.$$
(2.4)

Proof By lemma 2.2,we find

$$\left(\int_{-\infty}^{\infty}\frac{f(x)}{(|x|+|y|)(y^2+2axy+x^2)}dx\right)^p$$

$$=\left(\int_{-\infty}^{\infty}\frac{1}{(|x|+|y|)(y^2+2axy+x^2)}|y|^{1/p}f(x)|y|^{-1/p}dx\right)^p$$

$$\le \int_{-\infty}^{\infty}\frac{|y|}{(|x|+|y|)(y^2+2axy+x^2)}f^p(x)dx\left(\int_{-\infty}^{\infty}\frac{|y|^{1-q}}{(|x|+|y|)(y^2+2axy+x^2)}dx\right)^{p-1}$$

$$=\left(\frac{\pi}{2\sqrt{1-a^2}}\right)^{p-1}|y|^{-2p+1}\int_{-\infty}^{\infty}\frac{|y|}{(|x|+|y|)(y^2+2axy+x^2)}f^p(x)dx,$$

$$J\le\left(\frac{\pi}{2\sqrt{1-a^2}}\right)^{p-1}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\frac{|y|}{(|x|+|y|)(y^2+2axy+x^2)}f^p(x)dx\right]dy$$

$$=\left(\frac{\pi}{2\sqrt{1-a^2}}\right)^{p-1}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\frac{|y|}{(|x|+|y|)(y^2+2axy+x^2)}dy\right]f^p(x)dx$$

$$=\left(\frac{\pi}{2\sqrt{1-a^2}}\right)^p\int_{-\infty}^{\infty}|x|^{-1}f^p(x)dx$$
(2.5)

3. Main results

Theorem. If both functions, $f(x)$ and $g(x)$ are nonnegative measurable functions, and satisfy

$$0<\int_{-\infty}^{\infty}|x|^{pr-1}f^p(x)dx<\infty,\text{and } 0<\int_{-\infty}^{\infty}|x|^{-q-1}g^q(x)dx<\infty,\text{ then}$$

$$I^*:=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(x)g(y)}{(|x|+|y|)(y^2+2axy+x^2)}dxdy$$

$$<\frac{\pi}{2\sqrt{1-a^2}}\left(\int_{-\infty}^{\infty}|x|^{-1}f^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{-q-1}g^q(x)dx\right)^{1/q},$$
(3.1)

and

$$J=\int_{-\infty}^{\infty}|y|^{2p-1}\left(\int_{-\infty}^{\infty}\frac{f(x)}{(|x|+|y|)(y^2+2axy+x^2)}dx\right)^p dy<\left(\frac{\pi}{2\sqrt{1-a^2}}\right)^p\int_{-\infty}^{\infty}|x|^{-1}f^p(x)dx$$

(3.2)

Inequalities (3.1)and (3.2) are equivalent, and the constant factors in the two forms are all the best possible.

Proof If (2.5) takes the form of equality for some $y\in(-\infty,0)\cup(0,\infty)$,then there exist constants M and N , such that they are not all zero, and

$$M|y|f^p(x)=N|y|^{-q-1}\quad\text{a.e. in }(-\infty,\infty)\times(-\infty,\infty),$$

Hence, there exists a constant C , such that

$$Mf^p(x)=N|y|^{-q}=C\quad\text{a.e. in }(-\infty,\infty)\times(-\infty,\infty).$$

We claim that M =0. In fact, if $M\neq0$, then $|x|^{-1}f^p(x)=C/(M|x|)$ a.e. in $(-\infty,\infty)$ which contradicts the fact that $0<\int_{-\infty}^{\infty}|x|^{-1}f^p(x)dx<\infty$. .In the same way, we claim that N =0. This is too a contradiction and hence by (2.7), we have (3.2).
By Holder's inequality with weight and (3.2), we have,

$$I^*=\int_{-\infty}^{\infty}\left[|y|^{1+1/q}\int_{-\infty}^{\infty}\frac{f(x)}{(|x|+|y|)(y^2+2axy+x^2)}dx\right]\left[|y|^{-1-1/q}g(y)\right]dy$$

$$\leq(J)^{1/p}\left(\int_{-\infty}^{\infty}|y|^{-q-1}g^q(y)dy\right)^{1/q}$$

(3.3)

Using (3.2),we have (3.1).

Setting $g(y)=|y|^{2p-1}\left(\int_{-\infty}^{\infty}\frac{f(x)}{(|x|+|y|)(y^2+2axy+x^2)}dx\right)^{p-1}$,then $J=\int_{-\infty}^{\infty}|y|^{-q-1}g^q ydy$ by (2.7)

we have $J<\infty$.if $J=0$ then (3.2) is proved;if $0<J<\infty$,by (3.1),we obtain

$$0<\int_{-\infty}^{\infty}|y|^{-q-1}g^q(y)dy=J=I^*<\frac{\pi}{2\sqrt{1-a^2}}\left(\int_{-\infty}^{\infty}|x|^{-1}f^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{-q-1}g^q(x)dx\right)^{1/q}$$

$$\left(\int_{-\infty}^{\infty}|x|^{-q-1}g^q(x)dx\right)^{1/p}=J^{1/p}<\frac{\pi}{2\sqrt{1-a^2}}\left(\int_{-\infty}^{\infty}|x|^{-1}f^p(x)dx\right)^{1/p}$$

Inequalities (3.1)and (3.2) are equivalent.

If the constant factor $\frac{\pi}{2\sqrt{1-a^2}}$ in (3.1) is not the best possible, then there exists a positive h (with $h<\frac{\pi}{2\sqrt{1-a^2}}$),such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(x)g(y)}{(|x|+|y|)(y^2+2axy+x^2)}dxdy<h\left(\int_{-\infty}^{\infty}|x|^{-1}f^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{-q-1}g^q(x)dx\right)^{1/q}$$

(3.4)

For$\varepsilon>0$,by (3.4),using lemma 2.3,we have

$$\tilde{I}(\varepsilon)=k+o(1)<\varepsilon h\left(\int_{-\infty}^{\infty}|x|^{-1}f^p(x)dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{-q-1}g^q(x)dx\right)^{1/q}=h$$

(3.5)

Hence we find, $\dfrac{\pi}{2\sqrt{1-a^2}}+o(1)<h$ for $\varepsilon \to 0^+$, it follows that $\dfrac{\pi}{2\sqrt{1-a^2}}\le h$ which contradicts the fact that $h<\dfrac{\pi}{2\sqrt{1-a^2}}$ Hence the constant h in (3.1) is the best possible.As (3.1) and (3.2) are equivalent,if the constant factor in (3.2) is not the best possible, then by using (3.2), we can get a contradiction that the constant factor in (3.1) is not the best possible.

Thus we complete the prove of the theorem.

Remark For $\alpha=\dfrac{\pi}{4}, \beta=\dfrac{\pi}{3}$ in (3.1), we have the following particular result:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(x)g(y)}{(|x|+|y|)(y^2+2xy\cos\alpha+x^2)}dxdy<\frac{\pi}{2\sin\alpha}\left(\int_{-\infty}^{\infty}|x|^{-1}f^p(x)\,dx\right)^{1/p}\left(\int_{-\infty}^{\infty}|x|^{-q-1}g^q(x)\,dx\right)^{1/q}.$$

Where the constant factor $\dfrac{\pi}{2\sin\alpha}$ also is the best possible.

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