

The Last Theorem of Pierre de Fermat, in elementary way

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Abstract: In this paper the author works only through the factorization in factors, with the proceeding for absurd, that is if x, y, z are prime among them, under the hypothesis that the tern of integers (x, y, z) were a solution of the equation

$$x^n + y^n = z^n$$

Then he obtains that the first and the second term of an equivalent relation are odd (the first) and even (the second). Three cases are separated:

- 1) n is power of 2;
- 2) n is odd;
- 3) n is product of a power of 2 for an odd number.

For a better understanding also the Pythagorean set of three numbers is reported.

1. Introduction

Pierre de Fermat, (born at Beaumont de Lomagne in 1601 and dead at Castres in 1665), belonging to "noblesse de robe", who conferred him the noble symbol "de" in front of his surname, through a mirabilis proceeding, proved, about 1637, that the equation

$$x^n + y^n = z^n$$

has not any integer positive solutions, when " n " is more than two.

Unfortunately his demonstration has not passed on to posterity.

In fact, in the first edition, made by Bachet de Meziriac, of the book "Arithmetica" about the works of Diophanto, a Great Greek mathematician of the third century before Christ, Pierre de Fermat, did not think to quote it because it was too long, but in the margin of the same pages he referred to have a mirabilis proceeding of proof. Nor did it the son Samuel, who, in 1670, edited the reprint of the book on Diophanto titled *Varia Opera Mathematica*, even if he added some notes, he did not report such a demonstration.

During these four centuries the most renowned mathematicians have tried, without success, such demonstrations, for this they questioned the assertion of Fermat not to have succeeded.

J. Andrew Wiles tried the Theorem but in complicated way.

This work is the proof that the elementary demonstration is there and I am sure that Pierre had formulated it in a plainer way.

In the short demonstration, that has been reported here, the various passages have an elementary nature, therefore they were perhaps also in the mirabilis proceeding of Fermat; so, this work could be considered on archaeological type.

However it is probable that Pierre has used his knowledge on the effective powers of the numbers also with many digits, or those on the spheres and hyperspheres, or those on the parabolas of the type $y = z^n$ and $y = (z - h)^n$, as he was great at geometry.

1.1. Preliminary observation

Even if it is obvious and perhaps, for a mathematician as Pierre de Fermat really trivial and not important for the proof, here one introduces the

Zero Theorem: For the possible integer positive solutions of the equation

$$x^n + y^n - z^n \tag{1}$$

must be

$$x + y - z \geq 4$$

$$\forall z \geq 5 \text{ and } \forall n \geq 3.$$

Proof: Just to give everything more understanding also to people, who have no familiarity with numbers, one reports the demonstration.

We suppose x, y, z , prime among them, because, if (x, y, z) is a solution, also (hx, hy, hz) is still a solution and on the contrary: we observe that if two of three values are not prime, that is they have a common factor, then also the third has the same factor.

Then x, y, z cannot be all three odd at the same time: one of the three numbers has to be even. Therefore

$$x + y - z$$

is even.

Immediately we notice that only for $n=1$ appear

$$x + y - z = 0$$

For the predicate you have only to prove that, if one supposes

$$x + y - z = 2,$$

then the equation

$$(\alpha) \quad x^n + y^n = z^n \quad \forall n \geq 3$$

has no integer positive solution (x, y, z) .

The proof is symmetric with respect to x and y .

Let

$$x + y - z = 2.$$

If it is $x=1$, therefore $y=z+1$ and one should not have written

$$1^n + (z+1)^n = z^n,$$

because it is impossible.

If it is $x=2$, therefore $y=z$ and one should not have written

$$2^n + z^n = z^n$$

because it is impossible.

If it is $x=3$, therefore $y=z-1$ and the equation

$$3^n + (z-1)^n = z^n \quad \forall z \geq 5 \text{ and } \forall n \geq 3$$

is impossible, because we have obviously

$$(\beta) \quad 3^n + (z-1)^n < z^n.$$

For example, when $z=5$, it is

$$3^n + 4^n < 5^n \quad \forall n \geq 3.$$

Clearly $(z-1)^n$ increases less than z^n , whether z increases or n increases: one concludes that (β) is true $\forall z \geq 5$.

Therefore it is allowed the condition

$$x > 3, (y > 3).$$

Let

$$x = h + 2, \text{ therefore } y = z - h \text{ with } h \geq 2$$

and the equation becomes

$$(h+2)^n + (z-h)^n = z^n \quad \forall z \geq 5 \text{ and } \forall n \geq 3$$

which is never true, because it is

$$(\gamma) \quad (h+2)^n + (z-h)^n < z^n \quad \forall z \geq 5 \text{ and } \forall n \geq 3.$$

For $h=2$, the equation

$$4^n + (z-2)^n = z^n \quad \forall z \geq 5 \text{ and } \forall n \geq 3$$

has got no solution since

$$(\delta) \quad 4^n + (z-2)^n < z^n \quad \forall z \geq 5 \text{ and } \forall n \geq 3$$

For example, when $z=5$, we have

$$4^n + 3^n < 5^n \quad \forall n \geq 3.$$

Clearly $(z-2)^n$ increases less than z^n , whether z increases or n increases: one concludes that (δ) is true $\forall z \geq 5$.

For $h=3$ the equation

$$5^n + (z-3)^n = z^n \quad \forall z \geq 6 \text{ and } \forall n \geq 3$$

has got no solution since

$$(\epsilon) \quad 5^n + (z-3)^n < z^n \quad \forall z \geq 6 \text{ and } \forall n \geq 3.$$

For example, when $z=6$, we have

$$5^n + 3^n < 6^n \quad \forall n \geq 3.$$

Clearly $(z-3)^n$ increases less than z^n , whether z increases or n increases: one concludes that (ϵ) is true $\forall z \geq 6$.

Generally $(z-h)^n$ increases less than z^n , whether z increases or n increases: one concludes that (γ) is true $\forall z \geq 5$ and $\forall n \geq 3$, and the equation (α) has got no primitive integer positive solution.

So, the search for any possible solutions of

$$x^n + y^n = z^n$$

is restricted to the case

$$x + y - z \geq 4.$$

q.e.d.

It is easy to proof, but it is not necessary and here we do not quote, that the search for any possible solutions of (1) is restricted to the case

$$x + y - z = 6 \quad \forall n \geq 3$$

Generally, we do not quote the case

$$x + y - z = 2h,$$

also if it is probably the beginning of the way of Pierre de Fermat.

2. The Last Theorem of Fermat for n power of 2

2.1. The Pythagorean set of three numbers and the case n power of 2

The equation

$$x^2 + y^2 = z^2 \tag{1}$$

has got as solutions the Pythagorean set of three numbers.

We suppose x, y, z , prime among them, that is $\gcd(x, y, z)=1$, because, if (x, y, z) is a solution, also (hx, hy, hz) is still solution and viceversa.

We observe that if two of three values are not prime among them, that is, they have a common factor, then also the third has the same factor.

Then x, y, z cannot be all three odd at the same time: only one of the three numbers has to be even.

First Theorem: The Pythagorean set of three numbers is as

$$x = \frac{1}{2}(v^2 - u^2), \quad y = uv, \quad z = \frac{1}{2}(v^2 + u^2) \text{ with } u < v$$

and we have those, having components prime among them, when u and v are prime among them and both odd.

Proof: In the hypothesis that x, y, z are prime among them, x and y cannot be both even: we suppose y odd and let $z = x + k$ with k prime with x .

From

$$y^2 = z^2 - x^2 = (z - x)(z + x) = k(2x + k),$$

we have that k divides y^2 , and its factors are square, that is, k takes up in the form of

$$k = u^2.$$

In fact, if a factor of k is not square and divides y , then it divides also $2x$, that is, x or 2 , against the hypothesis that x, y, z are prime among them and y odd.

Therefore we have that also $2x + u^2$ is a square and also it is

$$v^2 = 2x + u^2 \text{ with } u \text{ and } v \text{ prime among them.}$$

In the end

$$x = \frac{1}{2}(v^2 - u^2), \quad y = uv, \quad z = \frac{1}{2}(v^2 + u^2), \text{ with } u < v$$

q.e.d.

We note that (if) y is odd, then x is even and z is odd, because $v - u$ and $v + u$ are even and

$$x = \frac{1}{2}(v - u)(v + u)$$

Second Theorem: The equation

$$x^4 + y^4 = z^4 \tag{2}$$

has no integer positive solution.

Proof: We could suppose, that (x, y, z) is a solution with x, y, z prime among them and y odd.

The equation can be written

$$(x^2)^2 + (y^2)^2 = (z^2)^2$$

so that (x^2, y^2, z^2) , will be a Pythagorean set of three numbers, that is

$$x^2 = \frac{1}{2}(v^2 - u^2), \quad y^2 = uv, \quad z^2 = \frac{1}{2}(v^2 + u^2)$$

with u and v prime among them, then

$$x^2 = \frac{1}{2}(v - u)(v + u)$$

is even, but y^2 and z^2 are odd, that is x even, y and z odd.

So,

$$x^2 + z^2 = v^2 \text{ and } x^2 + u^2 = z^2, \tag{3}$$

that is (x, z, v) and (x, u, z) are Pythagorean set of three numbers, as x is even, so z, u, v are odd and prime among them, then exist a, b prime among them and odd, so c, d and we have:

$$x = \frac{1}{2}(b^2 - a^2), \quad z = ab, \quad v = \frac{1}{2}(b^2 + a^2), \tag{4}$$

$$x = \frac{1}{2}(d^2 - c^2), \quad u = cd, \quad z = \frac{1}{2}(d^2 + c^2)$$

The proof is based on the first digit, or digit of the units, of a number. We observe that

$$y^2 = uv$$

is square and since u, v are odd and prime among them, so u and v are square, that is

$$u = \tau^2, v = \sigma^2, u^2 = \tau^4, v^2 = \sigma^4.$$

We indicate with $fd(x)$ the first digit of a number x and we note that the first digit of the fourth powers of an odd number is **1** or **5**, whereas of an even number is **0** or **6** and that the first digit of the second powers of an odd number is **1** or **9** or **5**, whereas of an even number is **4** or **6** or **0**. We have:

$$fd(u^2) = fd(\sigma^4) = \begin{cases} 1 \\ 5 \end{cases}, \quad fd(v^2) = fd(\tau^4) = \begin{cases} 1 \\ 5 \end{cases}$$

But u and v are prime among them, therefore $fd(u^2) = fd(v^2) = 5$ is impossible, because x, y, z are prime among them.

It is

$$fd(x^2) = \begin{cases} 0 \\ 4 \\ 6 \end{cases}, \quad fd\left(\frac{1}{2}(\sigma^4 - \tau^4)\right) \in \{0, 2, 3, 5, 7, 8\}$$

but

$$fd(x^2) = fd\left(\frac{1}{2}(\sigma^4 - \tau^4)\right),$$

so

$$fd(x^2) = 0, \quad fd(x) = 0$$

and

$$fd(\sigma^4) = fd(\tau^4) = 1.$$

By

$$fd(z^2) = \begin{cases} 1 \\ 5 \\ 9 \end{cases}, \quad fd\left(\frac{1}{2}(\sigma^4 + \tau^4)\right) = \begin{cases} 1 \\ 6 \end{cases}$$

we obtain

$$fd(z^2) = 1 \text{ and } fd(y^2) = \begin{cases} 1 \\ 9 \end{cases}$$

Now, by (4) and

$$fd(x) = 0,$$

so

$$fd(d^2) = fd(c^2) = \begin{cases} 1 \\ 9 \end{cases}, \quad fd(b^2) = fd(a^2) = \begin{cases} 1 \\ 9 \end{cases}, \quad fd(z) = \begin{cases} 1 \\ 9 \end{cases}, \quad fd(y) = \begin{cases} 1 \\ 9 \end{cases}$$

Therefore we have to check four cases:

- I $fd(x) = 0, fd(y) = fd(z) = 1$
- II $fd(x) = 0, fd(y) = 9, fd(z) = 1$
- III $fd(x) = 0, fd(y) = 1, fd(z) = 9$
- IV $fd(x) = 0, fd(y) = 9, fd(z) = 9.$

In the **I** case, two numbers, with the first digit equal **1**, have the difference multiple of 10, so we can write:

$$x = 10^s t, y = 10Y + 1, z = y + m10^r = 10Y + 1 + m10^r,$$

with $fd(t) > 0$,

we have

$$r = 4s,$$

because, by

$$x^4 = z^4 - y^4 = (z - y)(z + y)(z^2 + y^2),$$

we have

$$10^4 s t^4 = m10^r (m10^r + 2 \cdot 10Y + 2)(m^2 \cdot 10^{2r} + 2 \cdot 10^2 \cdot Y^2 + 2 + 2m10^{r+1}Y + 2m10^r + 4 \cdot 10Y);$$

we can write again

$$t^4 = m(2 + m10^r + 2 \cdot 10Y)(2 + m^2 10^{2r} + 2 \cdot 10^2 Y^2 + 2m10^{r+1}Y + 2m10^r + 4 \cdot 10Y) \quad (5)$$

This is absurd because

$$fd(t^4) = \begin{cases} 1 \\ 5 \end{cases} \text{ and } fd(\text{second term}) = 4m,$$

that is

$$1 = 4m \text{ or } 5 = 4m$$

but it is impossible, because the second term is even and the first is 1 or 5: only if

$$t = 0 \text{ and } m = 0$$

the relation **(5)** is possible, that is

$$a = b \quad \text{or} \quad u = v \quad \text{or} \quad x = 0,$$

but this is impossible because x is positive.

In the **II** case we have:

$$x = 10^s t, y = 10Y + 9, z = y + m10^{r-1} + 2 = 10Y + 9 + m10^{r-1} + 2 = 10(Y + 1) + m10^{r-1} + 1$$

and we return to the **I** case.

It is similar in the cases **III** and **IV**.

We conclude that the equation **(2)** has no integer positive solution.

q.e.d.

Third Theorem: The equation

$$x^{2^n} + y^{2^n} = z^{2^n} \quad \forall n \geq 2$$

has no integer positive solution.

Proof: We can write

$$((x^{2^{n-2}})^2)^2 + ((y^{2^{n-2}})^2)^2 = ((z^{2^{n-2}})^2)^2$$

If we lay down

$$X = x^{2^{n-2}}, Y = y^{2^{n-2}}, Z = z^{2^{n-2}}$$

therefore we have

$$X^4 + Y^4 = Z^4$$

and there are no solution for the second theorem.

q.e.d.

3. The Last Theorem of Fermat for $2n+1$

3.1. The case $2n+1$, that is the Last Theorem of Pierre de Fermat

Here is the

Last Theorem of Pierre de Fermat and Andrew Wiles: The equation

$$x^{2n+1} + y^{2n+1} = z^{2n+1} \quad (1)$$

has no integer positive solution.

Proof: We could suppose that (x, y, z) is solution with x, y, z prime among them, in particular x and y cannot be both multiple of $2n+1$: let y not be multiple of $2n+1$.

Let

$$z = x + k \text{ with } k \text{ prime with } x.$$

It follows that

$$\begin{aligned} y^{2n+1} = z^{2n+1} - x^{2n+1} &= (z-x)(z^{2n} + z^{2n-1}x + \dots + zx^{2n-1} + x^{2n}) = \\ &= k[(2n+1)x^{2n} + n(2n+1)x^{2n-1}k + \dots + k^{2n}] \end{aligned}$$

Every factor of k must divide y^{2n+1} , then every factor of k will be a power of exponent $2n+1$, otherwise it should divide also $(2n+1)x^{2n}$, that is x or $2n+1$, against the hypothesis; so k will be in the form of

$$k = u^{2n+1}$$

and also

$$v^{2n+1} = (2n+1)x^{2n} + n(2n+1)x^{2n-1}k + \dots + k^{2n},$$

that is

$$v^{2n+1} = (2n+1)x^{2n} + n(2n+1)x^{2n-1}u^{2n+1} + \dots + u^{2n}(2n+1), \quad (2)$$

where u and v are prime among them.

This result, that is

$$y = uv \text{ and } k = u^{2n+1},$$

can be obtained also by the relations of Waring.

We observe that v is always odd because x and z cannot be even at the same time.

By the relation (2), working out the term

$$(2n + 1)x^{2n} = v^{2n+1} - \dots - u^{2n} (2n + 1)$$

we obtain that

- if x is even, then u is odd;
- if x is odd, then u is either even or odd.

In other words one has two cases:

- the first one: x even, y and z odd;
- the second: x and y odd, z even.

Therefore

$$z = x + u^{2n+1}$$

is either odd or even.

At the same time, we have

$$z^{2n+1} = x^{2n+1} + y^{2n+1} = (x + y)(x^{2n} - x^{2n-1}y + \dots + y^{2n}).$$

Edward Waring, professor of Cambridge University (Shrewsbury, Shropshire, 1734-Pleasley, 1798) worked out

$$x^{2n} - x^{2n-1}y + \dots + y^{2n}$$

in a polynomial of $x+y$ and xy .

We remember that x and y are prime among them, so $x+y$ is odd if x is even and divides z^{2n+1} then the factors of z , that divide $x+y$, cannot divide xy , that is

$$x^{2n} - x^{2n-1}y + \dots + y^{2n},$$

such factor of $x^{2n+1} + y^{2n+1}$ we indicate with

$$p^{2n+1}.$$

One concludes:

$$x + y = p^{2n+1} \quad z = pq,$$

that is

and

$$z^{2n+1} = p^{2n+1}q^{2n+1}, \text{ with } \mathbf{p} \text{ and } \mathbf{q} \text{ prime among them and } \mathbf{q} \text{ odd}$$

and

$$q^{2n+1} = x^{2n} - x^{2n-1}y + \dots + y^{2n}.$$

Finally, we have

$$x = pq - u^{2n+1} \text{ and } x = p^{2n+1} - uv.$$

We remember that when z is odd, then \mathbf{p} and \mathbf{q} are odd. Therefore it is

$$z + y = pq + uv = x + y + z - x = p^{2n+1} + u^{2n+1} = (p + u)(p^{2n} - p^{2n-1}u + \dots + u^{2n}).$$

The factors p , q , v and u are prime among them and $p+u$ divides $z+y$, so

$$\exists c : c = \frac{z+y}{u+p} \text{ with } v < c < q,$$

or

$$pq+uv=c(u+p)$$

that is

$$u(c-v)=p(q-c),$$

but u and p are prime among them, so exists a such that

$$c-v=ap \text{ and } q-c=au$$

or

$$c = v + ap = q - au ;$$

where

$$a = \frac{q-v}{u+p}$$

and

$$q - au = v + ap = u^{2n} - u^{2n-1}p + \dots + p^{2n} ;$$

By those relations, one has:

$$(I) \quad p^{2n} - q = u(p^{2n-1} - up^{2n-2} + \dots + u^{2n-2}p - u^{2n-1} - a) = ul$$

and

$$(II) \quad v - u^{2n} = p(p^{2n-1} - up^{2n-2} + \dots + u^{2n-2}p - u^{2n-1} - a) = pl.$$

We observe that in the relation (I) the first term is even when p is odd, in that case ul is even and also a is even.

Likewise, placed

$$t = p^{2n} - q = ul \text{ and } s = v - u^{2n} = pl,$$

we have:

$$\begin{aligned} x + y - z = p^{2n+1} - pq &= p(p^{2n} - q) = pt = pul = us = \\ &= y - (z - x) = uv - u^{2n+1} = u(v - u^{2n}) = ulp \end{aligned}$$

We observe that l is prime with u , p , v and q ; looking l :

l is prime with u and p ;

l is prime with v :

in fact, if l were not prime with v , we would have a common factor of v , but

$$u(v) - pu(l) = y - (x + y - z) = z - x = u^{2n+1},$$

and u is prime with v ;

l is prime with q : so, if l were not prime with q , they would have a common factor of q , but it is

$$p(q) + up(l) = z + (x + y - z) = x + y = p^{2n+1}$$

and p is prime with q .

We observe that all the factors of u , v , p , q , uv , up , uq , vp , vq , pq do not divide $uv+pq$ and $u+v$, $p+q$.

For example, we proof that up does not divide $uv+pq$.

In fact, although it is obvious, if up divides $uv+pq$, then

$$p(uv + pq) - v(up) = p^2 q \text{ and } u(uv + pq) - q(up) = u^2 v,$$

that is p^2q and u^2v must have a common factor, against the hypothesis that pq and uv are prime among them.

We observe that l is the factor of x and of $z-y$ in fact it is

$$ulp = x + y - z = x - (z - y) \quad .$$

Then we have $l < x$: in fact, if it were $l=x$, we would have the absurd:

$$z - y = x - upx = x(1 - up) \leq 0 .$$

Now, we take:

$$x^{2n+1} - y^{2n+1} = z^{2n+1} - y^{2n+1} = (z - y)(z^{2n} + z^{2n-1}y + \dots + y^{2n}) ;$$

a factor of $z-y$, (for Waring), cannot divide zy , because z and y are prime among them; therefore, we have:

$$z - y = l^{2n+1} \text{ and } x = lf, \text{ so } f^{2n+1} = z^{2n} + z^{2n-1}y + \dots + y^{2n} ;$$

with l and f prime among them and f is odd.

At the same time , we have:

$$x^{2n+1} + z^{2n+1} = (x + z)(x^{2n} - x^{2n-1}z + \dots + z^{2n})$$

and

$$x + z = x + y + z - y = p^{2n+1} + l^{2n+1} = lf + pq = (p + l)(p^{2n} - p^{2n-1}l + \dots + l^{2n}) ,$$

from here, we obtain that

$$p + l \text{ divides } lf + pq ,$$

it is:

$$\exists c = \frac{lf + pq}{p + l} = q - bl = f + bp, \text{ with } b = \frac{q - f}{p + l}$$

because

$$c(p+l) = lf + pq$$

or

$$l(c-f) = p(q-c)$$

in which p and l are prime among them;

then

$$q - bl = f + bp = p^{2n} - p^{2n-1}l + \dots + l^{2n} .$$

We have:

$$(III) \quad p^{2n} - q = l(p^{2n-1} - p^{2n-2}l + \dots - l^{2n-1} - b) = li = lu ,$$

$$(IV) \quad f - l^{2n} = p(p^{2n-1} - p^{2n-2}l + \dots - l^{2n-1} - b) = pi = pu ,$$

but, $i=u$ because

$$\begin{aligned} x + y - z &= lf - l^{2n+1} = l(f - l^{2n}) = upl = ipl = \\ &= y - (z - x) = uv - u^{2n+1} = u(v - u^{2n}) = ulp \end{aligned}$$

We observe that in the relation (III), the first term is even when p is odd, in that case lu is even, moreover also b , as a , is even.

We conclude:

$$\begin{aligned} x &= lf, y = uv, z = pq \\ \text{and} \end{aligned}$$

$$lu = p^{2n} - q, pu = f - l^{2n}, pl = v - u^{2n}. \quad (3)$$

Now we consider

$$x - y = z - y - (z - x) = l^{2n+1} - u^{2n+1} = lf - uv = (l - u)(l^{2n} + l^{2n-1}u + \dots + u^{2n}).$$

It is:

$$\exists c = \frac{lf - uv}{l - u} = f - du = v - dl \quad \text{with } d = \frac{v - f}{l - u} \quad \text{with } d = v - f,$$

because

$$c(l - u) = lf - uv$$

or

$$l(c - f) = u(c - v).$$

Then, we have:

$$f - du = v - dl = l^{2n} + l^{2n-1}u + \dots + u^{2n}$$

and

$$(V) \quad f - l^{2n} = u(l^{2n-1} + l^{2n-2}u + \dots + u^{2n-1} + d) = ue = up,$$

$$(VI) \quad v - u^{2n} = l(l^{2n-1} + l^{2n-2}u + \dots + u^{2n-1} + d) = le = lp,$$

because

$$x + y - z = lf - l^{2n+1} = l(f - l^{2n}) = lue = u(v - u^{2n}) = ulp.$$

We observe that d is even if u is odd and p even.

Finally, we have:

$$x = p^{2n+1} - uv, y = uv, z = (x + y) - (x + y - z) = p^{2n+1} - ulp,$$

which verify the (2), that is

$$(p^{2n+1} - uv)^{2n+1} + (uv)^{2n+1} = (p^{2n+1} - ulp)^{2n+1},$$

or

$$(p^{2n+1} - uv + uv)(x^{2n} - x^{2n-1}y + \dots + y^{2n}) = (p^{2n+1} - ulp)^{2n+1},$$

dividing for p^{2n+1} , we have:

$$x^{2n} - x^{2n-1}y + \dots + y^{2n} = (p^{2n} - ul)^{2n+1},$$

that is

$$q^{2n+1} = q^{2n+1}.$$

From this verification , we are sure on the developments.
 For the conclusion of theorem we take again the relations:

$$\frac{uv + pq}{u + p} = v + ap = q - au \tag{4}$$

$$\frac{lf + pq}{p + l} = q - bl = f + bp \tag{5}$$

$$\frac{lf - uv}{l - u} = f - du = v - dl \tag{6}$$

where, between **a , b, d** , at least two are even .
 Now, we remember:

$$ulp = x + y - z$$

is even, so among **u, l, p**, only one is even.

By **(I), (III), (V)** we have

$$\begin{aligned} p^{2n-1} - up^{2n-2} + \dots u^{2n-2} p - u^{2n-1} - a &= l \\ p^{2n-1} - lp^{2n-2} + \dots l^{2n-2} p - l^{2n-1} - b &= u \\ l^{2n-1} + ul^{2n-2} + \dots u^{2n-2} l + u^{2n-1} + d &= p \end{aligned}$$

or

$$\begin{aligned} p^{2n-1} - up^{2n-2} + \dots u^{2n-2} p - u^{2n-1} - l &= a \\ p^{2n-1} - lp^{2n-2} + \dots l^{2n-2} p - l^{2n-1} - u &= b \\ p - (l^{2n-1} + ul^{2n-2} + \dots u^{2n-2} l + u^{2n-1}) &= d \end{aligned} \tag{7}$$

Now we prefer to work in this way: by

$$\begin{aligned} z - y &= l^{2n+1} , \quad z - x = u^{2n+1} , \quad x + y = p^{2n+1} , \quad x = z - y + (x + y - z) = l^{2n+1} + ulp , \\ y &= z - x + (x + y - z) , \quad z = x + y - (x + y - z) , \end{aligned}$$

we have

$$\begin{cases} x = lf = 2^{-1}(l^{2n+1} - u^{2n+1} + p^{2n+1}) = l^{2n+1} + upl \\ y = uv = 2^{-1}(-l^{2n+1} + u^{2n+1} + p^{2n+1}) = u^{2n+1} + upl \\ z = pq = 2^{-1}(l^{2n+1} + u^{2n+1} + p^{2n+1}) = p^{2n+1} - upl \\ x + y - z = upl = 2^{-1}(-l^{2n+1} - u^{2n+1} + p^{2n+1}) \end{cases} \tag{8}$$

By the relations **(8)**, we have

$$\begin{aligned} l \text{ divides } p^{2n+1} - u^{2n+1} \\ u \text{ divides } p^{2n+1} - l^{2n+1} \end{aligned}$$

$$p \text{ divides } u^{2n+1} + l^{2n+1}$$

and exist c_1, c_2, c_3 such that

$p^{2n+1} - u^{2n+1} = lc$, $p^{2n+1} - l^{2n+1} = uc$, $u^{2n+1} + l^{2n+1} = pc$, with c_2 and c_3 odd if l even.

c_1 and c_2 odd if p even

It turns out

$$c_1 = l^{2n} + 2up, \quad c_2 = u^{2n} + 2lp, \quad c_3 = p^{2n} - 2ul$$

with $c_1 = 2C_1$ and C_1 odd, if l is even, because $c_1 = 2\left(\frac{l^{2n}}{2} + up\right)$;

with $c_3 = 2C_3$ and C_3 odd, if p is even, because $c_3 = 2\left(\frac{p^{2n}}{2} - ul\right)$

In fact, it is

$$p^{2n+1} - u^{2n+1} - l^{2n+1} = (x+y) - (z-x) - (z-y) = 2(x+y-z) = 2ulp$$

and

$$p^{2n+1} - u^{2n+1} = l(l^{2n} + 2up), \quad p^{2n+1} - l^{2n+1} = u(u^{2n} + 2lp), \quad u^{2n+1} + l^{2n+1} = p(p^{2n} - 2ul)$$

In addition, we conclude that since, from (8), $f = l^{2n} + up$, $v = u^{2n} + lp$, $q = p^{2n} - ul$

$$c_1 = f + up, \quad c_2 = v + lp, \quad c_3 = q - ul. \quad (9)$$

We observe that

$$(2^\alpha x_1 + 1)^{2n} = 2^{\alpha(2n-1)} x_1^{2\alpha+1} + \dots + (2n+1) 2^\alpha x_1 + 1 = 2^\alpha x_3 \text{ with } h \geq 1$$

and

$$(2^\alpha x_1 + 1)^{2n+1} = 2^{\alpha(2n+1)} x_1^{2n+1} + \dots + (2n+1) \cdot 2^\alpha x_1 + 1 = 2^\alpha x_3 + 1$$

With x_1 , x_2 and x_3 odd.

We use this symbology on the following transformations.

The problem of Fermat is symmetric with respect to x and y , therefore, for simplicity, we consider two cases:

the first: p odd and among l and u ,
 l even;

the second: only p even

p even.

We go on in the first case and we use the above-mentioned symbology on the following transformations.

Let

$$\begin{aligned} l &= 2^\alpha L & l^2 &= 2^{2\alpha} L^2 & l^3 &= 2^{3\alpha} L^3 \\ u &= 2^\beta U_1 + 1 & u^2 &= 2^{\beta+1} U_2 + 1 & u^3 &= 2^\beta U_3 + 1 \\ p &= 2^\gamma P_1 + 1 & p^2 &= 2^{\gamma+1} P_2 + 1 & p^3 &= 2^\gamma P_3 + 1 \\ f &= 2^\delta F_1 + 1 & v &= 2^\epsilon V_1 + 1 & q &= 2^\rho Q_1 + 1 \end{aligned}$$

$$c_1 = 2C_1 \quad c_2 = 2^\tau C_2 + 1 \quad c_3 = 2^\sigma C_3 + 1$$

with $L, U_1, U_2, U_3, P_1, P_2, P_3, F_1, V_1, Q_1, C_1, C_2, C_3$ odd

We begin to examine the first of **(8)**

$$2^\alpha L(2^\delta F_1 + 1) = 2^{-1}(2^{(2n+1)\alpha} L^{2n+1} - 2^\beta U_3 + 2^\gamma P_3) =$$

$$= 2^{(2n+1)\alpha} L^{2n+1} + 2^\alpha L(2^\beta U_1 + 1)(2^\gamma P_1 + 1)$$

or

$$2^\delta F_1 + 1 = 2^{2n\alpha-1} L^{2n} + L^{-1}(2^{\gamma-\alpha-1} P_3 - 2^{\beta-\alpha-1} U_3) = 2^{2n\alpha} L^{2n} + (2^\beta U_1 + 1)(2^\gamma P_1 + 1);$$

we have two possibilities:

- (a)** $\gamma - \alpha - 1 = 0$ if $\gamma < \beta$
 - (b)** $\beta - \alpha - 1 = 0$ if $\beta < \gamma$
- that is

- (a)** $\gamma = \alpha + 1$, $\delta = \gamma = \alpha + 1$ if $\gamma < \beta$
- (b)** $\beta = \alpha + 1$, $\delta = \gamma = \alpha + 1$ if $\beta < \gamma$

Note down: the case $\gamma = \beta$ is impossible because, in such hypothesis

$2^\delta F_1 + 1$ is odd and

$$2^{\gamma-\alpha-1} P_3 - 2^{\beta-\alpha-1} U_3 = 2^{\beta-\alpha-1} (P_3 - U_3) \text{ is even.}$$

Now we examine the second of **(8)**

$$(2^\beta U_1 + 1)(2^\varepsilon V_1 + 1) = 2^{-1}(-2^{(2n+1)\alpha} L^{2n+1} + 2^\beta U_3 + 2^\gamma P_3 + 2) = 2^\beta U_3 + 1 +$$

$$+ 2^\alpha L(2^\beta U_1 + 1)(2^\gamma P_1 + 1)$$

or

$$(2^\beta U_1 + 1)(2^\varepsilon V_1 + 1) = (-2^{(2n+1)\alpha-1} L^{2n+1} + 2^{\beta-1} U_3 + 2^{\gamma-1} P_3 + 1) = 2^\beta U_3 + 1 +$$

$$+ 2^\alpha L(2^\beta U_1 + 1)(2^\gamma P_1 + 1)$$

that is

$$(2^\beta U_1 + 1)(2^\varepsilon V_1 + 1) = 2^{-1}[(2^\beta U_1 + 1)^{2n+1} + (2^\beta U_1 + 1)(2^\tau C_2 + 1)] =$$

$$= (2^\beta U_1 + 1)^{2n+1} + 2^\alpha L(2^\beta U_1 + 1)(2^\gamma P_1 + 1)$$

or

$$2^\varepsilon V_1 + 1 = 2^{-1}(2^{\beta+h} U_2 + 1 + 2^\tau C_2 + 1) = 2^{\beta+h} U_2 + 1 + 2^\alpha L(2^\gamma P_1 + 1)$$

and

$$2^\varepsilon V_1 + 1 = 2^{\beta+h-1} U_2 + 2^{\tau-1} C_2 + 1 = 2^{\beta+h} U_2 + 1 + 2^\alpha L(2^\gamma P_1 + 1)$$

We have

- (a)** $\gamma = \alpha + 1$, $\delta = \alpha + 1$, $\varepsilon = \gamma - 1 = \alpha$, $\tau - 1 = \alpha$
- (b)** $\beta = \alpha + 1$, $\delta = \alpha + 1$, $\varepsilon = \beta - 1 = \alpha$, $\tau - 1 = \alpha$.

Now we examine the third of **(8)**

$$(2^\gamma P_1 + 1)(2^\rho Q_1 + 1) = 2^{-1}(2^{(2n+1)\alpha} L^{2n+1} + 2^\beta U_3 + 2^\gamma P_3 + 2) =$$

$$= 2^\gamma P_3 + 1 - 2^\alpha L(2^\beta U_1 + 1)(2^\gamma P_1 + 1)$$

or

$$(2^\gamma P_1 + 1)(2^\rho Q_1 + 1) = 2^{(2n+1)\alpha-1} L^{2n+1} + 2^{\beta-1} U_3 + 2^{\gamma-1} P_3 + 1 = \\ = 2^\gamma P_3 + 1 - 2^\alpha L(2^\beta U_1 + 1)(2^\gamma P_1 + 1)$$

that is

$$(2^\gamma P_1 + 1)(2^\rho Q_1 + 1) = 2^{-1}[(2^\gamma P_1 + 1)^{2n+1} + (2^\sigma C_3 + 1)(2^\gamma P_1 + 1)] = \\ = (2^\gamma P_1 + 1)^{2n+1} - 2^\alpha L(2^\beta U_1 + 1)(2^\gamma P_1 + 1)$$

or

$$2^\rho Q_1 + 1 = 2^{-1}(2^{\gamma+h} P_2 + 1 + 2^\sigma C_3 + 1) = 2^{\gamma+h} P_2 + 1 - 2^\alpha L(2^\beta U_1 + 1)$$

and

$$2^\rho Q_1 + 1 = 2^{\gamma+h-1} P_2 + 2^{\sigma-1} C_3 + 1 = 2^{\gamma+h} P_2 + 1 - 2^\alpha L(2^\beta U_1 + 1).$$

We have

(a) $\gamma = \alpha + 1, \delta = \alpha + 1, \varepsilon = \alpha, \tau = \alpha + 1, \rho = \alpha, \sigma = \alpha + 1, (\gamma < \beta)$

(b) $\beta = \alpha + 1, \delta = \alpha + 1, \varepsilon = \alpha, \tau = \alpha + 1, \rho = \alpha, \sigma = \alpha + 1.$

The problem is symmetric with respect to γ and β , therefore on the cases **(a)**, **(b)**, we study only the case **(a)**.

We obtain, in the case **(a)**,

$$\begin{aligned} c_1 &= 2C_1 & c_2 &= 2^{\alpha+1}C_2 + 1 & c_3 &= 2^{\alpha+1}C_3 + 1 \\ l &= 2^\alpha L & u &= 2^\beta U_1 + 1 & p &= 2^{\alpha+1}P_1 + 1 \\ f &= 2^{\alpha+1}F_1 + 1 & v &= 2^\alpha V_1 + 1 & q &= 2^\alpha Q_1 + 1 \end{aligned}$$

Now we consider the hypothesis **(a)** $\beta > \gamma = \alpha + 1$, for relations **(9)**

$$\begin{aligned} 2C_1 &= 2^{\alpha+1}F_1 + 1 + (2^\beta U_1 + 1)(2^{\alpha+1}P_1 + 1), \\ 2^{\alpha+1}C_2 + 1 &= 2^\alpha V_1 + 1 + 2^\alpha L(2^{\alpha+1}P_1 + 1), \\ 2^{\alpha+1}C_3 + 1 &= 2^\alpha Q_1 + 1 - (2^\beta U_1 + 1)2^\alpha L. \end{aligned}$$

that is

$$\begin{aligned} C_1 &= 2^\alpha F_1 + 1 + 2^{\alpha+\beta} U_1 P_1 + 2^{\beta-1} U_1 + 2^\alpha P_1, \\ 2C_2 &= V_1 + L(2^{\alpha+1} P_1 + 1), \\ 2C_3 &= Q_1 - (2^\beta U_1 + 1)L. \end{aligned}$$

We conclude that we can write

$$C_1 = 2^\alpha \Gamma_1 + 1, \text{ with } \Gamma_1 \text{ odd.}$$

By $c_1 = f + up$, we obtain

$$2^{\alpha+1} \Gamma_1 + 2 = (2^{\alpha+1} F_1 + 1) + (2^\beta U_1 + 1)(2^{\alpha+1} P_1 + 1)$$

that is

$$2^{\alpha+1} \Gamma_1 = 2^{\alpha+1} F_1 + 2^{\alpha+\beta+1} U_1 P_1 + 2^{\alpha+1} P_1 + 2^\beta U_1$$

or

$$\Gamma_1 = F_1 + 2^\beta U_1 P_1 + P_1 + 2^{\beta-\alpha-1} U_1;$$

this relation is an absurd because the first term, $\forall \alpha > 0$ and $\forall \beta > \alpha + 1$, is odd and the second is even.

We conclude that, in the cases **(a)** and **(b)**, no integer positive primitive solution of (1) is possible, so no integer positive primitive solution of (1) is possible.

Now we consider the second case: it is similar to first one, exchanging the roles of l and p. We deal the second case using the showed symbology on the following transformations.

Let

$$\begin{aligned}
 l &= 2^\alpha L_1 + 1 & l^{2^n} &= 2^{\alpha+1} L_2 + 1 & l^{2^{n+1}} &= 2^\alpha L_3 + 1 \\
 u &= 2^\beta U_1 + 1 & u^{2^n} &= 2^{\beta+1} U_2 + 1 & u^{2^{n+1}} &= 2^\beta U_3 + 1 \\
 p &= 2^\gamma P & p^{2^n} &= 2^{2\gamma} P^{2^n} & p^{2^{n+1}} &= 2^{(2n+1)\gamma} P^{2^{n+1}} \\
 f &= 2^\delta F_1 + 1 & v &= 2^\varepsilon V_1 + 1 & q &= 2^\rho Q_1 + 1 \\
 & & c_1 &= 2^\tau C_1 + 1 & c_2 &= 2^\sigma C_2 + 1 & c_3 &= 2C_3
 \end{aligned}$$

with $L, U_1, U_2, U_3, P_1, P_2, P_3, F_1, V_1, Q_1, C_1, C_2, C_3$ odd

We begin to examine the third of (8)

$$\begin{aligned}
 2^\gamma P(2^\rho Q_1 + 1) &= 2^{-1}(2^\alpha L_3 + 2^\beta U_3 + 2^{(2n+1)\gamma} P^{2^{n+1}} + 2) = \\
 &= 2^{(2n+1)\gamma} P^{2^{n+1}} - 2^\gamma P(2^\beta U_1 + 1)(2^\alpha L_1 + 1)
 \end{aligned}$$

or

$$2^\rho Q_1 + 1 = 2^{2n\gamma-1} P^{2^n} + P^{-1} 2^{-\gamma} (2^{\alpha-1} L_3 + 2^{\beta-1} U_3 + 1) = 2^{2n\gamma} P^{2^n} - (2^\beta U_1 + 1)(2^\alpha L_1 + 1);$$

we have two possibilities:

- (a) $\alpha - \gamma - 1 = 0$ if $\alpha < \beta$
- (b) $\beta - \gamma - 1 = 0$ if $\beta < \alpha$

that is

- (a) $\alpha = \gamma + 1, \rho = \alpha = \gamma + 1$ if $\alpha < \beta$
- (b) $\beta = \gamma + 1, \rho = \beta = \gamma + 1$ if $\beta < \alpha$

Note down: the case $\alpha = \beta$ is impossible because, in such hypothesis

$2^\delta Q_1 + 1$ is odd and

$$2^{\alpha-\gamma-1} L_3 - 2^{\beta-\gamma-1} U_3 + 1 = 2^{\alpha-\gamma-1} (L_3 + U_3) + 1 \text{ could not be divisible for } P \text{ even.}$$

Now we examine the second of (8)

$$\begin{aligned}
 (2^\beta U_1 + 1)(2^\varepsilon V_1 + 1) &= 2^{-1}(-2^\alpha L_3 + 2^\beta U_3 + 2^{(2n+1)\gamma} P^{2^{n+1}}) = 2^\beta U_3 + 1 + \\
 &+ (2^\alpha L_1 + 1)(2^\beta U_1 + 1)(2^\gamma P)
 \end{aligned}$$

or

$$\begin{aligned}
 (2^\beta U_1 + 1)(2^\varepsilon V_1 + 1) &= (2^{(2n+1)\gamma-1} P^{2^{n+1}} + 2^{\beta-1} U_3 - 2^{\alpha-1} L_3) = 2^\beta U_3 + 1 + \\
 &+ (2^\alpha L_1 + 1)(2^\beta U_1 + 1)(2^\gamma P)
 \end{aligned}$$

that is

$$\begin{aligned}
 (2^\beta U_1 + 1)(2^\varepsilon V_1 + 1) &= 2^{-1}[(2^\beta U_1 + 1)^{2^{n+1}} + (2^\beta U_1 + 1)(2^\sigma C_2 + 1)] = \\
 &= (2^\beta U_1 + 1)^{2^{n+1}} + (2^\alpha L_1 + 1)(2^\beta U_1 + 1)(2^\gamma P)
 \end{aligned}$$

or

$$2^\varepsilon V_1 + 1 = 2^{-1}(2^{\beta+h} U_2 + 1 + 2^\sigma C_2 + 1) = 2^{\beta+h} U_2 + 1 + (2^\alpha L_1 + 1)(2^\gamma P)$$

and

$$2^\varepsilon V_1 + 1 = 2^{\beta+h-1} U_2 + 2^{\sigma-1} C_2 + 1 = 2^{\beta+h} U_2 + 1 + (2^\alpha L_1 + 1)(2^\gamma P)$$

We have

$$(a) \alpha = \gamma + 1, \rho = \gamma + 1, \varepsilon = \gamma = \alpha - 1, \sigma - 1 = \gamma$$

$$(b) \beta = \gamma + 1, \rho = \gamma + 1, \varepsilon = \gamma = \beta - 1, \sigma - 1 = \gamma.$$

Now we examine the first of (8)

$$(2^\alpha L_1 + 1)(2^\delta F_1 + 1) = 2^{-1}(2^\alpha L_3 - 2^\beta U_3 + 2^{(2n+1)\gamma} P^{2n+1}) = 2^\alpha L_3 + 1 + (2^\alpha L_1 + 1)(2^\beta U_1 + 1)(2^\gamma P)$$

or

$$(2^\alpha L_1 + 1)(2^\delta F_1 + 1) = 2^{(2n+1)\gamma-1} P^{2n+1} - 2^{\beta-1} U_3 + 2^{\alpha-1} L_3 + 1 = 2^\alpha L_3 + 1 + 2^\gamma P(2^\beta U_1 + 1)(2^\alpha L_1 + 1)$$

that is

$$(2^\alpha L_1 + 1)(2^\delta F_1 + 1) = 2^{-1}[(2^\alpha L_1 + 1)^{2n+1} + (2^\tau C_1 + 1)(2^\alpha L_1 + 1)] = (2^\alpha L_1 + 1)^{2n+1} + (2^\alpha L_1 + 1)(2^\beta U_1 + 1)(2^\gamma P)$$

or

$$2^\delta F_1 + 1 = 2^{-1}(2^{\alpha+h} L_2 + 1 + 2^\tau C_1 + 1) = 2^{\alpha+h} L_2 + 1 + 2^\gamma P(2^\beta U_1 + 1)$$

and

$$2^\delta F_1 + 1 = 2^{\alpha+h-1} L_2 + 2^{\tau-1} C_1 + 1 = 2^{\alpha+h} L_2 + 1 - 2^\gamma P(2^\beta U_1 + 1)$$

We have

$$(a) \alpha = \gamma + 1, \rho = \gamma + 1, \varepsilon = \gamma, \sigma = \gamma + 1, \delta = \gamma, \tau = \gamma + 1, (\alpha < \beta)$$

$$(b) \beta = \gamma + 1, \rho = \gamma + 1, \varepsilon = \gamma, \sigma = \gamma + 1, \delta = \gamma, \tau = \gamma + 1.$$

The problem is symmetric with respect to \mathbf{a} and $\mathbf{\beta}$, therefore on the cases (a), (b), we study only the case (a).

We obtain, in the case (a),

$$\begin{aligned} c_1 &= 2^{\gamma+1} C_1 + 1 & c_2 &= 2^{\gamma+1} C_2 + 1 & c_3 &= 2C_3 \\ l &= 2^{\gamma+1} L_1 + 1 & u &= 2^\beta U_1 + 1 & p &= 2^\gamma P \\ f &= 2^{\gamma+1} F_1 + 1 & v &= 2^\gamma V_1 + 1 & q &= 2^{\gamma+1} Q_1 + 1 \end{aligned}$$

Now we consider the hypothesis (a) $\beta > \alpha = \gamma + 1$, for relations (9)

$$2C_3 = 2^{\gamma+1} Q_1 + 1 - (2^\beta U_1 + 1)(2^{\gamma+1} L_1 + 1),$$

$$2^{\gamma+1} C_1 + 1 = 2^\gamma F_1 + 1 + 2^\gamma P(2^\beta U_1 + 1),$$

$$2^{\gamma+1} C_2 + 1 = 2^\gamma V_1 + 1 + (2^\beta U_1 + 1)2^\gamma P.$$

that is

$$C_3 = 2^\gamma Q_1 + 1 - 2^{\gamma+\beta} U_1 L_1 - 2^{\beta-1} U_1 - 2^\gamma L_1,$$

$$2C_1 = F_1 + P(2^\beta U_1 + 1),$$

$$2C_2 = V_1 + (2^\beta U_1 + 1)P.$$

We conclude that we can write

$$C_3 = 2^\gamma \Gamma_1 + 1, \text{ with } \Gamma_1 \text{ odd.}$$

By $c_3 = q - ul$ we obtain

$$2^{\gamma+1}\Gamma_1 + 2 = (2^{\gamma+1}Q_1 + 1) - (2^\beta U_1 + 1)(2^{\gamma+1}L_1 + 1)$$

that is

$$2^{\gamma+1}\Gamma_1 + 2 = 2^{\gamma+1}Q_1 - 2^{\gamma+\beta+1}U_1L_1 - 2^{\gamma+1}L_1 - 2^\beta U_1$$

or

$$2^\gamma\Gamma_1 + 1 = 2^\gamma Q_1 - 2^{\beta+\gamma}U_1L_1 - 2^\gamma L_1 - 2^{\beta-1}U_1$$

this relation is an absurd because the first term, $\forall \gamma > 0$ and $\forall \beta > \gamma + 1$, is odd and the second is even.

We conclude that, in the cases (a) and (b), no integer positive primitive solution of (1) is possible, so no integer positive primitive solution of (1) is possible.

We conclude that, in both cases, no integer positive primitive solution of (1) is possible, so no integer positive primitive solution of (1) is possible. **q.e.d.**

Final note:

also the equation $p^{2n+1} - u^{2n+1} - l^{2n+1} = 2upl$ solves the problem.

4. The Last Theorem of Fermat for $n = 2^s (2t + 1)$

4.1. The case $n = 2^s (2t + 1)$

Here is the last

Theorem: The equation

$$x^{2^s (2t + 1)} + y^{2^s (2t + 1)} = z^{2^s (2t + 1)} \tag{1}$$

has no integer positive solution.

Proof: It is obvious that it is interpreted with the preceding theorems.

In fact the equation can be written

$$(x^{2^s})^{2t + 1} + (y^{2^s})^{2t + 1} = (z^{2^s})^{2t + 1}.$$

If we lay down

$$X = x^{2^s}, Y = y^{2^s}, Z = z^{2^s},$$

we have

$$X^{2t + 1} + Y^{2t + 1} = Z^{2t + 1} \tag{2}$$

and there is no solution for the proof of $2n+1$, therefore (x,y,z) , with

$$x = X^{\frac{1}{2^s}}, y = Y^{\frac{1}{2^s}}, z = Z^{\frac{1}{2^s}}$$

is not solution of (1), so $(X; Y; Z)$ of (2). **q.e.d.**

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