Modules with $\otimes(\otimes'$ or $\otimes'')$ Condition

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Abstract. In this paper we introduce and study modules with $\otimes$ ($\otimes'$ or $\otimes''$) condition. We give several properties of these types of modules and some relationships between them.

Introduction

Let M be an R -module, where R is a commutative ring with unity. Recall that a nonzero submodule N of M is called strongly hollow (briefly, SH-submodule) if whenever $L_1, L_2 \leq M$, $N \subseteq L_1 + L_2$ implies either $N \subseteq L_1$ or $N \subseteq L_2$, [1]. A submodule N of M is called strongly irreducible (SI-submodule) if whenever $L_1, L_2 \leq M$, $N \supseteq L_1 \cap L_2$ implies $N \supseteq L_1$ or $N \supseteq L_2$, [4]. The sets $\{K: K \text{ is a SH-submodule of } M\}$, $\{K: K \text{ is a proper SH-submodule of } M\}$ and $\{K: K \text{ is a nonzero proper SI-submodule of } M\}$ are denoted by $\overset{1SH}{Spec}(M)$, $\overset{2SH}{Spec}(M)$ and $\overset{SI}{Spec}(M)$ respectively, [8]. In [8] we studied and topologized these sets by setting that for any $L \leq M$

$$\begin{aligned} \overset{1SH}{V}(L) &= \{K \in \overset{1SH}{Spec}(M), K \subseteq L\}, \quad \overset{1SH}{\chi}(L) = \overset{1SH}{Spec}(M) - \overset{1SH}{V}(L), \quad \overset{1SH}{\zeta}(M) = \{\overset{1SH}{V}(L); L \leq M\}, \quad \overset{1SH}{\tau}(M) = \{\overset{1SH}{\chi}(L); L \leq M\}. \\ \overset{2SH}{V}(L) &= \{K \in \overset{2SH}{Spec}(M), K \supseteq L\}, \quad \overset{2SH}{\chi}(L) = \overset{2SH}{Spec}(M) - \overset{2SH}{V}(L), \quad \overset{2SH}{\zeta}(M) = \{\overset{2SH}{V}(L); L \leq M\}, \quad \overset{2SH}{\tau}(M) = \{\overset{2SH}{\chi}(L); L \leq M\}. \\ \overset{SI}{V}(L) &= \{K \in \overset{SI}{Spec}(M), K \subseteq L\}, \quad \overset{SI}{\chi}(L) = \overset{SI}{Spec}(M) - \overset{SI}{V}(L), \quad \overset{SI}{\zeta}(M) = \{\overset{SI}{V}(L); L \leq M\}, \quad \overset{SI}{\tau}(M) = \{\overset{SI}{\chi}(L); L \leq M\}. \end{aligned}$$

We prove that $(\overset{1SH}{Spec}(M), \overset{1SH}{\tau}(M))$ is a topological space (see [8, Th.2.1.9]). Also we see that $\overset{2SH}{\zeta}(M)$ is not closed under finite union, however all other axioms of closed sets of a topological space are valid (see [8, Th.2.4.1]). This led us to call an R -module M a $\overset{2SH}{top}$ -module if $\overset{2SH}{\zeta}(M)$ is closed under finite union. Equivalently M is a $\overset{2SH}{top}$ -module if $(\overset{2SH}{Spec}(M), \overset{2SH}{\tau}(M))$ is a topological space.

Beside these we see that $\overset{SI}{\zeta}(M)$ is not closed under finite union, however all other axioms of closed sets of a topological space are valid (see [8, Th.3.2.1]). This led us to call an R -module M is a $\overset{SI}{top}$ -module if $\overset{SI}{\zeta}(M)$ is closed under a finite union. Equivalently M is a $\overset{SI}{top}$ -module if $(\overset{SI}{Spec}(M), \overset{SI}{\tau}(M))$ is a topological space.

We notice that, for any $L_1 \leq M, L_2 \leq M$, if $\overset{1SH}{V}(L_1) = \overset{1SH}{V}(L_2)$ (or $\overset{2SH}{V}(L_1) = \overset{2SH}{V}(L_2)$ or $\overset{SI}{V}(L_1) = \overset{SI}{V}(L_2)$) then it is not necessarily that $L_1 = L_2$, as the following examples show.

- (1) Consider the Z -module Z , $\overset{1SH}{V}(3Z) = \overset{1SH}{V}(\{0\}) = \phi$ but $3Z \neq \{0\}$.
- (2) For the Z -module Z_{12} , $\overset{2SH}{V}(Z_{12}) = \phi = \overset{2SH}{V}(\langle \bar{2} \rangle)$ but $Z_{12} \neq \langle \bar{2} \rangle$.
- (3) For the Z -module Z_{12} , $\overset{SI}{V}(\langle \bar{6} \rangle) = \phi = \overset{SI}{V}(\{0\})$ but $\langle \bar{6} \rangle \neq \{0\}$.

These observations lead us to introduce the following conditions:

$$\circledast : \overset{1SH}{V}(L_1) = \overset{1SH}{V}(L_2) \Rightarrow L_1 = L_2, \text{ for each } L_1, L_2 \leq M.$$

$$\circledast' : \overset{2SH}{V}(L_1) = \overset{2SH}{V}(L_2) \Rightarrow L_1 = L_2, \text{ for each } L_1, L_2 \leq M.$$

$$\circledast'' : \overset{SI}{V}(L_1) = \overset{SI}{V}(L_2) \Rightarrow L_1 = L_2, \text{ for each } L_1, L_2 \leq M.$$

This paper is devoted to study modules with $\circledast$ ($\circledast'$, $\circledast''$ respectively). Also we shall study the behaviour $\overset{1SH}{\text{Spec}}(M)$, $\overset{2SH}{\text{Spec}}(M)$ and $\overset{SI}{\text{Spec}}(M)$ respectively when M satisfies $\circledast$ ($\circledast'$, $\circledast''$).

S.1 Modules with the Condition $\circledast$

We start this by the following remarks and examples.

Remarks and Examples 1.1:

- (1) The Z -module Z does not satisfies $\circledast$ since for each $L, N \leq Z$, $L \neq Z$, $\overset{1SH}{V}(L_1) = \overset{1SH}{V}(L_2) = \phi$
- (2) Every simple module satisfies $\circledast$.
- (3) Let M, M' be two isomorphic R -modules. Then M satisfies $\circledast$ if and only if M' satisfies $\circledast$.
- (4) Let M_1, M_2 be R -modules. If M_1, M_2 satisfies $\circledast$ condition, then $M_1 \oplus M_2$ may not be satisfy $\circledast$, as an example: Let $M_1 = Z_3$ as a Z -module and $M_2 = Z_4$ as a Z -module. Each of M_1 and M_2 satisfies $\circledast$. However $Z_3 \oplus Z_4 \cong Z_{12}$ and Z_{12} does not satisfy $\circledast$.
- (5) Let M be an R -module. Then M satisfies $\circledast$ as R -module if and only if M satisfies $\circledast$ as $\bar{R}$ -module where $\bar{R} = R/\text{ann } M$.

Proposition 1.2. Let M be an R -module such that every nonzero submodules is SH. Then M satisfies $\circledast$.

Proof: First note that $\overset{1SH}{V}(<0>) = \phi \neq \overset{1SH}{V}(N)$ for each $N \neq <0>$. Let $L, N \leq M$, $L \neq (0)$, $N \neq (0)$ such that $\overset{1SH}{V}(L) = \overset{1SH}{V}(N)$. Since $L \subseteq L$ and L is a SH-submodule by hypothesis, $L \in \overset{1SH}{V}(L) = \overset{1SH}{V}(N)$. It follows that $L \subseteq N$. Similarly, $N \in \overset{1SH}{V}(N) = \overset{1SH}{V}(L)$ and hence $N \subseteq L$. Thus $L = N$.

Recall that an R -module M is called chained if the lattice of its submodules is linearly ordered by inclusion [10].

Corollary 1.3. Let M be a chained R -module. Then M satisfies $\circledast$.

The following theorem gives a characterization of modules with the condition $\circledast$.

Theorem 1.4:

Let M be a nonzero R -module. Then M satisfies $\circledast$ if and only if every nonzero submodule of M can be represented as sum of SH-submodules.

Proof: $(\Rightarrow)$ Let $(0) \neq K \leq M$. Then $\overset{1SH}{V}(K) \neq \phi$, since if $\overset{1SH}{V}(K) = \phi$, then $\overset{1SH}{V}(K) = \overset{1SH}{V}(0)$ and hence $K =$

(0) (by $\circledast$), which is a contradiction. Set $N = \sum_{W \in \overset{1SH}{V}(K)} W$ and let $L \in \overset{1SH}{V}(N)$. Then $L \subseteq \sum_{W \in \overset{1SH}{V}(K)} W$

But for each $W \in \overset{1SH}{V}(K)$, $W \subseteq K$, so $N \subseteq K$. Thus $L \subseteq K$ and hence $L \in \overset{1SH}{V}(K)$. Thus $\overset{1SH}{V}(N) \subseteq \overset{1SH}{V}(K)$

Now, let $L \in \overset{1SH}{V}(K)$, then $L \subseteq N$ (by definition of N). Hence $L \in \overset{1SH}{V}(K)$ and so $\overset{1SH}{V}(N) \subseteq \overset{1SH}{V}(K)$. Thus $\overset{1SH}{V}(N) \subseteq \overset{1SH}{V}(K)$ and by $\otimes$, $K=N$. Thus K is a sum of SH-submodules.

($\Leftarrow$) Let $N \leq M$, $N \neq 0$. Then $N = \sum_{i \in \Lambda} T_i$ where T_i is a SH-submodule of M , $\forall i \in \Lambda$ (by hypothesis).

Since for each $i \in \Lambda$, $T_i \subseteq N$, we have $T_i \in \overset{1SH}{V}(N)$. Hence $N = \sum_{T_i \in \overset{1SH}{V}(N)} T_i$. Now, assume $N_1, N_2 \leq M$ such that $\overset{1SH}{V}(N_1) = \overset{1SH}{V}(N_2)$. But $N_1 = \sum_{T_i \in \overset{1SH}{V}(N_1)} T_i$, $N_2 = \sum_{S_i \in \overset{1SH}{V}(N_2)} S_i$, by the previous argument. Then $N_1 = N_2$. Thus M satisfies $\otimes$.

Corollary 1.5. Let M be a semisimple R -module such that every simple submodule is SH. Then M satisfies $\otimes$.

Recall that an R -module is called comultiplication if for each $N \leq M$, $N = \overset{\text{ann } I}{M}$ for some ideal I of R , [11] where $\overset{\text{ann } I}{M} = \{m \in M \mid Im = (0)\}$.

An R -module is called distributive if for each $N, L, W \leq M$, $N \cap (L + W) = (N \cap L) + (N \cap W)$, [6]

Lemma 1.6: [8, Cor.1.1.9], [8, Prop.1.1.11] Let M be a comultiplication (or distributive) R -module. Then every simple submodule of M is a SH-submodule.

Corollary 1.7. Let M be a semisimple comultiplication (or distributive). Then M satisfies $\otimes$.

Proof: It follows by Lemma 1.6 and Cor. 1.5.

Remarks 1.8:

(1) The condition “every simple submodule is a SH-submodule” is necessary in Cor.1.5, as for example: It is clear that the vector space $\mathbb{R}^2$ over $\mathbb{R}$ is semisimple, but $N_1 = \mathbb{R}(1,0)$ is a simple submodule of $\mathbb{R}^2$ and it is not SH. However $\mathbb{R}^2$ does not satisfies $\otimes$ since $\overset{1SH}{V}(N_1) = \overset{1SH}{V}(N_2) = \emptyset$, where $N_2 = \mathbb{R}(0,1)$.

(2) The converse of Cor.1.5 is not true in general, for example: consider the $\mathbb{Z}$ -module $\mathbb{Z}_4$, $\mathbb{Z}_4$ satisfies $\otimes$ and every nonzero submodule of $\mathbb{Z}_4$ is SH. However $\mathbb{Z}_4$ is not semisimple. Before giving the next result, we give the following lemma.

Lemma 1.9. Let M be an R -module. Then $\overset{1SH}{\text{Spec}}(M)$ is a T_1 -space if and only if every SH-submodule is minimal SH in $\overset{1SH}{\text{Spec}}(M)$.

Proof: ($\Rightarrow$) If $\overset{1SH}{\text{Spec}}(M)$ is T_1 . Suppose $\overset{1SH}{\text{Spec}}(M) = \emptyset$, then nothing to prove. Let $\overset{1SH}{\text{Spec}}(M) \neq \emptyset$. Since $\overset{1SH}{\text{Spec}}(M)$ is T_1 , then for any $N \in \overset{1SH}{\text{Spec}}(M)$, N is closed; that is $\{N\} = \overset{1SH}{V}(L)$ for some $L \leq M$. If N is not minimal SH, then there exists $K \in \overset{1SH}{\text{Spec}}(M)$ and $K \subsetneq N$. Hence $K \subseteq L$; that is $K \in \overset{1SH}{V}(L) = \{N\}$ and $K = N$ which is a contradiction. Therefore N is a minimal SH-submodule.

($\Leftarrow$) Let $N \in \overset{1SH}{\text{Spec}}(M)$. Then N is SH and so $N \in \overset{1SH}{V}(N)$. Assume there exists $L \leq M$, $L \neq N$ such that $L \in \overset{1SH}{V}(N)$. It follows that L is SH and $L \subseteq N$. Hence N is not a minimal SH-submodule which contradicts the hypothesis.

Theorem 1.10. Let M be a comultiplication R -module. Then $\overset{1SH}{\text{Spec}}(M)$ is T_1 and M satisfies $\otimes$ if and only if M is semisimple and every SH-submodule is minimal SH.

Proof: $(\Rightarrow)$ Since $\text{Spec}^{\text{SH}}(M)$ is T_1 , then by lemma 1.9, every SH-submodule is minimal SH. But M satisfies $\otimes$, so by Th.1.4, every submodule is a sum of SH-submodules. On the other hand, M is comultiplication and $\text{Spec}^{\text{SH}}(M)$ is T_1 imply $S(M) = \text{Spec}^{\text{SH}}(M)$ by [8, Cor.2.2.19] where $S(M)$ = set of all simple submodules of M . Thus every submodule of M is a sum of simple submodules; that is M is semisimple.

$(\Leftarrow)$ It follows by Cor.1.7 and Lemma 1.9.

Recall that an R -module M is called multiplication if for each $N \leq M$, there exists $I \leq R$ such that $N = IM$, [3].

Lemma 1.11. Let M be a faithful finitely generated multiplication over a comultiplication ring and let $(0) \neq N \leq M$. Then N is a minimal SH-submodule of M if and only if N is simple.

Proof: $(\Rightarrow)$ Since M is multiplication, $N = IM$ for some ideal I of R . Then by [8, Prop.1.2.1] I is a SH ideal of R . We claim that I is a simple ideal of R . If I is not simple, then there exists a simple

ideal J of R such that $J \subsetneq I$ since R is comultiplication. Moreover by Lemma 1.6, J is a SH ideal of R and so by [8, Prop.1.2.1] JM is a SH-submodule. But M is a faithful finitely generated

multiplication, so by [5, Th.3.1] $JM \subsetneq IM = N$. Thus N is not a minimal SH-submodule of M , which is a contradiction. Thus I is a simple ideal of R and so N is a simple submodule of M .

$(\Leftarrow)$ Let N be a simple submodule of M . Since M is multiplication then $N = IM$ for some ideal I of R . It is easy to check that I is a simple ideal of R . Then by Lemma 1.6, I is a SH ideal and so [8, Prop.1.2.1], N is a SH-submodule. Thus N is a minimal SH-submodule of M .

By using Lemma 1.11, we have the following immediate result.

Corollary 1.12. Let R be a comultiplication ring and let $J \leq R$. Then J is a minimal SH ideal if and only if J is a simple (minimal) ideal.

Now we have the following:

Theorem 1.13. Let M be a faithful finitely generated multiplication over a comultiplication ring R .

$\text{Spec}^{\text{SH}}(M)$ is T_1 and M satisfies $\otimes$ if and only if M is semisimple and every SH-submodule of M is minimal SH.

Proof: $(\Rightarrow)$ By Lemma 1.9, every SH-submodule of M is minimal SH and by Lemma 1.11, every minimal SH-submodule is a simple submodule. But by Th.1.4, every submodule of M is a sum of SH-submodules. Thus every submodule of M is a sum of simple submodules. Therefore M is semisimple.

$(\Leftarrow)$ Since M is semisimple, every submodule is a sum of simple submodule. But by Lemma 1.11, every simple submodule is minimal SH. Thus every submodule of M is a sum of SH-submodules and so that by Th.1.4, M satisfies $\otimes$.

On the other hand, since every SH-submodule of M is minimal SH, then by Lemma 1.9, $\text{Spec}^{\text{SH}}(M)$ a T_1 -space.

Next we have the following:

Theorem 1.14. Let M be a faithful finitely generated multiplication R -module. Then M satisfies $\otimes$ if and only if R satisfies $\otimes$.

Proof: $(\Rightarrow)$ Let I and J be ideal of R such that $\text{V}^{\text{SH}}(I) = \text{V}^{\text{SH}}(J)$. We claim that $\text{V}^{\text{SH}}(I) = \text{V}^{\text{SH}}(J)$. To see this,

let $K \in \text{V}^{\text{SH}}(IM)$. Then K is a SH-submodule and $K \subseteq IM$. But by [8, Prop.1.2.1], there exists a SH ideal T of R such that $K = TM$. Thus $TM \subseteq IM$ and so by [5, Th.3.1], $T \leq I$. It follows that $T \in$

$\text{V}^{\text{SH}}(I) = \text{V}^{\text{SH}}(J)$ that is $T \subseteq J$ which implies $K = TM \subseteq JM$. Thus $K \in \text{V}^{\text{SH}}(JM)$ and hence $\text{V}^{\text{SH}}(IM) \subseteq \text{V}^{\text{SH}}(JM)$

Similarly $\overset{1SH}{V}(IM) \subseteq \overset{1SH}{V}(JM)$ Therefore $\overset{1SH}{V}(IM) \subseteq \overset{1SH}{V}(JM)$ and so $IM = JM$ and then by [5,Th.3.1], $I = J$. Thus R satisfies $\circledast$. ($\Leftarrow$) The proof is Similarly.

Remark 1.15. The condition (M is faithful) in Th.1.14 can't be dropped, as for example. The Z -module Z_8 is finitely generated multiplication but not faithful. However Z_8 satisfies $\circledast$, but Z does not satisfy $\circledast$.

Corollary 1.16. Let M be a finitely generated R -module. Then the following statements are equivalent:

- (1) M satisfies $\circledast$ as R -module.
- (2) M satisfies $\circledast$ as R -module.
- (3) $R/\text{ann } M$ satisfies $\circledast$.

where $R = R/\text{ann } M$.

Recall that a submodule N of an R -module is called second if for each ideal I of R , either $IK = K$ or $IK = (0)$, [13].

To give our next result, first we introduce the following Lemma.

Lemma 1.17. Let M be an R -module such that every SH-submodule is second. Then

$$\overset{1SH}{V}(N) = \overset{1SH}{V}(0 : I)_M = \overset{1SH}{V}(N + (0 : I)_M) = \overset{1SH}{V}(N : I)_M \text{ For any } N \leq M, I \leq R.$$

Proof: By [8,Prop.2.1.9], $\overset{1SH}{V}(N) \cup \overset{1SH}{V}(0 : I)_M = \overset{1SH}{V}(N + (0 : I)_M)$. Now $N + (0 : I)_M \subseteq (N : I)_M$, hence

$\overset{1SH}{V}(N + (0 : I)_M) \subseteq \overset{1SH}{V}(N : I)_M$. Now, let $K \in \overset{1SH}{V}(N : I)_M$. Then $IK \subseteq N$ and K is a SH-submodule. Hence by hypothesis K is second, so that either $IK = K$ or $IK = (0)$. It follows that either $K \subseteq N$ or $IK = (0)$

$\subseteq N$, hence either $K \in \overset{1SH}{V}(N)$ or $K \in \overset{1SH}{V}(0 : I)_M$; that $K \in \overset{1SH}{V}(N) \cup \overset{1SH}{V}(0 : I)_M$

Recall that a submodule N of an R -module M is called copure if for each $I \leq R$, $N + (0 : I)_M = (N : I)_M$, [12]

Theorem 1.18. Let M be an R -module with $\circledast$ condition such that every SH-submodule is second. Then every submodule of M is copure.

Proof: Let $N \leq M$. By Lemma 1.17 $\overset{1SH}{V}(N + (0 : I)_M) = \overset{1SH}{V}(N : I)_M$ for any $I \leq R$. Hence by condition $\circledast$, $N + (0 : I)_M = (N : I)_M$ for any $I \leq R$; that is N is copure.

Proposition 1.19. Let M be an R -module which satisfies $\circledast$. Then

- (1) M is Noetherian (Artinian) if and only if $\overset{1SH}{\text{Spec}}(M)$ satisfies a.c.c (d.c.c) on closed set.
- (2) M is Noetherian (Artinian) if and only if $\overset{1SH}{\text{Spec}}(M)$ satisfies d.c.c (a.c.c) on open sets.

S.2 Modules with the Condition $\circledast'$

In this section, we study modules that satisfy $\circledast'$. Some properties of these modules are analogous to that of modules with the condition $\circledast$.

As we mention in the introduction, a module with condition $\circledast'$ if it satisfies the condition $\circledast'$, where

$\circledast'$: if for each $L, N \leq M$, $\overset{1SH}{V}(L) = \overset{1SH}{V}(N)$ implies $L = N$

Remarks and Examples 2.1

- (1) Every simple module satisfies $\otimes'$.
- (2) If every proper nonzero submodule of M is SH, then M satisfies $\otimes'$ for each proper nonzero submodules of M .
- (3) The $\mathbb{Z}$ -module $\mathbb{Z}_4$ does not satisfy $\otimes'$ since $\overset{2SH}{V}(\bar{0}) = \{\langle \bar{2} \rangle\} = \overset{2SH}{V}\{\langle \bar{2} \rangle\}$ but $\langle \bar{0} \rangle \neq \langle \bar{2} \rangle$
- (4) If M_1 and M_2 are R -modules such that $M_1 \cong M_2$ then M_1 satisfies $\otimes'$ if and only if M_2 satisfies $\otimes'$.
- (5) Let M be an R -module. Then M satisfies $\otimes'$ as R -module if and only if M satisfies $\otimes'$ as $R/\text{ann } M$ -module.

Theorem 2.2. Let M be a nonzero R -module. Then M satisfies $\otimes'$ if and only if every proper submodule of M is an intersection of SH-submodules.

Proof: $(\Rightarrow)$ Let $K \subsetneq M$. Then $\overset{2SH}{V}(K) \neq \phi$, because if $\overset{2SH}{V}(K) = \phi = \overset{2SH}{V}(M)$, and hence by $\otimes'$, $K = M$

which is a contradiction. Put $N = \bigcap_{W \in \overset{2SH}{V}(K)} W$. Let $L \in \overset{2SH}{V}(N)$. Then L is SH and $L \supseteq N = \bigcap_{W \in \overset{2SH}{V}(K)} W$. But $W \in \overset{2SH}{V}(K)$ means W is SH and $W \supseteq K$. Then implies $L \supseteq K$; that is $L \in \overset{2SH}{V}(K)$. Thus $\overset{2SH}{V}(N) \subseteq \overset{2SH}{V}(K)$ (1)

Now let $L \in \overset{2SH}{V}(K)$. Since $N = \bigcap_{W \in \overset{2SH}{V}(K)} W$, then $L \supseteq N$; that is $L \in \overset{2SH}{V}(N)$. Hence $\overset{2SH}{V}(K) = \phi = \overset{2SH}{V}(M)$ (2)

Thus by (1) and (2), $\overset{2SH}{V}(N) \subseteq \overset{2SH}{V}(K)$ and by $\otimes'$, $K = N = \bigcap_{W \in \overset{2SH}{V}(K)} W$, i.e. K is an intersection of SH-submodules

$(\Leftarrow)$ Let $N \leq M$. Then $N = \bigcap_{i \in \Lambda} N_i$ where N_i is a SH-submodule of M . It follows that for each $i \in \Lambda$,

$N_i \supseteq N$, and so that $N_i \in \overset{2SH}{V}(N)$. Thus $N = \bigcap_{N_i \in \overset{2SH}{V}(N)} N_i$. Now let $L \leq M$, hence $L = \bigcap_{L_i \in \overset{2SH}{V}(L)} L_i$.

Assume $\overset{2SH}{V}(L) = \overset{2SH}{V}(N)$. It follows $N = L$.

Recall that an R -module M is called cosemisimple if every submodule of M is an intersection of maximal submodules, [2].

Proposition 2.3. Let M be a cosemisimple R -module. If $\text{Max}(M) \subseteq \overset{2SH}{\text{Spec}}(M)$ then M satisfies $\otimes'$, where $\text{Max}(M)$ is the set of all maximal submodules in M .

Proof: Let $N \leq M$. Since M is cosemisimple, then $N = \bigcap_{i \in \Lambda} W_i$, where $W_i \in \text{Max}(M)$. Hence by hypothesis W_i is a SH-submodule, and this implies that N is an intersection of SH-submodules. Then by Theorem 2.2, M satisfies $\otimes'$.

Remark 2.4. The condition " $\text{Max}(M) \subseteq \overset{2SH}{\text{Spec}}(M)$ " is necessary in Prop.2.3, as for example:

$\mathbb{Z}_{30}$ as a $\mathbb{Z}$ -module does not satisfy $\otimes'$ since $\overset{2SH}{V}\langle \bar{2} \rangle = \overset{2SH}{V}\langle \bar{2} \rangle_{\mathbb{Z}_{30}} = \phi$ but $\langle \bar{2} \rangle \neq \mathbb{Z}_{30}$. But $\mathbb{Z}_{30}$ is cosemisimple, also $\text{Max}(\mathbb{Z}_{30}) \subseteq \overset{2SH}{\text{Spec}}(M)$, since $\langle \bar{2} \rangle$ is a maximal submodule but not SH. It is known that every semisimple module is cosemisimple. Hence we have:

Corollary 2.5. Every semisimple module M with $\text{Max}(M) \subseteq {}^{2\text{SH}}\text{Spec}(M)$ satisfies $\otimes'$.
However for a ring R , we have:

Theorem 2.6. Every semisimple ring R satisfies $\otimes'$.

Proof: Let $I, J \leq R$ such that ${}^{2\text{SH}}V(I) = {}^{2\text{SH}}V(J)$. Since R is semisimple ring, I, J are direct summands of R . Hence $I = \langle e \rangle$, $J = \langle f \rangle$ for some idempotent elements $e, f \in R$. It follows that ${}^{2\text{SH}}V(\langle e \rangle \oplus \langle 1-e \rangle) = {}^{2\text{SH}}V(R) = \phi$, ${}^{2\text{SH}}V(\langle f \rangle \oplus \langle 1-f \rangle) = {}^{2\text{SH}}V(R) = \phi$, hence ${}^{2\text{SH}}V(\langle e \rangle) \cap {}^{2\text{SH}}V(\langle 1-e \rangle) = \phi$, ${}^{2\text{SH}}V(\langle f \rangle) \cap {}^{2\text{SH}}V(\langle 1-f \rangle) = \phi$. Then ${}^{2\text{SH}}V(\langle f \rangle) \cap {}^{2\text{SH}}V(\langle 1-e \rangle) = \phi$, ${}^{2\text{SH}}V(\langle e \rangle) \cap {}^{2\text{SH}}V(\langle 1-f \rangle) = \phi$ (since ${}^{2\text{SH}}V(\langle e \rangle) = {}^{2\text{SH}}V(\langle f \rangle)$), which imply that ${}^{2\text{SH}}V(\langle f \rangle + \langle 1-e \rangle) = {}^{2\text{SH}}V(R)$ and ${}^{2\text{SH}}V(\langle e \rangle + \langle 1-f \rangle) = {}^{2\text{SH}}V(R)$. But R is comultiplication, since R is semisimple. Then by [8, Prop 2.5.14] $\langle e \rangle + \langle 1-f \rangle = R$ and $\langle f \rangle + \langle 1-e \rangle = R$. Now to prove $\langle e \rangle = \langle f \rangle$. Let $x \in \langle f \rangle$, then $x = cf$ for some $c \in R$ and $x = cf = r_1e + r_2(1-f)$ for some $r_1, r_2 \in R$. It follows that $cf^2 = cef + r_2f(1-f)$, hence $x = cf = cef$. Thus $x \in \langle e \rangle$, and so $\langle f \rangle \subseteq \langle e \rangle$. Similarly $\langle e \rangle \subseteq \langle f \rangle$. Thus $I = \langle e \rangle = \langle f \rangle = J$ and R satisfies $\otimes'$.
To give the next result we need the following lemmas. First compare the first lemma with lemma 1.9.

Lemma 2.7. Let M be an R -module. Then ${}^{2\text{SH}}\text{Spec}(M)$ is a T_1 -space if and only if every proper SH-submodule is maximal SH in $\text{Spec}(M)$.

Proof: It is analogous to the proof of lemma 1.9, so is omitted.

Recall that a topological space (X, τ) is called cofinite if the only closed subsets of X are finite sets or X . Equivalently $\tau = \{U : U \subseteq X \text{ and } X - U \text{ is a finite set}\} \cup \{\phi\}$.

Lemma 2.8. Let M be an R -module. Then ${}^{2\text{SH}}\text{Spec}(M)$ is a cofinite topological space if and only if every proper SH-submodule is maximal SH in ${}^{2\text{SH}}\text{Spec}(M)$ and for any $N \leq M$, either ${}^{2\text{SH}}V(N) = {}^{2\text{SH}}\text{Spec}(M)$ or $V(N)$ is a finite set.

Proof: ($\Rightarrow$) Since ${}^{2\text{SH}}\text{Spec}(M)$ is a cofinite topological space, then ${}^{2\text{SH}}\text{Spec}(M)$ is a T_1 -space. Hence by Lemma 2.7, every proper SH-submodule is a maximal SH-submodule. Moreover for any $N \leq M$ it is clear that either ${}^{2\text{SH}}V(N) = {}^{2\text{SH}}\text{Spec}(M)$ or $V(N)$ is a finite.

($\Leftarrow$) For any $N \leq M$. $\chi(N) = {}^{2\text{SH}}\text{Spec}(M) - V(N)$. But $V(N)$ is either finite or ${}^{2\text{SH}}\text{Spec}(M)$ so $\chi(N)$ is either finite or ϕ . Thus ${}^{2\text{SH}}\text{Spec}(M)$ is cofinite.
Compare the following Lemma with Lemma 1.11.

Lemma 2.9. Let M be a faithful generated multiplication over a comultiplication ${}^{2\text{SH}}\text{top}$ top ring R , if $N \not\leq M$, then N is a maximal SH-submodule of M if and only if N is a maximal submodule of M .

Proof: ($\Rightarrow$) Let $N \not\leq M$ and N is a maximal SH-submodule. Then by [8, Prop.1.2.1] there exists a SH-ideal I of R such that $N = IM$. Suppose N is not a maximal submodule, so there exists a maximal submodule W of M such that $W \supsetneq N$, since M is multiplication. Also $W = JM$ for some maximal ideal J of R , because M is multiplication. But by [8, Cor.2.5.6] J is a SH-ideal and hence by [8, Prop.1.2.1], W is a SH-submodule. But N is a maximal SH-submodule, so $N = W$; that is N is a maximal submodule.

($\Leftarrow$) Let N be a maximal submodule of M . Then $N = IM$ for some maximal ideal I of R . Hence by [8, Cor.2.5.6], I is a SH-ideal of R , hence by [8, Prop.1.2.1] N is a SH-submodule. Thus N is a maximal SH-submodule of M .

Now we can give the following theorem (compare with theorem 1.13).

2SH
top

Theorem 2.10. Let M be a faithful finitely generated multiplication over a comultiplication ring.

Then $\text{Spec}^{\text{2SH}}(M)$ is a T_1 -space and M satisfies $\otimes'$ if and only if M is cosemisimple and every proper SH-submodule of M is a maximal SH of M .

Proof: $(\Rightarrow)$ By lemma 2.7, every proper SH-submodule is a maximal SH-submodule and by Lemma 2.9, every maximal SH-submodule is a maximal submodule. Thus every proper SH-submodule is a maximal submodule. Moreover by Th.2.2, every proper submodule of M is an intersection of SH-submodule. Thus every proper submodule is an intersection of maximal submodules of M ; that is M is cosemisimple.

$(\Leftarrow)$ By lemma 2.7, $\text{Spec}^{\text{2SH}}(M)$ is T_1 . But M is cosemiple, so every submodule is an intersection of maximal submodules. Hence by Lemma 2.9, every proper submodule is an intersection of SH-submodules. Hence by Th.2.2, M satisfies $\otimes'$.

Compare the following result with Th.1.14.

Theorem 2.11. Let M be a finitely generated faithful multiplication R -module. Then M satisfies $\otimes'$ if and only if R satisfies $\otimes'$.

Proof: It is similar to the proof of Th.1.14, so is omitted.

Remark 2.12. The condition “ M is faithful” is necessary in Th.2.11, as for example:

The Z -module Z_6 is a finitely generated not faithful multiplication Z -module and satisfies $\otimes'$. But

the ring Z does not satisfies $\otimes'$, since $\bigvee^{\text{2SH}}(I) = \bigvee^{\text{2SH}}(J) = \phi$ for any $I, J \leq R$.

Corollary 2.13. Let M be a finitely generated multiplication R -module. Then the following statements are equivalent:

- (1) M satisfies $\otimes'$ as R -module.
- (2) M satisfies $\otimes'$ as R^- -module.
- (3) R^- satisfies $\otimes'$.

where $R^- = R/\text{ann } M$.

Recall that a proper submodule N of an R -module M is called prime if whenever $r \in R, x \in M, rx \in N$ implies $x \in N$ or $r \in [N:M]$, [9].

Equivalently a proper submodule N of an R -module M is prime if for any ideal I of R and for any $K \leq M, IK \subseteq N$ implies $K \subseteq N$ or $I \subseteq [N:M]$, [9].

Compare the following Lemma with Lemma 1.12.

Lemma 2.14. Let M be an R -module such that every proper SH-submodule is prime. Then for any I

$$\leq R, N \leq M, \bigvee^{\text{2SH}}(N) \cup \bigvee^{\text{2SH}}(IM) = \bigvee^{\text{2SH}}(IM \cap N) = \bigvee^{\text{2SH}}(IN).$$

Proof: It is clear that $\bigvee^{\text{2SH}}(N) \cup \bigvee^{\text{2SH}}(IM) \subseteq \bigvee^{\text{2SH}}(IM \cap N) \subseteq \bigvee^{\text{2SH}}(IN)$. Now let $K \in \bigvee^{\text{2SH}}(IN)$. Then K is a SH-submodule and $K \supseteq IN$. But by hypothesis K is prime, so either $N \subseteq K$ or $I \subseteq [K:M]$; that is

either $K \in \bigvee^{\text{2SH}}(N)$ or $K \in \bigvee^{\text{2SH}}(IM)$. Thus $K \in (\bigvee^{\text{2SH}}(N) \cup \bigvee^{\text{2SH}}(IM))$. Hence the result is obtained.

Compare the following Theorem with Th.1.19.

Theorem 2.15. Let M be an R -module such that M satisfies $\otimes'$ and every proper SH-submodule is prime. Then M is F-regular (i.e. every submodule of M is pure; that is $IM \cap N = IN, \forall I \leq R$).

Proof: It follows by Lemma 2.14 and definition of modules with the condition $\otimes'$.

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Proposition 2.16. Let M be a F-regular top^{2SH} -module, then every proper SH-submodule is prime.

Proof: Since M is a F -regular, then for each $N \leq M$, $IM \cap N = IN$ for each $N \leq M$. Hence

$\overset{2SH}{V}(IM \cap N) = \overset{2SH}{V}(IN)$. But M is a $\overset{2SH}{top}$ -module, so $\overset{2SH}{V}(IM) \cup \overset{2SH}{V}(N) = \overset{2SH}{V}(IM \cap N)$. Thus $\overset{2SH}{V}(IM) \cup \overset{2SH}{V}(N) = \overset{2SH}{V}(IM \cap N)$. Now let K be a proper SH -submodule of M such that $IN \leq K$, $I \leq R$, $N \leq M$. Hence $K \in \overset{2SH}{V}(IM) \cup \overset{2SH}{V}(N)$

Hence either $K \in \overset{2SH}{V}(IM)$ or $K \in \overset{2SH}{V}(N)$. Thus $K \supseteq IM$ or $K \supseteq N$; that is either $I \subseteq \overset{[K : M]}{R}$ or $N \subseteq K$. Therefore K is prime.

Proposition 2.17. Let M be an R -module, if M satisfies $\circledast'$. Then

- (1) If $\overset{2SH}{Spec}(M)$ satisfies d.c.c (a.c.c) on closed sets, then M is Noetheian (Artinian).
- (2) If $\overset{2SH}{Spec}(M)$ satisfies a.c.c (d.c.c) on open sets, then M is Noetheian (Artinian).

Proof: It is easy so is omitted.

S.3 Modules with the Condition $\circledast''$

In this section, we introduce modules that satisfy $\circledast''$, where $\circledast''$: for each $N, L \leq M$, $\overset{SI}{V}(L) = \overset{SI}{V}(N)$ implies $N = L$.

Many results about these modules are similar to that of module with $\circledast$ condition. Also we give some relations modules with $\circledast$ and modules with condition $\circledast''$.

Remark and Examples 3.1

- (1) Every simple module M does not satisfy $\circledast''$, since $\overset{SI}{V}(M) = \overset{SI}{V}(\langle 0 \rangle)$ but $M \neq \langle 0 \rangle$.
- (2) The Z -module Z_4 does not satisfy $\circledast''$, since $\overset{SI}{V}(\langle \bar{2} \rangle) = \overset{SI}{V}(Z_4) = \{\langle \bar{2} \rangle\}$ but $Z_4 \neq \langle 2 \rangle$.
- (3) The Z -module Z_6 satisfies $\circledast''$.

The following theorem is similar to Th.1.4

Theorem 3.2. Let M be a nonzero R -module. Then M satisfies $\circledast''$ if and only if every proper nonzero proper submodule can be represented as sum of SI -submodules
Compare the following Lemma with Lemma 1.9.

Lemma 3.3. Let M be an R -module. Then $\overset{SI}{Spec}(M)$ is T_1 -space if and only if every SI -submodule is a minimal SI -submodule in $\overset{SI}{Spec}(M)$

Proof: It is similar the proof of Lemma 1.9.

The following theorem is similar to Th.1.10.

Theorem 3.4 Let M be an R -module. Then $\overset{SI}{Spec}(M)$ is T_1 and M satisfies $\circledast''$ if and only if M is semisimple and every SI -submodule is minimal SI -submodule of M .

The following Lemma is similar to Lemma 1.11.

Lemma 3.5. Let M be a faithful finitely generated multiplication over comultiplication ring R , let $N \leq M$. If N is a minimal SI -submodule, then N is simple.

Note 3.6. The converse of Lemma 3.5 is true if R is top ring.

Compare the following theorem with Th.1.13.

Theorem 3.7. Let M be a finitely generated faithful multiplication over comultiplication $\overset{2SH}{top}$ ring R .

Then M semisimple and every SI -submodule is a minimal SI -submodule if and only if $\overset{SI}{Spec}(M)$ is a T_1 -space and M satisfies $\circledast''$.

Proof: It follows by Th.3.4, Lemma 3.5 and note 3.6.

The following result is similar to Th.1.14

Theorem 3.8. Let M be a finitely generated faithful multiplication R -module. Then M satisfies $\circledast''$ if and only if R satisfies $\circledast''$.

Next we give some relationships modules the condition $\circledast$, and modules with condition $\circledast''$.

Proposition 3.9. If M is a ${}^{2SH}_{top}$ -module and M satisfies $\circledast$ then M satisfies $\circledast''$.

Proof: Since M satisfies $\circledast$, then by Th.1.4 every nonzero submodule of M is a sum of SH-submodule. But M is a ${}^{2SH}_{top}$ -module, so by [8,Prop.2.4.4] every SH-submodule is a SI-submodule. Thus every nonzero submodule is a sum of SI-submodule. Thus by Th.3.2, M satisfies $\circledast''$.

Proposition 3.10. Let M be a ${}^{2SH}_{top}$ -module and M satisfies $\circledast''$ then M satisfies $\circledast$.

Proof: Since M satisfies $\circledast''$, every nonzero submodule of M is a sum of SI-submodule (by Th.3.2). But M is a ${}^{2SH}_{top}$, so by [8,Prop.3.2.3], every SI-submodule is a SH-submodule. Thus every submodule of M is a sum of SH-submodule. Thus M satisfies $\circledast$ by Th.1.4.

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