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Modules with ⊛(⊛'or ⊛'') Condition

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Abstract. In this paper we introduce and study modules with ⊛ (⊛′ or ⊛″) condition. We give several properties of these types of modules and some relationships between them.

Introduction

Let M be an R-module, where R is a commutative ring with unity. Recall that a nonzero submodule N of M is called strongly hollow (briefly, SH-submodule) if whenever L_1, $L_2 \le M$, $N \subseteq L_1 + L_2$ implies either $N \subseteq L_1$ or $N \subseteq L_2$, [1]. A submodule N of M is called strongly irreducible (SI-submodule) if whenever L_1, $L_2 \le M$, $N \supseteq L_1 \cap L_2$ implies $N \supseteq L_1$ or $N \supseteq L_2$, [4]. The sets {K:K is a SH-submodule of M} {K:K is a proper SH-submodule of M} and {K:K is a nonzero proper SI-submodule of M} are denoted by $\overset{1SH}{\mathrm{Spec}}(M)$, $\overset{2SH}{\mathrm{Spec}}(M)$ and $\overset{SI}{\mathrm{Spec}}(M)$ respectively,[8].In [8] we studied and topologized these sets by setting that for any $L \le M$

$$\overset{1SH}{V}(L) = \{K \in \overset{1SH}{\mathrm{Spec}}(M), K \subseteq L\},\ \overset{1SH}{\chi}(L) = \overset{1SH}{\mathrm{Spec}}(M) - \overset{1SH}{V}(L),\ \overset{1SH}{\zeta}(M) = \{\overset{1SH}{V}(L); L \le M\},\ \overset{1SH}{\tau}(M) = \{\overset{1SH}{\chi}(L); L \le M\}.$$

$$\overset{2SH}{V}(L) = \{K \in \overset{2SH}{\mathrm{Spec}}(M), K \supseteq L\},\ \overset{2SH}{\chi}(L) = \overset{2SH}{\mathrm{Spec}}(M) - \overset{2SH}{V}(L),\ \overset{2SH}{\zeta}(M) = \{\overset{2SH}{V}(L); L \le M\},\ \overset{2SH}{\tau}(M) = \{\overset{2SH}{\chi}(L); L \le M\}$$

$$\overset{SI}{V}(L) = \{K \in \overset{SI}{\mathrm{Spec}}(M), K \subseteq L\},\ \overset{SI}{\chi}(L) = \overset{SI}{\mathrm{Spec}}(M) - \overset{SI}{V}(L),\ \overset{SI}{\zeta}(M) = \{\overset{SI}{V}(L); L \le M\},\ \overset{SI}{\tau}(M) = \{\overset{SI}{\chi}(L); L \le M\}.$$

We prove that $(\overset{1SH}{\mathrm{Spec}}(M), \overset{1SH}{\tau}(M))$ is a topological space (see [8, Th.2.1.9)]. Also we see that $\overset{2SH}{\zeta}(M)$ is not closed under finite union, however all other axioms of closed sets of a topological space are valid (see [8, Th.2.4.1]). This leaded us to call an R-module M a $\overset{2SH}{\mathrm{top}}$-module if $\overset{2SH}{\zeta}(M)$ is closed under finite union. Equivalently M is a $\overset{2SH}{\mathrm{top}}$-module if $(\overset{2SH}{\mathrm{Spec}}(M), \overset{2SH}{\tau}(M))$ is a topological space. Beside these we see that $\overset{SI}{\zeta}(M)$ is not closed under finite union, however all other axioms of closed sets of a topological space are valid (see [8, Th.3.2.1]). This leaded us to call an R-module M is a $\overset{SI}{\mathrm{top}}$-module if $\overset{SI}{\zeta}(M)$ is closed under a finite union. Equivalently M is a $\overset{SI}{\mathrm{top}}$- module if $(\overset{2SH}{\mathrm{Spec}}(M), \overset{2SH}{\tau}(M))$ is a topological space.

We notice that, for any $L_1 \le M$, $L_2 \le M$, if $\overset{1SH}{V}(L_1) = \overset{1SH}{V}(L_2)$ (or $\overset{2SH}{V}(L_1) = \overset{2SH}{V}(L_2)$ or $\overset{SI}{V}(L_1) = \overset{SI}{V}(L_2)$) then it is not necessarily that $L_1 = L_2$, as the following examples show.

(1) Consider the Z-module Z, $\overset{1SH}{V}(3Z) = \overset{1SH}{V}(\{0\}) = \phi$ but $3Z \neq \{0\}$.

(2) For the Z-module Z_{12}, $\overset{2SH}{V}(Z_{12}) = \phi = \overset{2SH}{V}(<\bar{2}>)$ but $Z_{12} \neq <\bar{2}>$.

(3) For the Z-module Z_{12}, $\overset{SI}{V}(<\bar{6}>) = \phi = \overset{SI}{V}(\{0\})$ but $<6> \neq \{0\}$.

These observations lead us to introduce the following conditions:

$$\circledast : \overset{1SH}{V}(L_1) = \overset{1SH}{V}(L_2) \Rightarrow L_1 = L_2, \text{ for each } L_1, L_2 \leq M.$$

$$\circledast' : \overset{2SH}{V}(L_1) = \overset{2SH}{V}(L_2) \Rightarrow L_1 = L_2, \text{ for each } L_1, L_2 \leq M.$$

$$\circledast'' : \overset{SI}{V}(L_1) = \overset{SI}{V}(L_2) \Rightarrow L_1 = L_2, \text{ for each } L_1, L_2 \leq M.$$

This paper is devoted to study modules with $\circledast$ ($\circledast'$, $\circledast''$ respectively). Also we shall study the behaviour $\overset{1SH}{\text{Spec}}(M)$, $\overset{2SH}{\text{Spec}}(M)$ and $\overset{SI}{\text{Spec}}(M)$ respectively when M satisfies $\circledast$ ($\circledast'$, $\circledast''$).

S.1 Modules with the Condition $\circledast$

We start this by the following remarks and examples.

Remarks and Examples 1.1:

(1) The Z-module Z does not satisfies $\circledast$ since for each L, N $\leq$ Z, L $\neq$ Z, $\overset{1SH}{V}(L_1) = \overset{1SH}{V}(L_2) = \phi$

(2) Every simple module satisfies $\circledast$.

(3) Let M, M′ be two isomorphic R-modules. Then M satisfies $\circledast$ if and only if M′ satisfies $\circledast$.

(4) Let M_1, M_2 be R-modules. If M_1, M_2 satisfies $\circledast$ condition, then $M_1 \oplus M_2$ may not be satisfy $\circledast$, as an example: Let $M_1 = Z_3$ as a Z-module and $M_2 = Z_4$ as a Z-module. Each of M_1 and M_2 satisfies $\circledast$. However $Z_3 \oplus Z_4 \cong Z_{12}$ and Z_{12} does not satisfy $\circledast$.

(5) Let M be an R-module. Then M satisfies $\circledast$ as R-module if and only if M satisfies $\circledast$ as $\bar{R}$ –module where $\bar{R}$ = R/ann M.

Proposition 1.2. Let M be an R-module such that every nonzero submodules is SH. Then M satisfies $\circledast$.

Proof: First note that $\overset{1SH}{V}(<0>) = \phi \neq \overset{1SH}{V}(N)$ for each N $\neq$ <0>. Let L, N $\leq$ M, L $\neq$ (0), N $\neq$ (0) such that $\overset{1SH}{V}(L) = \overset{1SH}{V}(N)$. Since L $\subseteq$ L and L is a SH-submodule by hypothesis, $L \in \overset{1SH}{V}(L) = \overset{1SH}{V}(N)$. It follows that L $\subseteq$ N. Similarly, $N \in \overset{1SH}{V}(N) = \overset{1SH}{V}(L)$ and hence N $\subseteq$ L. Thus L = N.

Recall that an R-module M is called chained if the lattice of its submodules is linearly ordered by inclusion [10].

Corollary 1.3. Let M be a chained R-module. Then M satisfies $\circledast$.

The following theorem gives a characterization of modules with the condition $\circledast$.

Theorem 1.4:

Let M be an nonzero R-module. Then M satisfies $\circledast$ if and only if every nonzero submodule of M can be represented as sum of SH-submodules.

Proof: ($\Rightarrow$) Let (0) $\neq$ K $\leq$ M. Then $\overset{1SH}{V}(K) \neq \phi$, since if $\overset{1SH}{V}(K) = \phi$, then $\overset{1SH}{V}(K) = \overset{1SH}{V}(0)$ and hence K = (0) (by $\circledast$), which is a contradiction. Set $N = \sum_{W \in \overset{1SH}{V}(K)} W$ and let $L \in V(N)$. Then $L \subseteq N = \sum_{W \in \overset{1SH}{V}(K)} W$

But for each $W \in V(K)$, W $\subseteq$ K, so N $\subseteq$ K. Thus L $\subseteq$ K and hence $L \in \overset{1SH}{V}(K)$. Thus $\overset{1SH}{V}(N) \subseteq \overset{1SH}{V}(K)$

Now, let $L \in \overset{1SH}{V}(K)$, then $L \subseteq N$ (by definition of N). Hence $L \in \overset{1SH}{V}(K)$ and so $\overset{1SH}{V}(N) \subseteq \overset{1SH}{V}(K)$ Thus $\overset{1SH}{V}(N) \subseteq \overset{1SH}{V}(K)$ and by ⊛, K=N. Thus K is a sum of SH-submodules.

($\Leftarrow$) Let $N \leq M$, $N \neq 0$.Then $N = \sum_{i \in \Lambda} T_i$ where T_i is a SH-submodule of M, $\forall$ $i \in \Lambda$ (by hypothesis).

Since for each $i \in \Lambda$, $T_i \subseteq N$, we have $T_i \in \overset{1SH}{V}(N)$. Hence $N = \sum_{T_i \in \overset{1SH}{V}(N)} T_i$ Now, assume $N_1, N_2 \leq M$ such that $\overset{1SH}{V}(N_1) = \overset{1SH}{V}(N_2)$ But $N_1 = \sum_{T_i \in \overset{1SH}{V}(N_1)} T_i$, $N_2 = \sum_{S_i \in \overset{1SH}{V}(N_2)} S_i$) by the previous argument. Then $N_1 = N_2$.Thus M satisfies ⊛.

Corollary 1.5. Let M be a semisimple R-module such that every simple submodule is SH. Then M satisfies ⊛.

Recall that an R-module is called comultiplication if for each $N \leq M$, $N = \operatorname{ann}_M I$ for some ideal I of R,[11] where $\operatorname{ann}_M I = \{m \in M \mid Im = (0)\}$.

An R-module is called distributive if for each $N, L, W \leq M$, $N \cap (L + W) = (N \cap L) + (N \cap W)$, [6]

Lemma 1.6: [8,Cor.1.1.9], [8,Prop.1.1.11] Let M be a comultiplication (or distributive) R-module. Then every simple submodule of M is a SH-submodule.

Corollary 1.7. Let M be a semisimple comultiplication (or distributive). Then M satisfies ⊛.

Proof: It follows by Lemma 1.6 and Cor. 1.5.

Remarks 1.8:

(1) The condition "every simple submodule is a SH-submodule" is necessary in Cor.1.5, as for example: It is clear that the vector space $\mathbb{R}^2$ over $\mathbb{R}$ is semisimple, but $N_1 = \mathbb{R}(1,0)$ is a simple submodule of $\mathbb{R}^2$ and it is not SH. However $\mathbb{R}^2$ does not satisfies ⊛ since $\overset{1SH}{V}(N_1) = \overset{1SH}{V}(N_2) = \phi$, where $N_2 = \mathbb{R}(0,1)$.

(2) The converse of Cor.1.5 is not true in general, for example: consider the Z-module Z_4, Z_4 satisfies ⊛ and every nonzero submodule of Z_4 is SH. However Z_4 is not semisimple.

Before giving the next result, we give the following lemma.

Lemma 1.9. Let M be an R-module. Then $\overset{1SH}{\operatorname{Spec}}(M)$ is a T_1-space if and only if every SH-submodule is minimal SH in $\overset{1SH}{\operatorname{Spec}}(M)$

Proof: ($\Rightarrow$) If $\overset{1SH}{\operatorname{Spec}}(M)$ is T_1. Suppose $\overset{1SH}{\operatorname{Spec}}(M) = \phi$, then nothing to prove. Let $\overset{1SH}{\operatorname{Spec}}(M) \neq \phi$. Since $\overset{1SH}{\operatorname{Spec}}(M)$ is T_1, then for any $N \in \overset{1SH}{\operatorname{Spec}}(M)$, N is closed; that is $\{N\} = \overset{1SH}{V}(L)$ for some $L \leq M$. If N is not minimal SH, then there exists $K \in \overset{1SH}{\operatorname{Spec}}(M)$ and $K \subsetneq N$. Hence $K \subseteq L$; that is $K \in \overset{1SH}{V}(L) = \{N\}$ and $K = N$ which is a contradiction. Therefore N is a minimal SH-submodule.

($\Leftarrow$) Let $N \in \overset{1SH}{\operatorname{Spec}}(M)$. Then N is SH and so $N \in \overset{1SH}{V}(N)$. Assume there exists $L \leq M$, $L \neq M$ such that $L \in \overset{1SH}{V}(N)$. It follows that L is SH and $L \subseteq N$. Hence N is not a minimal SH-submodule which contradicts the hypothesis.

Theorem 1.10. Let M be a comultiplication R-module. Then $\overset{1SH}{\operatorname{Spec}}(M)$ is T_1 and M satisfies ⊛ if and only if M is semisimple and every SH-submodule is minimal SH.

Proof: ($\Rightarrow$) Since $\overset{1SH}{\text{Spec}(M)}$ is T_1, then by lemma 1.9, every SH-submodule is minimal SH. But M satisfies ⊛, so by Th.1.4, every submodule is a sum of SH-submodules. On the other hand, M is comultiplication and $\overset{1SH}{\text{Spec}(M)}$ is T_1 imply $S(M) = \overset{1SH}{\text{Spec}(M)}$ by [8, Cor.2.2.19] where S(M) =set of all simple submodules of M. Thus every submodule of M is a sum of simple submodules; that is M is semisimple.
($\Leftarrow$) It follows by Cor.1.7 and Lemma 1.9.

Recall that an R-module M is called multiplication if for each $N \leq M$, there exists $I \leq R$ such that $N = IM$, [3].

Lemma 1.11. Let M be a faithful finitely generated multiplication over a comultiplication ring and let $(0) \neq N \leq M$. Then N is a minimal SH-submodule of M if and only if N is simple.

Proof: ($\Rightarrow$) Since M is multiplication, N = IM for some ideal I of R. Then by [8, Prop.1.2.1] I is aSH ideal of R. We claim that I is a simple ideal of R. If I is not simple, then there exists a simple ideal J of R such that $J \underset{\neq}{\subset} I$ since R is comultiplication. Moreover by Lemmal 1.6, J is a SH ideal of R and so by [8, Prop.1.2.1] JM is a SH-submodule. But M is a faithful finitely generated multiplication, so by [5,Th.3.1] $JM \underset{\neq}{\subset} IM = N$. Thus N is not a minimal SH-submodule of M, which is a contradiction. Thus I is a simple ideal of R and so N is a simple submodule of M.
($\Leftarrow$) Let N be a simple submodule of M. Since M is multiplication then N = IM for some ideal I of R. It is easy to check that I is a simple ideal of R. Then by Lemma 1.6, I is a SH ideal and so [8, Prop.1.2.1], N is a SH-submodule. Thus N is a minimal SH-submodule of M.
By using Lemma 1.11, we have the following immediate result.

Corollary 1.12. Let R be a comultiplication ring and let $J \leq R$. Then J is a minimal SH ideal if and only if J is a simple (minimal) ideal.
Now we have the following:

Theorem 1.13. Let M be a faithful finitely generated multiplication over a comultiplication ring R. $\overset{1SH}{\text{Spec}(M)}$ is T_1 and M satisfies ⊛ if and only if M is semisimple and every SH-submodule of M is minimal SH.

Proof: ($\Rightarrow$) By Lemma 1.9, every SH-submodule of M is minimal SH and by Lemma 1.11, every minimal SH-submodule is a simple submodule. But by Th.1.4, every submodule of M is a sum of SH-submodules. Thus every submodule of M is a sum of simple submodules. Therefore M is semisimple.
($\Leftarrow$) Since M is semisimple, every submodule is a sum of simple submodule. But by Lemma 1.11, every simple submodule is minimal SH. Thus every submodule of M is a sum of SH-submodulesand so that by Th.1.4, M satisfies ⊛.

On the other hand, since every SH-submodule of M is minimal SH, then by Lemma 1.9, $\overset{1SH}{\text{Spec}(M)}$ a T_1-space.
Next we have the following:

Theorem 1.14. Let M be a faithful finitely generated multiplication R-module. Then M satisfies ⊛ if and only if R satisfies ⊛.

Proof: ($\Rightarrow$) Let I and J be ideal of R such that $\overset{1SH}{V}(I) = \overset{1SH}{V}(J)$. We claim that $\overset{1SH}{V}(I) = \overset{1SH}{V}(J)$. To see this, let $K \in \overset{1SH}{V}(IM)$ Then K is a SH-submodule and $K \subseteq IM$. But by [8,Prop.1.2.1], there exists a SH ideal T of R such that K = TM. Thus $TM \subseteq IM$ and so by [5,Th.3.1], $T \leq I$. It follows that $T \in \overset{1SH}{V}(I) = \overset{1SH}{V}(J)$ that is $T \subseteq J$ which implies $K=TM \subseteq JM$. Thus $K \in \overset{1SH}{V}(IM)$ and hence $\overset{1SH}{V}(IM) \subseteq \overset{1SH}{V}(JM)$

Similarly $\overset{1SH}{V}(IM) \subseteq \overset{1SH}{V}(JM)$ Therefore $\overset{1SH}{V}(IM) \subseteq \overset{1SH}{V}(JM)$ and so IM = JM and then by [5,Th.3.1], I = J. Thus R satisfies ⊛. (⇐) The proof is Similarly.

Remark 1.15. The condition (M is faithful) in Th.1.14 cann't be dropped, as for example. The Z-module Z_8 is finitely generated multiplication but not faithful. However Z_8 satisfies ⊛, but Z does not satisfy ⊛.

Corollary 1.16. Let M be a finitely generated R-module. Then the following statements are equivalent:

(1) M satisfies ⊛ as R-module.

(2) M satisfies ⊛ as $\bar{R}$-module.

(3) $\bar{R}$ satisfies ⊛.

where $\bar{R}$ = R/ann M.

Recall that a submodule N of an R-module is called second if for each ideal I of R, either IK = K or IK = (0), [13].

To give our next result, first we introduce the following Lemma.

Lemma 1.17. Let M be an R-module such that every SH-submodule is second. Then $\overset{1SH}{V}(N) = \overset{1SH}{V}(0 \underset{M}{:} I) = \overset{1SH}{V}(N+(0 \underset{M}{:} I)) = \overset{1SH}{V}(N \underset{M}{:} I)$ For any N ≤ M, I ≤ R.

Proof: By [8,Prop.2.1.9], $\overset{1SH}{V}(N) \cup \overset{1SH}{V}(0 \underset{M}{:} I) = \overset{1SH}{V}(N+0 \underset{M}{:} I)$. Now $N+(0 \underset{M}{:} I) \subseteq (N \underset{M}{:} I)$, hence $\overset{1SH}{V}(N+(0 \underset{M}{:} I)) \subseteq \overset{1SH}{V}(N \underset{M}{:} I)$ Now, let K $\in \overset{1SH}{V}(N \underset{M}{:} I)$. Then IK ⊆ N and K is a SH-submodule. Hence by hypothesis K is second, so that either IK = K or IK = (0). It follows that either K ⊆ N or IK =(0) ⊆ N, hence either K ∈ $\overset{1SH}{V}(N)$ or K ∈ $\overset{1SH}{V}(0 \underset{M}{:} I)$; that K ∈ $\overset{1SH}{V}(N) \cup \overset{1SH}{V}(0 \underset{M}{:} I)$

Recall that a submodule N of an R-module M is called copure if for each I ≤ R, $N+(0 \underset{M}{:} I) = (N \underset{M}{:} I)$, [12]

Theorem 1.18. Let M be an R-module with ⊛ condition such that every SH-submodule is second. Then every submodule of M is copure.

Proof: Let N ≤ M. By Lemma 1.17 $\overset{1SH}{V}(N+(0 \underset{M}{:} I)) = \overset{1SH}{V}(N \underset{M}{:} I)$ for any I ≤ R. Hence by condition ⊛, $N+(0 \underset{M}{:} I) = (N \underset{M}{:} I)$ for any I ≤ R; that is N is copure.

Proposition 1.19. Let M be an R-module which satisfies ⊛. Then

(1) M is Noetherian (Artinian) if and only if $\overset{1SH}{Spec}(M)$ satisfies a.c.c (d.c.c) on closed set.

(2) M is Noetherian (Artinian) if and only if $\overset{1SH}{Spec}(M)$ satisfies d.c.c (a.c.c) on open sets.

S.2 Modules with the Condition ⊛'

In this section, we study modules that satisfy ⊛′. Some properties of these modules are analogus to that of modules with the condition ⊛.

As we mention in the introduction, a module with condition ⊛′ if it satisfies the condition ⊛′, where

⊛′ : if for each L, N ≤ M, $\overset{1SH}{V}(L) = \overset{1SH}{V}(N)$ implies L = N

Remarks and Examples 2.1

(1) Every simple module satisfies ⊛′.

(2) If every proper nonzero submodule of M is SH, then M satisfies ⊛′ for each proper nonzero submodules of M.

(3) The Z-module Z_4 does not satisfy ⊛′ since $\overset{2SH}{V}(\bar{0}) = \{<\bar{2}>\} = \overset{2SH}{V}\{<\bar{2}>\}$ but $<\bar{0}> \neq <\bar{2}>$

(4) If M_1 and M_2 are R-modules such that $M_1 \cong M_2$ then M_1 satisfies ⊛′ if and only if M_2 satisfies ⊛′.

(5) Let M be an R-module. Then M satisfies ⊛′ as R-module if and only if M satisfies ⊛′ as R/ann M-module.

Theorem 2.2. Let M be a nonzero R-module. Then M satisfies ⊛’ if and only if every proper submodule of M is an intersection of SH-submodules.

Proof: (⇒) Let K ⪵ M. Then $\overset{2SH}{V}(K) \neq \phi$, because if $\overset{2SH}{V}(K) = \phi = \overset{2SH}{V}(M)$, and hence by ⊛’ , K = M which is a contradiction. Put N=$\bigcap_{W\in \overset{2SH}{V}(K)} W$ Let L $\in$ $\overset{2SH}{V}(N)$. Then L is SH and L $\supseteq$N$\bigcap_{W\in \overset{2SH}{V}(K)} W$ .But W $\in$ $\overset{2SH}{V}(K)$ means W is SH and W $\supseteq$ K..Then implies L $\supseteq$ K; that is L $\in$ $\overset{2SH}{V}(K)$.

Thus $\overset{2SH}{V}(N) \subseteq \overset{2SH}{V}(K)$ (1)

Now let L $\in$ $\overset{2SH}{V}(K)$ Since $N = \bigcap_{W\in \overset{2SH}{V}(K)} W$, then L ⊇ N; that is L $\in$ $\overset{2SH}{V}(N)$.

Hence $\overset{2SH}{V}(K) = \phi = \overset{2SH}{V}(M)$ (2)

Thus by (1) and (2), $\overset{2SH}{V}(N) \subseteq \overset{2SH}{V}(K)$ and by ⊛′, K = N=$\bigcap_{W\in \overset{2SH}{V}(K)} \overset{2SH}{V}(W)$, i.e. K is an intersection of SH-submodules

(⇐) Let N ≤ M. Then $N = \bigcap_{i\in\Lambda} N_i$ where N_i is a SH-submodule of M. It follows that for each I ∈ Λ, $N_i \supseteq N$, and so that $N_i \in \overset{2SH}{V}(N)$. Thus $N = \bigcap_{N_i\in \overset{2SH}{V}(N)} N_i$ Now let L δ M, hence L= $\bigcap_{L_i\in \overset{2SH}{V}(L)} L_i$.

Assume $\overset{2SH}{V}(L) = \overset{2SH}{V}(N)$ It follows N = L.

Recall that an R-module M is called cosemisimple if every submodule of M is an intersection of maximal submodules, [2].

Proposition 2.3. Let M be a cosemisimple R-module. If Max(M) ⊆ $\overset{2SH}{Spec(M)}$ then M satisfies ⊛′, where Max(M) is the set of all maximal submodules in M.

Proof: Let N ≤ M. Since M is cosemisimple, then $N = \bigcap_{i\in\Lambda} W_i$ where W_i ∈ Max(M). Hence by hypothesis W_i is a SH-submodule, and this implies that N is an intersection of SH-submodules. Then by Theorem 2.2, M satisfies ⊛′.

Remark 2.4. The condition “Max(M) ⊆ $\overset{2SH}{Spec(M)}$ ” is necessary in Prop.2.3, as for example:

Z_{30} as a Z-module does not satisfy ⊛′ since $\overset{2SH}{V} <\bar{2}> = \overset{2SH}{V} <Z_{30}> = \phi$ but $<\bar{2}> \neq Z_{30}$. But Z_{30} is cosemisimple, also Max(Z_{30})⊆ $\overset{2SH}{Spec(M)}$, since $<\bar{2}>$ is a maximal submodule but not SH.

It is known that every semisimple module is cosemisimple. Hence we have:

Corollary 2.5. Every semmisimple module M with Max(M) $\subseteq \overset{2SH}{\mathrm{Spec}(M)}$ satisfies ⊛′.
However for a ring R, we have:

Theorem 2.6. Every semisimple ring R satisfies ⊛′.

Proof: Let I, J ≤ R such that $\overset{2SH}{V}(I) = \overset{2SH}{V}(J)$. Since R is semisimple ring, I, J are direct summands of R. Hence I = <e>, J = <f> for some idempotent elements e, f∈R. It follows that $\overset{2SH}{V}(\langle e\rangle \oplus \langle 1-e\rangle) = \overset{2SH}{V}(R) = \phi$, $\overset{2SH}{V}(\langle f\rangle \oplus \langle 1-f\rangle) = \overset{2SH}{V}(R) = \phi$, hence $\overset{2SH}{V}(\langle e\rangle) \cap \overset{2SH}{V}(\langle 1-e\rangle) = \phi$, $\overset{2SH}{V}(\langle f\rangle) \cap \overset{2SH}{V}(\langle 1-f\rangle) = \phi$. Then $\overset{2SH}{V}(\langle f\rangle) \cap \overset{2SH}{V}(\langle 1-e\rangle) = \phi$, $\overset{2SH}{V}(\langle e\rangle) \cap \overset{2SH}{V}(\langle 1-f\rangle) = \phi$ (since $\overset{2SH}{V}(\langle e\rangle) = \overset{2SH}{V}(\langle f\rangle)$, which imply that $\overset{2SH}{V}(\langle f\rangle + \langle 1-e\rangle) = \overset{2SH}{V}(R)$ and $\overset{2SH}{V}(\langle e\rangle + \langle 1-f\rangle) = \overset{2SH}{V}(R)$.
But R is comultiplication, since R is semisimple. Then by [8,Prop 2.5.14] <e> + <1 – f > = R and < f > + <1 – e> = R. Now to prove <e> = <f>. Let x ∈ < f >, then x = cf for some c ∈ R and x = cf = $r_1e + r_2(1 - f)$ for some $r_1, r_2 \in$ R. It follows that $cf^2 = cef + r_2f(1 - f)$, hence x = cf = cef. Thus x ∈ <e>, and so < f > ⊆ <e>. Similary <e> ⊆ < f >. Thus I = <e> = < f > = J and R satisfies ⊛′.
To give the next result we need the following lemmas. First compare the first lemma with lemma 1.9.

Lemma 2.7. Let M be an R-module. Then $\overset{2SH}{\mathrm{Spec}(M)}$ is a T_1-space if and only if every proper SH-submodule is maximal SH in Spec(M) .
Proof: It is analogus to the proof of lemma 1.9, so is omitted.
Recall that a topological space (X,τ) is called cofinite if the only closed subsets of X are finite sets or X. Equivalently τ = {U: U ⊆ X and X – U is a finite set} ∪ {ϕ}.

Lemma 2.8. Let M be an R-module. Then $\overset{2SH}{\mathrm{Spec}(M)}$ is a cofinite topological space if and only if every proper SH-submodule is maximal SH in $\overset{2SH}{\mathrm{Spec}(M)}$ and for any N ≤ M, either $\overset{2SH}{V}(N) = \overset{2SH}{\mathrm{Spec}(M)}$ or V (N) is a finite set.

Proof: (⇒) Since $\overset{2SH}{\mathrm{Spec}(M)}$ is a cofinite topological space, then $\overset{2SH}{\mathrm{Spec}(M)}$ is a T_1-space. Hence by Lemma 2.7, every proper SH-submodule is a maximal SH-submodule. Moreover for any N ≤ M it is clear that either $\overset{2SH}{V}(N) = \overset{2SH}{\mathrm{Spec}(M)}$ or $\overset{2SH}{V}(N)$ is a finite.
(⇐) For any N ≤ M. $\overset{2SH}{\chi}(N) = \overset{2SH}{\mathrm{Spec}(M)} - \overset{2SH}{V}(N)$. But $\overset{2SH}{V}(N)$ is either finite or $\overset{2SH}{\mathrm{Spec}(M)}$ so $\overset{2SH}{\chi}(N)$ either finite or ϕ. Thus $\overset{2SH}{\mathrm{Spec}(M)}$ is cofinite.
Compare the following Lemma with Lemma 1.11.

Lemma 2.9. Let M be a faithful generated multiplication over a comultiplication $\overset{2SH}{\mathrm{top}}$ top ring R, if N ≨M, then N is a maximal SH-submodule of M if and only if N is a maximal submodule of M.

Proof: (⇒) Let N ≨ M and N is a maximal SH-submodule. Then by [8,Prop.1.2.1] there exists a SH-ideal I of R such tha N = IM. Suppose N is not a maximal submodule, so there exists a maximal submodule W of M such that W ⊇ N, since M is multiplication. Also W = JM for some maximal ideal J of R, because M is multiplication. But by [8,Cor.2.5.6] J is a SH-ideal and hence by [8,Prop.1.2.1], W is a SH-submodule. But N is a maximal SH-submodule, so N = W; that is N is a maximal submodule.
(⇐) Let N be a maximal submodule of M. Then N = IM for some maximal ideal I of R. Hence by [8, Cor.2.5.6], I is a SH-ideal of R, hence by [8,Prop.1.2.1] N is a SH-submodule. Thus N is a maximal SH-submodule of M.
Now we can give the following theorem (compare with theorem 1.13).

Theorem 2.10. Let M be a faithful finitely generated multiplication over a comultiplication $\overset{2SH}{top}$ ring.

Then $\overset{2SH}{\mathrm{Spec}(M)}$ is a T_1-space and M satisfies ⊛′ if and only if M is cosemisimple and every proper SH-submodule of M is a maximal SH of M.

Proof: (⇒) By lemma 2.7, every proper SH-submodule is a maximal SH-submodule and by Lemma 2.9, every maximal SH-submodule is a maximal submodule. Thus every proper SH-submodule is a maximal submodule. Moreover by Th.2.2, every proper submodule of M is an intersection of SH-submodule. Thus every proper submodule is an intersection of maximal submodules of M; that is M is cosemisimple.

(⇐) By lemma 2.7, $\overset{2SH}{\mathrm{Spec}(M)}$ is T_1. But M is cosemiple, so every submodule is an intersection of maximal submodules. Hence by Lemma 2.9, every proper submodule is an intersection of SH-submodules. Hence by Th.2.2, M satisfies⊛′.

Compare the following result with Th.1.14.

Theorem 2.11. Let M be a finitely generated faithful multiplication R-module. Then M satisfies ⊛′ if and only if R satisfies ⊛′.

Proof: It is similar to the proof of Th.1.14, so is omitted.

Remark 2.12. The condition "M is faithful" is necessary in Th.2.11, as for example:

The Z-module Z_6 is a finitely generated not faithful multiplication Z-module and satisfies ⊛′. But the ring Z does not satisfies ⊛′, since $\overset{2SH}{V}(I) = \overset{2SH}{V}(J) = \phi$ for any I, J ≤ R.

Corollary 2.13. Let M be a finitely generated multiplication R-module. Then the following statements are equivalent:

(1) M satisfies ⊛′ as R-module.

(2) M satisfies ⊛′ as $\bar{R}$-module.

(3) $\bar{R}$ satisfies ⊛′.

where $\bar{R}$ = R/ann M.

Recall that a proper submodule N of an R-module M is called prime if whenever r ∈ R, x ∈ M, rx ∈ N implies x ∈ N or r ∈ [N:M], [9].

Equivalently a proper submodule N of an R-module M is prime if for any ideal I of R and for any K ≤ M, IK ⊆ N implies K ⊆ N or I ⊆ [N:M], [9].

Compare the following Lemma with Lemma 1.12.

Lemma 2.14. Let M be an R-module such that every proper SH-submodule is prime. Then for any I

$$\leq R,\ N \leq M,\ \overset{2SH}{V}(N) \cup \overset{2SH}{V}(IM) = \overset{2SH}{V}(IM \cap N) = \overset{2SH}{V}(IN).$$

Proof: It is clear that $\overset{2SH}{V}(N) \cup \overset{2SH}{V}(IM) \subseteq \overset{2SH}{V}(IM \cap N) \subseteq \overset{2SH}{V}(IN)$ Now let K ∈ $\overset{2SH}{V}(IN)$. Then K is a SH-submodule and K ⊇ IN. But by hypothesis K is prime, so either N ⊆ K or I ⊆ [K : M] ; that is either K ∈ $\overset{2SH}{V}(N)$ or K ∈ $\overset{2SH}{V}(IM)$ Thus K ∈ $(\overset{2SH}{V}(N) \cup \overset{2SH}{V}(IM))$. Hence the result is obtained.

Compare the following Theorem with Th.1.19.

Theorem 2.15. Let M be an R-module such that M satisfies ⊛′ and every proper SH-submodule is prime. Then M is F-regular (i.e. every submodule of M is pure; that is IM ∩ N = IN, ∀ I ≤ R).

Proof: It follows by Lemma 2.14 and definition of modules with the condition ⊛′.

Proposition 2.16. Let M be a F-regular $\overset{2SH}{top}$ -module, then every proper SH-submodule is prime.

Proof: Since M is a F-regular, then for each N ≤ M, IM ∩ N = IN for each N ≤ M. Hence $\overset{2SH}{V}(IM\cap N) = \overset{2SH}{V}(IN)$. But M is a $\overset{2SH}{top}$-module, so $\overset{2SH}{V}(IM) \cup \overset{2SH}{V}(N) = \overset{2SH}{V}(IM\cap N)$ Thus $\overset{2SH}{V}(IM) \cup \overset{2SH}{V}(N) = \overset{2SH}{V}(IM\cap N)$ Now let K be a proper SH-submodule of M such that IN ≤ K, I ≤ R, N≤ M. Hence K ∈ $\overset{2SH}{V}(IM) \cup \overset{2SH}{V}(N)$

Hence either K ∈ $\overset{2SH}{V}(IM)$ or K ∈ $\overset{2SH}{V}(N)$ Thus K ⊇ IM or K ⊇ N; that is either I ⊆ $[K \underset{R}{:} M]$ or N ⊆ K. Therefore K is prime.

Proposition 2.17. Let M be an R-module, if M satisfies ⊛′. Then
2SH
(1) If $\overset{2SH}{Spec}(M)$ satisfies d.c.c (a.c.c) on closed sets, then M is Noetheian (Artinian).
2SH
(2) If $\overset{2SH}{Spec}(M)$ satisfies a.c.c (d.c.c) on open sets, then M is Noetheian (Artinian).

Proof: It is easy so is omitted.

S.3 Modules with the Condition ⊛″

In this section, we introduce modules that satisfy ⊛″, where ⊛″ : for each N, L ≤ M, $\overset{SI}{V}(L) = \overset{SI}{V}(N)$ implies N = L.

Many results about these modules are similar to that of module with ⊛ condition. Also we give some relations modules with ⊛ and modules with condition ⊛″.

Remark and Examples 3.1

(1) Every simple module M does not satisfy ⊛″, since $\overset{SI}{V}(M) = \overset{SI}{V}(<0>)$ but M ≠ <0>.

(2) The Z-module Z_4 does not satisfy ⊛″, since $\overset{SI}{V}(<\bar{2}>) = \overset{SI}{V}(Z_4) = \{<\bar{2}>\}$ but $Z_4 \neq <2>$.

(3) The Z-module Z_6 satisfies ⊛″.

The following theorem is similar to Th.1.4

Theorem 3.2. Let M be an nonzero R-module. Then M satisfies ⊛″ if and only if every proper nonzero proper submodule can be represented as sum of SI-submodules

Compare the following Lemma with Lemma 1.9.

Lemma 3.3. Let M be an R-module. Then $\overset{SI}{Spec}(M)$ is T_1-space if and only if every SI-submodule is a minimal SI-submodule in $\overset{SI}{Spec}(M)$

Proof: It is similar the proof of Lemma 1.9.

The following theorem is similar to Th.1.10.

Theorem 3.4 Let M be an R-module. Then $\overset{SI}{Spec}(M)$ is T_1 and M satisfies ⊛″ if and only if M is semisimple and every SI-submodule is minimal SI-submodule of M.

The following Lemma is similar to Lemma 1.11.

Lemma 3.5. Let M be a faithful finitely generated multiplication over comultiplication ring R, let N ≤ M. If N is a minimal SI-submodule, then N is simple.

Note 3.6. The converse of Lemma 3.5 is true if R is top ring.

Compare the following theorem with Th.1.13.

Theorem 3.7. Let M be a finitely generated faithful multiplication over comultiplication $\overset{2SH}{top}$ ring R.

Then M semisimple and every SI-submodule is a minimal SI-submodule if and only if $\overset{SI}{Spec}(M)$ is a T_1-space and M satisfies ⊛″.

Proof: It follows by Th.3.4, Lemma 3.5 and note 3.6.
The following result is similar to Th.1.14

Theorem 3.8. Let M be a finitely generated faithful multiplication R-module. Then M satisfies $\circledast''$ if and only if R satisfies $\circledast''$.

Next we give some relationships modules the condition $\circledast$, and modules with condition $\circledast''$.

Proposition 3.9. If M is a $\overset{2SH}{\text{top}}$-module and M satisfies $\circledast$ then M satisfies $\circledast''$.

Proof: Since M satisfies $\circledast$, then by Th.1.4 every nonzero submodule of M is a sum of SH-submodule. But M is a $\overset{2SH}{\text{top}}$ -module, so by [8,Prop.2.4.4] every SH-submodule is a SI-submodule. Thus every nonzero submodule is a sum of SI-submodule. Thus by Th.3.2, M satisfies $\circledast''$.

Proposition 3.10. Let M be a $\overset{2SH}{\text{top}}$ -module and M satisfies $\circledast''$ then M satisfies $\circledast$.

Proof: Since M satisfies $\circledast''$, every nonzero submodule of M is a sum of SI-submodule (by Th.3.2). But M is a $\overset{2SH}{\text{top}}$, so by [8,Prop.3.2.3], every SI-submodule is a SH-submodule. Thus every submodule of M is a sum of SH-submodule. Thus M satisfies $\circledast$ by Th.1.4.

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