Intuitionistic Fuzzy semiboundary and intuitionistic Fuzzy Product Related spaces

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Keywords: Intuitionistic fuzzy topology, intuitionistic fuzzy topological spaces, intuitionistic fuzzy semi open set, intuitionistic fuzzy semi closed set, intuitionistic fuzzy semi boundary, intuitionistic fuzzy product related spaces, intuitionistic fuzzy semi continuous function.

Abstract: In this paper intuitionistic fuzzy semi boundary is introduced and its properties are investigated. Intuitionistic fuzzy semi continuous functions are characterized via intuitionistic fuzzy semi boundary.

1. Introduction


In this paper we introduce intuitionistic fuzzy semi boundary and investigate some of their properties. Further intuitionistic fuzzy semi boundary in product related spaces is analysed. Finally necessary conditions for intuitionistic fuzzy semi continuous functions are obtained via intuitionistic fuzzy semi boundary. Throughout this paper X,Y are non-empty sets.

2. Preliminaries

Definition 2.1: [1] An intuitionistic fuzzy set (IFS) A in X is an object having the form

\[ A = \{(x, \mu_A(x), \gamma_A(x)) | x \in X\} \]

where the functions \( \mu_A: X \rightarrow [0,1] \) and \( \gamma_A: X \rightarrow [0,1] \) denote the degree of membership (namely, \( \mu_A(x) \)) and the degree of non-membership (namely, \( \gamma_A(x) \)) of each element \( x \in X \) to the set A respectively, and

\[ 0 \leq \mu_A(x) + \gamma_A(x) \leq 1 \quad \text{for each } x \in X \]

Definition 2.2: [1] Let A and B be IFS's of the forms

\[ A = \{(x, \mu_A(x), \gamma_A(x)) | x \in X\} \text{ and } B = \{(x, \mu_B(x), \gamma_B(x)) | x \in X\}. \]

Then,

(a) \( A \subseteq B \) if and only if \( \mu_A(x) \leq \mu_B(x) \) and \( \gamma_A(x) \geq \gamma_B(x) \) for all \( x \in X \)

(b) \( A = B \) if and only if \( A \subseteq B \) and \( B \subseteq A \)

(c) The complement of A is denoted by \( \bar{A} \) and is defined by

\[ \bar{A} = \{(x, \gamma_A(x), \mu_A(x)) | x \in X\} \]

(d) \( A \cap B = \{(x, \mu_A(x) \land \mu_B(x), \gamma_A(x) \lor \gamma_B(x)) | x \in X\} \)

(e) \( A \cup B = \{(x, \mu_A(x) \lor \mu_B(x), \gamma_A(x) \land \gamma_B(x)) | x \in X\} \)

The IFS's \( 0_{\circ} = \{ < x, 0, 1 > | x \in X \} \) and \( 1_{\circ} = \{ < x, 1, 0 > | x \in X \} \) are respectively the empty set and the whole set of X. For the sake of simplicity, we will use the notation \( A = < x, \mu_A, \gamma_A > \) instead of \( A = \{(x, \mu_A(x), \gamma_A(x)) | x \in X\} \).
Definition 2.3: [1] Let X and Y be IFTS’s and f:X→Y be a function. If B = \{(y, \mu_B(y), \nu_B(y)) | y \in Y\} is an IFS in Y, then the pre image of B under f, denoted by f^{-1}(B), is the IFS in X defined by

\[ f^{-1}(B) = \{(x : f^{-1}(\mu_B)(x), f^{-1}(\nu_B)(x)) | x \in X\} \]

and the image of A under f, denoted by f(A) = (y, f(\mu_A), f(\nu_A)) is an IFS of Y, where for each y \in Y

\[ f(\mu_A)(y) = \begin{cases} \sup_{x \in f^{-1}(y)}\mu_A(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \]

\[ f(\nu_A)(y) = \begin{cases} \inf_{x \in f^{-1}(y)}\nu_A(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 1 & \text{otherwise.} \end{cases} \]

It is noted that \( f^{-1}(\overline{B}) = f^{-1}(\overline{B}) \).

Definition 2.4: [5] An intuitionistic fuzzy topology (IFT) on X is a family \( \tau \) of IFS’s in X satisfying the following axioms:

1. \( 0, 1 \in \tau \),
2. \( G_1 \cap G_2 \in \tau \) for any \( G_1, G_2 \in \tau \),
3. \( \bigcap G_i \in \tau \) for any family \( \{ G_i : i \in J \} \).

In this case, the pair (X, \( \tau \)) is called an intuitionistic fuzzy topological space (IFTS) and any IFS in \( \tau \) is known as an intuitionistic fuzzy open set (IFOS) in X. The complement \( \overline{A} \) of an IFOS A in IFTS (X, \( \tau \)) is called an intuitionistic fuzzy closed set (IFCS) in X.

Definition 2.5: [5] Let (X, \( \tau \)) be an IFTS and let A = \( < x, \mu_A, \nu_A > \) be an IFS in X. Then the intuitionistic fuzzy interior and intuitionistic fuzzy closure of A are defined by

\[ \text{Int}(A) = \bigcup \{ G : G \text{ is an IFOS in X and } G \subseteq A \}, \]

\[ \text{Cl}(A) = \bigcap \{ K : K \text{ is an IFCS in X and } A \subseteq K \}. \]

Properties of closure and interior of a intuitionistic fuzzy set which are needed in the sequel, are summarized in the following lemma.

Lemma 2.6: For any IFS’s A and B in IFTS (X, \( \tau \)),

1. \( \text{Cl}(A) = \overline{\text{Int}(A)} \);
2. \( \text{Int}(A) = \overline{\text{Cl}(A)} \);
3. A is an intuitionistic fuzzy closed (IFC) \( \iff \) \( \text{Cl}(A) = A \) (resp. intuitionistic fuzzy open (IFO) \( \iff \) \( \text{Int}(A) = A \));
4. \( A \subseteq B \iff \text{Cl}(A) \subseteq \text{Cl}(B) \); \( \text{Int}(A) \subseteq \text{Int}(B) \);
5. \( \text{Cl} \text{ Cl}(A) = \text{Cl}(A) ;\) \( \text{Int} \text{ Int}(A) = \text{Int}(A) ;\)
6. \( \text{Cl}(A) \lor \text{Cl}(B) = \text{Cl}(A \lor B) ;\)
7. \( \text{Cl}(A) \land \text{Cl}(B) \supseteq \text{Cl}(A \land B) ;\)
8. \( \text{Int}(A) \lor \text{Int}(B) \subseteq \text{Int}(A \lor B) ;\)
9. \( \text{Int}(A) \land \text{Int}(B) = \text{Int}(A \land B) .\)

Definition 2.7: [9] Let A be an IFS in an IFTS (X, \( \square \)). Then the intuitionistic fuzzy boundary of A is defined as \( \text{IBd} A = \text{Cl} A \land \text{Cl} \overline{A} \). IBdA is a intuitionistic fuzzy closed set (IFCS).
3. Intuitionistic Fuzzy Semiboundary

**Definition 3.1:** [6] An IFS $A$ in an IFTS $(X, \tau)$ is called an intuitionistic fuzzy semi open set (IFSOS) if $A \subseteq \text{cl} (\text{int}(A))$ (or) if there exists $V \in \tau$ such that $V \subseteq A \subseteq \text{Cl}V$.

An IFS $A$ is called intuitionistic fuzzy semi closed set, if the complement of $A$ is an IFSOS.

**Definition 3.2:** [6] Let $A$ be an IFS in an IFTS $(X, \tau)$. Then Semi interior (SInt) and semi closure (SCl) of $A$ are given as

$$\text{SInt} A = \bigcup \{G / G \text{ is an IFSOS in } X, G \subseteq A\}$$

$$\text{SCl} A = \cap \{K / K \text{ is an IFSCS in } X, A \subseteq K\}$$

**Theorem 3.3:** For IFSs $A$ and $B$ in an IFTS $(X, \tau)$, then

1. $\text{SInt}(A \cup B) \supseteq \text{SInt}(A) \cup \text{SInt}(B)$
2. $\text{SInt}(A \cap B) = \text{SInt}(A) \cap \text{SInt}(B)$
3. $\text{SCl}(A \cup B) = \text{SCl}(A) \cup \text{SCl}(B)$
4. $\text{SCl}(A \cap B) \subseteq \text{SCl}(A) \cap \text{SCl}(B)$
5. $\text{SCl}(A) = \text{SCl}(A)$
6. $\text{SInt}(A) = \text{SInt}(A)$
7. $\text{SInt}(A) \subseteq \text{SInt}(A)$
8. $\text{SCl}(A) \supseteq \text{SCl}(A)$

**Proof:** (1) $\text{SInt}(A)$ and $\text{SInt}(B)$ are both intuitionistic fuzzy semiopen and $A \subseteq A \cup B$, $B \subseteq A \cup B \Rightarrow \text{SInt}(A) \subseteq \text{SInt}(A \cup B)$ and $\text{SInt}(B) \subseteq \text{SInt}(A \cup B)$. Hence, $\text{SInt}(A) \cup \text{SInt}(B) \subseteq \text{SInt}(A \cup B)$.

(2) $A \cap B \subseteq A$ and $A \cap B \subseteq B \Rightarrow \text{SInt}(A \cap B) \subseteq \text{SInt}(A) \cap \text{SInt}(B)$.

$\text{SInt}(A \cap B) \subseteq \text{SInt}(A) \cap \text{SInt}(B)$. Therefore $\text{SInt}(A \cap B) \subseteq \text{SInt}(A) \cap \text{SInt}(B)$.

Conversely, $\text{SInt}(A) \subseteq A$ and $\text{SInt}(B) \subseteq B \Rightarrow \text{SInt}(A) \cap \text{SInt}(B) \subseteq A \cap B$ and $\text{SInt}(A) \cap \text{SInt}(B)$ is intuitionistic fuzzy semiopen. But $\text{SInt}(A \cap B)$ is the largest intuitionistic fuzzy semiopen set contained in $A \cap B$. Therefore $\text{SInt}(A) \cap \text{SInt}(B) \subseteq \text{SInt}(A \cap B)$. Hence $\text{SInt}(A) \cap \text{SInt}(B) = \text{SInt}(A \cap B)$.

(3) It follows easily from (2).

(4) Since $A \cap B \subseteq A$, $A \cap B \subseteq B \Rightarrow \text{SCl}(A \cap B) \subseteq \text{SCl}(A)$, $\text{SCl}(A \cap B) \subseteq \text{SCl}(B)$ Hence, $\text{SCl}(A \cap B) \subseteq \text{SCl}(A) \cap \text{SCl}(B)$.

(5) – (8) proofs are straightforward.

In theorem 3.3 (1 and 4), the equality may not hold as seen in the following example.

**Example 3.4:** Let $X = \{a, b\}$ and let $A = \langle x, \frac{0.6}{a}, \frac{0.5}{b}, \frac{0.4}{a}, \frac{0.3}{b}\rangle$. Then

$$\tau = \{0_\infty, 1_\infty, A\} \text{ is an IFT on } X, \tau' = \{0_\infty, 1_\infty, \langle x, \frac{0.4}{a}, \frac{0.3}{b}, \frac{0.6}{a}, \frac{0.5}{b}\rangle\}.$$

IFSOS($X$) = $\{0_\infty, 1_\infty, \langle x, \frac{l_1}{a} + \frac{l_2}{b}, \frac{m_1}{a} + \frac{m_2}{b}\rangle \text{ where } 0 \leq l_1 \leq 1, 0.5 \leq l_2 \leq 1, 0 \leq m_1 \leq 0.4, 0 \leq m_2 \leq 0.3$.

$I\text{FCS}($ $X$) = $\{0_\infty, 1_\infty, \langle x, \frac{a_1}{a} + \frac{a_2}{b}, \frac{b_1}{a} + \frac{b_2}{b}\rangle \text{ where } 0 \leq a_1 \leq 0.4, 0 \leq a_2 \leq 0.3$.

$0.6 \leq b_1 \leq 1, 0.5 \leq b_2 \leq 1$.

$B = \langle x, \frac{0.6}{a}, \frac{0.2}{a}, \frac{0.4}{b}, \frac{0.5}{b}\rangle$ and $C = \langle x, \frac{0.3}{a}, \frac{0.5}{b}, \frac{0.6}{a}, \frac{0.3}{b}\rangle$.

Let
Definition 3.5: Let $A$ be a IFS in an IFTS $(X, \tau)$. Then the intuitionistic fuzzy semiboundary of $A$ is defined as

$$\text{ISBd } A = \text{SCl } A \cap \text{SCl } \bar{A}.$$ 

ISBd $A$ is an intuitionistic fuzzy semiclosed set (IFSCS).

Remark 3.6: In classical topology, for an arbitrary set $A$ of a topological space $X$, $A \cup \text{ISBd } A = \text{SCl } A$, but $A \cup \text{ISBd } A \leq \text{SCl } A$ for an arbitrary intuitionistic fuzzy set $A$ in an IFTS $(X, \tau)$, where the equality may not hold as is seen in the following example.

Example 3.7: Let $X = [a, b]$ and let $A = (\frac{0.2}{a} + \frac{0.2}{b}, \frac{0.4}{a} + \frac{0.6}{b})$. Then $\tau = \{0_\tau, 1_\tau, A\}$ is an IFTS on $X$, where $\text{IFSO}(X) = (0_\tau, 1_\tau, (\frac{l_1}{a} + \frac{l_2}{b} + \frac{m_3}{a} + \frac{m_4}{b})$ where $0.2 \leq l_1 \leq 0.4, 0.3 \leq l_2 \leq 0.6, 0.2 \leq m_3 \leq 0.4, 0.3 \leq m_4 \leq 0.6$.

IFSCS$(X) = (0_\tau, 1_\tau, (\frac{a_1}{a} + \frac{a_2}{b} + \frac{a_3}{a} + \frac{a_4}{b})$ where $0.2 \leq a_1 \leq 0.4, 0.3 \leq a_2 \leq 0.6, 0.2 \leq a_3 \leq 0.4, 0.3 \leq a_4 \leq 0.6$.

Choose, $P = (\frac{0.2}{a} + \frac{0.4}{b}, \frac{0.6}{a} + \frac{0.4}{b})$ be IFS on $X$.

Then $\text{SCl } P = (\frac{0.4}{a} + \frac{0.3}{b} + \frac{0.4}{a} + \frac{0.6}{b})$.

ISBd $P = \text{SCl } P \cap \text{SCl } \bar{P} = (\frac{0.4}{a} + \frac{0.3}{b} + \frac{0.4}{a} + \frac{0.6}{b})$.

Hence, $P \cup \text{ISBd } P = (\frac{0.2}{a} + \frac{0.6}{b} + \frac{0.4}{a} + \frac{0.3}{b}) = 1 = \text{SCl } P$.

Proposition 3.8: Let $A$ and $B$ be IFS’s in an IFTS $(X, \tau)$. Then the following conditions hold.

1. $\text{ISBd } A = \text{ISBd } \bar{A}$
2. If $A$ is IFSC, then $\text{ISBd } A \leq A$
3. If $A$ is IFSO, then $\text{ISBd } A \leq \bar{A}$
4. Let $A \leq B$ and $B \in \text{IFSCS}(X)$ (resp., $B \in \text{IFSCS}(X)$). Then, $\text{ISBd } A \leq B$ (resp., $\text{ISBd } A \leq B)$;
5. $\text{ISBd } \bar{A} = \text{SCl } A \cup \text{SCl } \bar{A}$.
6. $\text{ISBd } A \leq \text{IBd } A$
7. $\text{SCl } (\text{ISBd } A) \leq \text{IBd } A$

Proof: (1) $\text{ISBd } A = \text{SCl } A \cap \text{SCl } \bar{A} = \text{SCl } \bar{A} \cap \text{SCl } A$

$= \text{SCl } \bar{A} \cap \text{SCl } \bar{A} = \text{ISBd } \bar{A}$.
Hence, $\overline{\text{IFS}} \implies \overline{\text{IFS}}$. By (1), (2) \( \overline{\text{IFS}} \leq \text{IFS} \). Hence, \( \overline{\text{IFS}} \leq \overline{\text{IFS}} \).

(4) Since $A \subseteq B$ implies $\overline{A} \not\subseteq \overline{B}$

\[
\overline{\text{IFS}} A = \overline{\text{IFS}} (A \wedge \overline{\text{IFS}} A) \leq \text{IFS} B \wedge \overline{\text{IFS}} A \leq \text{IFS} B = B. \quad \text{Hence,} \quad \overline{\text{IFS}} A \leq B.
\]

(5) $\overline{\text{IFS}} A = (\text{IFS} A \wedge \overline{\text{IFS}} A) = (\text{IFS} A) \vee (\overline{\text{IFS}} A) = \text{SInt} A \vee \text{SInt} \overline{A}$.

(6) Since $\overline{\text{IFS}} A \leq \text{IFS} A$ and $\overline{\text{IFS}} A \leq \overline{\text{IFS}} A$.

Then $\overline{\text{IFS}} A = \overline{\text{IFS}} (\text{IFS} A \wedge \overline{\text{IFS}} A) \leq \text{IFS} A \wedge \overline{\text{IFS}} A \leq \overline{\text{IFS}} A$. Hence, \( \overline{\text{IFS}} A \leq \overline{\text{IFS}} A \).

(7) $\overline{\text{IFS}}(\overline{\text{IFS}} A) = \overline{\text{IFS}} (\text{IFS} A \wedge \overline{\text{IFS}} A) \leq \text{IFS} \text{IFS} A \wedge \overline{\text{IFS}} A = \text{IFS} A \wedge \overline{\text{IFS}} A$

Hence, $\overline{\text{IFS}}(\overline{\text{IFS}} A) \leq \overline{\text{IFS}} A$.

The converse of (2), (3) and the equality may hold in (6), (7) of Proposition 3.8 is not true as seen in the following example.

**Example 3.9:** Choose IFS $\mathcal{P} = \langle \chi, \frac{0.5}{a} + \frac{0.6}{b}, \frac{0.4}{a} + \frac{0.3}{b} \rangle$ in the IFTS $(X, \tau)$ defined in example 3.7.

Then (1) $\overline{\text{IFS}} \mathcal{P} = \langle \chi, \frac{0.4}{a} + \frac{0.3}{b}, \frac{0.4}{a} + \frac{0.3}{b} \rangle \leq \mathcal{P}$, but $\mathcal{P}$ is not intuitionistic fuzzy semi closed.

(2) $\overline{\text{IFS}} \overline{\mathcal{P}} = \langle \chi, \frac{0.2}{a} + \frac{0.4}{b}, \frac{0.4}{a} + \frac{0.4}{b} \rangle$, $\overline{\text{IFS}} \overline{\mathcal{P}} = \langle \chi, \frac{0.4}{a} + \frac{0.3}{b}, \frac{0.4}{a} + \frac{0.3}{b} \rangle \leq \mathcal{Q}$.

Hence, \( \overline{\text{IFS}} P \not\subseteq \text{IFS} P \).

(4) $\overline{\text{IFS}}(\overline{\text{IFS}} P) = \langle \chi, \frac{0.4}{a} + \frac{0.2}{b}, \frac{0.4}{a} + \frac{0.6}{b} \rangle \not\subseteq \overline{\text{IFS}} P$.

**Proposition 3.10:** Let $A$ be a intuitionistic fuzzy set in an IFTS $(X, \tau)$. Then,

(1) $\overline{\text{IFS}} A = \text{IFS} A \wedge (\text{SInt} A)$;

(2) $\overline{\text{IFS}} (\text{SInt} A) \leq \overline{\text{IFS}} A$;

(3) $\overline{\text{IFS}} (\text{IFS} A) \leq \overline{\text{IFS}} A$;

(4) $\text{SInt} A \leq A \wedge (\overline{\text{IFS}} A)$.

The following example shows that the equality may not hold in (2), (3) and (4) of Proposition 3.10.

**Example 3.11:** Choose IFS $\mathcal{M} = \langle \chi, \frac{0.5}{a} + \frac{0.2}{b}, \frac{0.5}{a} + \frac{0.6}{b} \rangle$ in the IFTS $(X, \tau)$ defined in example 3.7.

Then (1) $\text{IFS} \mathcal{M} = \text{IFS} \mathcal{M}$, $\text{IFS} \overline{\mathcal{M}} = \text{IFS} \overline{\mathcal{M}}$, $\text{SInt} \mathcal{M} = \text{SInt} \overline{\mathcal{M}} = 0$. \( \text{SInt} \overline{\mathcal{M}} = 0 \).

\( \overline{\text{IFS}} \mathcal{M} = \overline{\text{IFS}} \mathcal{M} = \overline{\text{IFS}} \mathcal{M} = 1 \).

\( \overline{\text{IFS}} (\text{SInt} \mathcal{M}) = \overline{\text{IFS}} (\text{SInt} \mathcal{M}) \wedge \text{IFS} (\overline{\text{SInt} \mathcal{M}}) = \text{IFS} (\overline{\text{SInt} \mathcal{M}}) \wedge \text{IFS} (\text{SInt} \mathcal{M}) = 0 \).
Hence,  \( \text{ISBD}(\text{SInt } M) = 0_\infty \leq \text{ISBD } M = 1_\infty \).

(2)  \( \text{ISBD}(\text{SCI } M) = \text{SCI}(\text{SCI } M) \land \text{SCI}(\text{SCI } M) = \text{SCI}(1_\infty) \land \text{SCI}(0_\infty) = 0_\infty \).

Hence,  \( \text{ISBD}(\text{SCI } M) \leq \text{ISBD } M \).

(3)  \( \text{SInt } M = 0_\infty \leq M \land \text{ISBD } M = (x, \frac{0.5}{a} + \frac{0.2}{b} \cdot \frac{0.5}{a} + \frac{0.6}{b}) \).

**Remark 3.12:** In general topology, the following conditions hold.

\[
\text{SBD } A \cap \text{SInt } A = \emptyset ; \text{SInt } A \cup \text{SBD } A = \text{SCI } A ; \text{SInt } A \cup \text{SInt } A = \text{SBD } A = X.
\]

In intuitionistic fuzzy topology, these may not hold as seen in the following example.

Choose IFS

**Example 3.13:** Choose IFS  \( P = (x, \frac{0.5}{a} + \frac{0.6}{b} \cdot \frac{0.4}{a} + \frac{0.3}{b}) \) in the IFTS \( (X \tau) \) defined in example 3.7.

(1)  \( \text{ISBD } P = (x, \frac{0.5}{a} + \frac{0.2}{b} \cdot \frac{0.4}{a} + \frac{0.6}{b}) \), \( \text{SInt } P = (x, \frac{0.4}{a} + \frac{0.6}{b} \cdot \frac{0.4}{a} + \frac{0.3}{b}) \).

Hence,  \( \text{ISBD } P \land \text{SInt } P = 0_\infty \).

(2)  \( \text{SInt } P \lor \text{ISBD } P = (x, \frac{0.4}{a} + \frac{0.6}{b} \cdot \frac{0.4}{a} + \frac{0.3}{b}) \neq 1_\infty = \text{SCI } P \).

(3)  \( \text{SInt } P \lor \text{ISBD } P = (x, \frac{0.4}{a} + \frac{0.6}{b} \cdot \frac{0.4}{a} + \frac{0.3}{b}) \neq 1_\infty \).

**Theorem 3.14:** Let  \( A \) and  \( B \) be IFS's in an IFTS \( (X \tau) \). Then,  \( \text{ISBD}(A \lor B) \leq \text{ISBD } A \lor \text{ISBD } B \).

**Proof:**  \( \text{ISBD}(A \lor B) = \text{SCI}(A \lor B) \land \text{SCI}(A \lor B) \leq (\text{SCI } A \lor \text{SCI } B) \land (\text{SCI } \overline{A} \lor \text{SCI } \overline{B}) \leq \text{ISBD } A \lor \text{ISBD } B \).

In theorem 3.14, the equality may not hold as seen in the following example.

**Example 3.15:** Choose IFS's  \( P = (x, \frac{0.5}{a} + \frac{0.6}{b} \cdot \frac{0.4}{a} + \frac{0.3}{b}) \) in the IFTS \( (X \tau) \) defined in example 3.7.

Then  \( \text{SCI } P = 1_\infty \), \( \overline{P} = (x, \frac{0.4}{a} + \frac{0.3}{b} \cdot \frac{0.4}{a} + \frac{0.6}{b}) \).

\( \text{ISBD } P = (x, \frac{0.4}{a} + \frac{0.2}{b} \cdot \frac{0.4}{a} + \frac{0.6}{b}) \).

\( \text{SCI } Q = 1_\infty \), \( \overline{Q} = 1_\infty \), \( \text{ISBD } Q = 1_\infty \).

\( P \lor Q = (x, \frac{0.5}{a} + \frac{0.6}{b} \cdot \frac{0.4}{a} + \frac{0.3}{b}) \), \( \text{SCI } P \lor Q = 1_\infty \), \( \text{SCI } (P \lor Q) = (x, \frac{0.4}{a} + \frac{0.3}{b} \cdot \frac{0.4}{a} + \frac{0.6}{b}) \).

\( \text{ISBD } (P \lor Q) = (x, \frac{0.4}{a} + \frac{0.3}{b} \cdot \frac{0.4}{a} + \frac{0.6}{b}) \neq 1_\infty = \text{ISBD } P \lor \text{ISBD } Q \).

Hence,  \( \text{ISBD}(P \lor Q) \neq \text{ISBD } P \lor \text{ISBD } Q \).

**Remark 3.16:** Let  \( P, Q \) be IFS's in an IFTS \( (X \tau) \), then the IFS's  \( \text{ISBD } (P \land Q) \) and  \( \text{ISBD } P \land \text{ISBD } Q \) behave arbitrarily is seen in the following example.

**Example 3.17:** Choose IFS's  \( P = (x, \frac{0.5}{a} + \frac{0.6}{b} \cdot \frac{0.4}{a} + \frac{0.3}{b}) \), \( Q = (x, \frac{0.5}{a} + \frac{0.2}{b} \cdot \frac{0.5}{a} + \frac{0.6}{b}) \), \( R = (x, \frac{0.4}{a} + \frac{0.8}{b} \cdot \frac{0.6}{a} + \frac{0.2}{b}) \) and
Theorem 3.18: For any intuitionistic fuzzy sets $A$ and $B$ in an IFTS $(X, \tau)$, 
\[ \text{ISBD} (A \land B) \leq (\text{ISBD} A \land \text{SCL} B) \lor (\text{ISBD} B \land \text{SCL} A) \]

Proof:
\[ \text{ISBD} (A \land B) = \text{SCL}(A \land B) \lor \text{SCL}(\overline{A} \lor \text{SCL} B) \]
\[ \leq (\text{SCL} A \land \text{SCL} B) \lor (\text{SCL} \overline{A} \lor \text{SCL} B) \]
\[ = [(\text{SCL} A \land \text{SCL} B) \lor \text{SCL} \overline{A}] \lor [(\text{SCL} A \land \text{SCL} B) \land \text{SCL} \overline{B}] \]
\[ = [(\text{SCL} A \land \text{SCL} \overline{A}) \lor \text{SCL} B] \lor [(\text{SCL} B \land \text{SCL} \overline{B}) \land \text{SCL} A] \]

Hence, 
\[ \text{ISBD} (A \land B) \leq (\text{ISBD} A \land \text{SCL} B) \lor (\text{ISBD} B \land \text{SCL} A) . \]

Corollary 3.19: For any intuitionistic fuzzy sets $A$ and $B$ in an IFTS $(X, \tau)$, then 
\[ \text{ISBD} (A \land B) \leq \text{ISBD} A \lor \text{ISBD} B . \]

The equality in the above theorem may not hold as seen in the following example.

Example 3.20: Choose IFS’s 
\[ R = \langle x, \frac{\alpha}{a \otimes b} + \frac{\beta}{a \otimes b}, \frac{\eta}{a \otimes b} + \frac{\gamma}{a \otimes b} \rangle \]
and 
\[ S = \langle x, \frac{\alpha}{a \otimes b} + \frac{\beta}{a \otimes b}, \frac{\eta}{a \otimes b} + \frac{\gamma}{a \otimes b} \rangle \]
in the IFTS $(X, \tau)$ of example 3.7.

Then, 
\[ \text{ISBD} R = 1, \quad \text{ISBD} S = \langle x, \frac{\alpha}{a \otimes b} + \frac{\beta}{a \otimes b}, \frac{\eta}{a \otimes b} + \frac{\gamma}{a \otimes b} \rangle \]
\[ (\text{ISBD} R \land \text{SCL} S) \lor (\text{ISBD} S \land \text{SCL} R) = 1 \lor \langle x, \frac{\alpha}{a \otimes b} + \frac{\beta}{a \otimes b}, \frac{\eta}{a \otimes b} + \frac{\gamma}{a \otimes b} \rangle = 1. \]

\[ \text{ISBD} (R \land S) = \langle x, \frac{\alpha}{a \otimes b} + \frac{\beta}{a \otimes b}, \frac{\eta}{a \otimes b} + \frac{\gamma}{a \otimes b} \rangle \]

Hence, 
\[ \text{ISBD} (R \land S) \not\leq (\text{ISBD} R \land \text{SCL} S) \lor (\text{ISBD} S \land \text{SCL} R) . \]

Proposition 3.21: For any IFS $A$ in an IFTS $X$, then, 
(1) \[ \text{ISBD} (\text{ISBD} A) \leq \text{ISBD} A . \]

(2) \[ \text{ISBD} (\text{ISBD} (\text{ISBD} A)) \leq \text{ISBD} (\text{ISBD} A) . \]
Proof:

(1) \( \text{ISBd} (\text{ISBd} A) = \text{Scl} (\text{ISBd} A) \land \text{Scl} (\text{ISBd} A) \)

\[ \leq \text{Scl} (\text{ISBd} A) \leq \text{ISBd} A \] of proposition 3.8

Hence, \( \text{ISBd} (\text{ISBd} A) \leq \text{ISBd} A \).

(2) \( \text{ISBd} (\text{ISBd} (\text{ISBd} A)) = \text{Scl} (\text{ISBd} (\text{ISBd} A)) \land \text{Scl} (\text{ISBd} (\text{ISBd} A)) \)

\[ \leq \text{ISBd} (\text{ISBd} A) \land \text{Scl} (\text{ISBd} (\text{ISBd} A)) \leq \text{ISBd} (\text{ISBd} A) \]

Hence, \( \text{ISBd} (\text{ISBd} (\text{ISBd} A)) \leq \text{ISBd} (\text{ISBd} A) \).

Remark 3.22: The equality may not hold in proposition 3.21 (1) as seen in the following example. We could not find an example to show that equality in proposition 3.21 (2) may not hold.

Example 3.23: Choose IFS \( M = \langle x, 0.5, a, 0.2, b, 0.5, a, 0.6, b \rangle \) in the IFTS \( (X, \tau) \) defined in example 3.7. Then, \( \text{Scl} M = 1_\sim, \text{Scl} M = 1_\sim, \text{ISBd} M = 1_\sim, \text{ISBd} M = 0_\sim \)

\( \text{ISBd} (\text{ISBd} M) = \text{Scl} (\text{ISBd} M) \land \text{Scl} (\overline{\text{ISBd} M}) = \text{Scl} 1_\sim \land \text{Scl} 0_\sim = 1_\sim \land 0_\sim = 0_\sim \)

Hence, \( \text{ISBd} (\text{ISBd} M) \neq \text{ISBd} M \).

4. Intuitionistic Fuzzy product related spaces


**Definition 4.1:** [7] If \( A = \langle x, \mu_A(x), \gamma_A(x) \rangle \) and \( B = \langle y, \mu_B(y), \gamma_B(y) \rangle \) be IFSs of \( X \) and \( Y \) respectively. Then the product of intuitionistic fuzzy sets \( A \) and \( B \) be defined by,

\[ A \times B = \langle (x, y), \mu_{AB}(x, y), \gamma_{AB}(x, y) \rangle \]

is IFS in \( X \times Y \);

where \( \mu_{(A \times B)}(x, y) = \min\{\mu_A(x), \mu_B(y)\} \) for each \( (x, y) \in X \times Y \);

\( \gamma_{(A \times B)}(x, y) = \max\{\gamma_A(x), \gamma_B(y)\} \) for each \( (x, y) \in X \times Y \)

and \( 0 \leq \mu_{AB}(x, y) + \gamma_{AB}(x, y) \leq 1 \). It may be notice that

\( \overline{A \times B} = \langle (x, y), \gamma_{AB}(x, y), \mu_{AB}(x, y) \rangle \). The following lemma is needed in sequel.

**Lemma 4.2:** Let \( A, B, C \) and \( D \) be a IFS in IFTS \( X \), then

\( (A \land B) \times (C \land D) = (A \times D) \land (B \times C) \)
Proof: \[
\mu_{(A\land B)\times(C\land D)}(x,y) = \min\{\mu_{(A\land B)}(x), \mu_{(C\land D)}(y)\}
= \min\{\min\{\mu_{A}(x), \mu_{B}(x)\}, \min\{\mu_{C}(y), \mu_{D}(y)\}\}
= \min\{\min\{\mu_{A}(x), \mu_{D}(y)\}, \min\{\mu_{B}(x), \mu_{C}(y)\}\}
= \min\{\mu_{(A\land D)}(x,y), \mu_{(B\land C)}(x,y)\}
\]

\[
\nu_{(A\land B)\times(C\land D)}(x,y) = \max\{\nu_{(A\land B)}(x), \nu_{(C\land D)}(y)\}
= \max\{\max\{\nu_{A}(x), \nu_{B}(x)\}, \max\{\nu_{C}(y), \nu_{D}(y)\}\}
= \max\{\max\{\nu_{A}(x), \nu_{D}(y)\}, \max\{\nu_{B}(x), \nu_{C}(y)\}\}
= \max\{\nu_{(A\land D)}(x,y), \nu_{(B\land C)}(x,y)\}
\]

\[
\nu_{(A\land B)\times(C\land D)}(x,y) = \nu_{(A\land D)\land(B\land C)}(x,y).
\]
Hence, \((A\land B)\times(C\land D) = (A\land D) \land (B\land C)\).

Definition 4.3: [7] An IFTS \((X, \tau_X)\) is product related to another IFTS \((Y, \tau_Y)\) if for any intuitionistic fuzzy sets \(U\) of \(X\) and \(V\) of \(Y\), whenever \(\bar{A} \geq U\) and \(\bar{B} \geq V\) imply \(\bar{A} \times 1_\ldots \lor 1_\ldots \times \bar{B} \geq U \times V\), where \(A \in \tau_X\) and \(B \in \tau_Y\), there exist \(A_1 \in \tau_X\) and \(B_1 \in \tau_Y\) such that \(\bar{A}_1 \geq U \lor \bar{B}_1 \geq V\) and \(\bar{A}_1 \times 1_\ldots \lor 1_\ldots \times \bar{B}_1 = \bar{A} \times 1_\ldots \lor 1_\ldots \times \bar{B}\).

Lemma 4.4: [7] If \(A\) is an IFS of \(X\) and \(B\) is an IFS of \(Y\), then
(1) \((A \times 1_\ldots) \cap (1_\ldots \times B) = A \times B\);
(2) \(-\bar{A} \times \bar{B} = (\bar{A} \times 1_\ldots) \cup (1_\ldots \times \bar{B})\);
(3) \((A \times 1_\ldots) \cup (1_\ldots \times B) = \bar{A} \times \bar{B}\).

Proof: Let \(= (x, \mu_A(x), \nu_A(x))\), \(B = (y, \mu_B(y), \nu_B(y))\).

(1) Since \(A \times 1_\ldots = (x, \min(\mu_A, 1_\ldots), \max(\nu_A, 0_\ldots)) = (x, \mu_A(x), \nu_A(x)) = A\).
Similarly \(1_\ldots \times B = (y, \min(1_\ldots, \mu_B), \max(0_\ldots, \nu_B)) = (y, \mu_B(x), \nu_B(x)) = B\).
We have \((A \times 1_\ldots) \cap (1_\ldots \times B) = A(x) \cap B(y) = \{(x, y), \mu_A(x) \land \mu_B(y), \nu_A(x) \lor \nu_B(y)\}\)
Hence, \((A \times 1_\ldots) \cap (1_\ldots \times B) = A \times B\).

(2) Since \(\bar{A} = (x, \nu_A(x), \mu_A(x))\), \(\bar{B} = (y, \nu_B(y), \mu_B(y))\)
\begin{align*}
\bar{A} \times 1_\ldots &= ((x, y), \min(\nu_A(x), 1_\ldots(y)), \max(\mu_A(x), 0_\ldots(y))) = ((x, y), \nu_A(x), \mu_A(x)) \\
1_\ldots \times \bar{B} &= ((x, y), \min(1_\ldots(x), \mu_B(y)), \max(0_\ldots(x), \nu_B(y))) = ((x, y), \nu_B(y), \mu_B(y)) \\
(\bar{A} \times 1_\ldots) \cup (1_\ldots \times \bar{B}) &= ((x, y), \max(\nu_A(x), \nu_B(y)), \min(\mu_A(x), \mu_B(y))) = \bar{A} \times \bar{B}
\end{align*}
(3) Follows from (1) and (2).
Lemma 4.5: Let $A$ and $B$ be IFSCS’s in IFTS’s $X$ and $Y$ respectively. Then $A \times B$ is an IFSCS in the intuitionistic fuzzy product topological space (IFPTS) of $X \times Y$.

Proof: Let $A = \{x, \mu_A(x), \gamma_A(x)\}$, $B = \{y, \mu_B(y), \gamma_B(y)\}$. From Lemma 4.4 we have $(A \times B)(x,y) = ((A \times 1_\sim) \cup (1_\sim \times B))(x,y)$. Since $\overline{A \times 1_\sim}$ and $1_\sim \times \overline{B}$ are IFSOS’s in $X$ and $Y$ respectively, hence $(\overline{A \times 1_\sim}) \cup (1_\sim \times \overline{B})$ is IFSOS of $X \times Y$. Hence $\overline{A \times B}$ is an IFSOS of $X \times Y$ and consequently $A \times B$ is an IFSCS of $X \times Y$.

Theorem 4.6: If $A$ and $B$ are IFS’s of IFTS’s $X$ and $Y$ respectively, then

1. $\text{scl} A \times \text{scl} B \geq \text{scl}(A \times B)$;

2. $\text{Sint} A \times \text{Sint} B \leq \text{Sint}(A \times B)$.

Proof: (1) Since $A \leq \text{scl}(A)$ and $B \leq \text{scl}(B)$, hence $A \times B \leq \text{scl}(A) \times \text{scl}(B)$ implies $\text{scl}(A \times B) \leq \text{scl}((\text{scl}(A) \times \text{scl}(B)))$ and from Lemma 4.5 we have $\text{scl}(A \times B) \leq \text{scl}(\text{scl}(A) \times \text{scl}(B))$.

(2) Follows from (1) and using the facts that $\overline{\text{scl}(C)} = \text{sint}(\overline{C})$ and $\overline{\text{sint}(C)} = \text{scl}(\overline{C})$.

Theorem 4.7: Let $(X, \tau)$ and $(Y, \delta)$ be product related IFTSs. Then for a IFS $A$ of $X$ and a IFS $B$ of $Y$, then

1. $\text{scl}(A \times B) = \text{scl} A \times \text{scl} B$;

2. $\text{Sint}(A \times B) = \text{Sint} A \times \text{Sint} B$.

Proof: (1) For intuitionistic fuzzy sets $A_i'$ of $X$ and $B_j'$ of $Y$ then note that

(i) $\text{inf}\{A_i, B_j\} = \min\{\text{inf} A_i, \text{inf} B_j\}$;

(ii) $\text{inf}\{A_i \times 1_\sim\} = (\text{inf} A_i) \times 1_\sim$;

(iii) $\text{inf}\{1_\sim \times B_j\} = 1_\sim \times (\text{inf} B_j)$

By theorem 4.6, sufficient to show that $\text{scl} (A \times B) \geq \text{scl} A \times \text{scl} B$. Let $A_i \in \text{IFSO}(X)$ and $B_j \in \text{IFSO}(Y)$. Then,

$$\text{scl}(A \times B) = \inf\{(A_i \times B_j) : (A_i \times B_j) \geq A \times B\}$$

$$= \inf\{\overline{A_i} \times 1_\sim \vee 1_\sim \times \overline{B_j} : \overline{A_i} \times 1_\sim \vee 1_\sim \times B_j \geq A \times B\}$$

$$= \inf\{\overline{A_i} \times 1_\sim \vee 1_\sim \times B_j : A_i \geq A \ (\text{or}) \ B_j \geq B\}$$

$$= \min\{\text{inf}\{\overline{A_i} \times 1_\sim \vee 1_\sim \times B_j : A_i \geq A\}, \text{inf}\{\overline{A_i} \times 1_\sim \vee 1_\sim \times B_j : B_j \geq B\}\}$$
Hence, (2) This follows from (1) and using the facts that \( \text{and} \) and \( \text{Theorem 4.8:} \) Let \( \text{, be a family of product related intuitionistic fuzzy topological spaces. If each } \mathcal{A}_i \text{ is a IFS in } \mathcal{X}_i, \text{ then} \)

\[
\text{Proof:} \text{ It suffices to prove this for } n=2.
\]

\[
\text{ISBd } (\mathcal{A}_1 \times \mathcal{A}_2) = \text{Scl}(\mathcal{A}_1 \times \mathcal{A}_2) \cap \text{Scl}(\overline{\mathcal{A}_1 \times \mathcal{A}_2})
\]

\[
= \text{Scl}(\mathcal{A}_1 \times \mathcal{A}_2) \cap (\text{SInt}(\mathcal{A}_1 \times \mathcal{A}_2))
\]

\[
= (\text{Scl } \mathcal{A}_1 \times \text{Scl } \mathcal{A}_2) \cap (\text{SInt } \mathcal{A}_1 \times \text{SInt } \mathcal{A}_2)
\]

\[
= (\text{Scl } \mathcal{A}_1 \times \text{Scl } \mathcal{A}_2) \cap (\text{SInt } \mathcal{A}_1 \times (\text{Scl } \mathcal{A}_2 \times \text{Scl } \mathcal{A}_1))
\]

\[
= (\text{Scl } \mathcal{A}_1 \times \text{Scl } \mathcal{A}_2) \cap (\text{Scl } \mathcal{A}_1 \times \mathcal{A}_2) \times (1 \times \text{Scl } \mathcal{A}_2)
\]

\[
= (\text{Scl } \mathcal{A}_1 \times \text{Scl } \mathcal{A}_2) \times (\text{Scl } \mathcal{A}_1 \times \mathcal{A}_2) \times (1 \times \text{Scl } \mathcal{A}_2)
\]

by Lemma 4.4 of (2)

\[
= (\text{Scl } \mathcal{A}_1 \times \text{Scl } \mathcal{A}_2) \times (\text{Scl } \mathcal{A}_1 \times (1 \times \text{Scl } \mathcal{A}_2)) \times (1 \times \text{Scl } \mathcal{A}_2)
\]

by Lemma 4.2

\[
=[(\text{Scl } \mathcal{A}_1 \times \text{Scl } \mathcal{A}_2) \times (\text{Scl } \mathcal{A}_1 \times (1 \times \text{Scl } \mathcal{A}_2)) \times (\text{Scl } \mathcal{A}_1 \times \text{Scl } \mathcal{A}_2) \times (1 \times \text{Scl } \mathcal{A}_2)]
\]

Hence, \( \text{ISBd } (\mathcal{A}_1 \times \mathcal{A}_2) = (\text{ISBd } \mathcal{A}_1 \times \text{Scl } \mathcal{A}_2) \times (\text{Scl } \mathcal{A}_1 \times \text{ISBd } \mathcal{A}_2). \)

5. Intuitionistic Fuzzy continuous function and Irresolute function:

Definition 5.1: \([6]\) Let \((X, \tau)\) and \((Y, \delta)\)be two IFTS's. A function \(f: (X, \tau) \rightarrow (Y, \delta)\) is said to be intuitionistic fuzzy semi continuous, if the pre image of each IFOS in Y is IFSOS in X.

Theorem 5.2:[11] Let \((X, \tau)\) and \((Y, \delta)\) be two IFTS’s and let \(f: X \rightarrow Y\) be a mapping. Then, the following conditions are equivalent
(1) $f$ is intuitionistic fuzzy semi continuous,
(2) $f^{-1}(B)$ is an IFSCS in $X$ for each IFCS $B$ in $Y$,
(3) $(S\text{Cl} f^{-1}(B)) \subseteq (f^{-1}(B))$, for each intuitionistic fuzzy set $B$ in $Y$,
(4) $f^{-1}(\text{Int}(B)) \subseteq S\text{Int}(f^{-1}(B))$, for each intuitionistic fuzzy set $B$ in $Y$,
(5) $f^{-1}(B) \subseteq \text{Cl} (S\text{Int}(f^{-1}(B)))$, for each intuitionistic fuzzy set $B$ in $Y$.

**Theorem 5.3**: Let $X$ be a IFTS and $f: X \rightarrow Y$ be a intuitionistic fuzzy semi continuous function. Then, $\text{ISBd} f^{-1}(B) \subseteq (\text{IBd} B)$, for any IFS $B$ in $Y$.

**Proof**: Let be intuitionistic fuzzy semi continuous and $B$ be a IFS in $Y$. Then, $\text{Cl} B$ is intuitionistic fuzzy closed in $Y$. Which implies that $f^{-1}(\text{Cl} B)$ is intuitionistic fuzzy semi closed in $X$.

Therefore,

$$\text{ISBd} f^{-1}(B) = S\text{Cl} (f^{-1}(B)) \wedge S\text{Cl} (f^{-1}(B))$$

$$\leq f^{-1}(\text{Cl} B) \wedge S\text{Cl} f^{-1}(B) \leq f^{-1}(\text{Cl} B) \wedge f^{-1}(\text{Cl} B)$$

$$\leq f^{-1}(\text{Cl} B \wedge \text{Cl} B) = f^{-1}(\text{IBd} B)$$

Hence, $\text{ISBd} f^{-1}(B) \leq f^{-1}(\text{IBd} B)$.

**Definition 5.4**: [11] Let $(X, \tau)$ and $(Y, \delta)$ be two IFTS’s. A function $f: (X, \tau) \rightarrow (Y, \delta)$ is said to be intuitionistic fuzzy irresolute if and only if the pre image of each IFSOS in $Y$ is and IFSOS in $X$.

**Theorem 5.5**: [11] Let $(X, \tau)$ and $(Y, \delta)$ be two IFTS’s and let $f: X \rightarrow Y$ be a mapping. Then, the following conditions are equivalent,

(1) $f$ is intuitionistic fuzzy irresolute,
(2) $f^{-1}(B)$ is an IFSCS in $X$ for each IFCS $B$ in $Y$,
(3) $f^{-1}(S\text{Cl} A) \subseteq S\text{Cl} (f^{-1}(A))$ for each intuitionistic fuzzy set $A$ in $X$,
(4) $S\text{Cl}(f^{-1}(B)) \subseteq f^{-1}(S\text{Cl} B)$ for each intuitionistic fuzzy set $B$ in $Y$,
(5) $f^{-1}(S\text{Int} B) \subseteq \text{Int}(f^{-1}(B))$ for each intuitionistic fuzzy set $B$ in $Y$.

The following theorem gives a necessary condition of intuitionistic fuzzy irresolute functions in terms of intuitionistic fuzzy semi boundary.

**Theorem 5.6**: Let $f: X \rightarrow Y$ be a intuitionistic fuzzy irresolute function.

Then, $\text{ISBd} f^{-1}(B) \subseteq f^{-1}(\text{ISBd} B)$ for any IFS $B$ in $Y$.

**Proof**: Let $B$ be a IFS in $Y$. Then, $S\text{Cl}$ is intuitionistic fuzzy closed in $Y$. Which implies that $f^{-1}(S\text{Cl} B)$ is intuitionistic fuzzy semi closed in $X$.

Therefore,

$$\text{ISBd} f^{-1}(B) = S\text{Cl} (f^{-1}(B)) \wedge S\text{Cl} (f^{-1}(B))$$

$$\leq f^{-1}(S\text{Cl} B) \wedge S\text{Cl} (f^{-1}(B)) \leq f^{-1}(S\text{Cl} B) \wedge f^{-1}(S\text{Cl} B)$$

$$\leq f^{-1}(S\text{Cl} B \wedge S\text{Cl} B) = f^{-1}(\text{ISBd} B)$$

Hence, $\text{ISBd} f^{-1}(B) \leq f^{-1}(\text{ISBd} B)$.

**Acknowledgement**: The authors are highly grateful to referees for valuable comments and suggestions for improving the paper.
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