RECENT PROGRESS ABOUT THE CONJECTURE OF ERDÖS- STRAUS

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Keywords: Conjecture of Erdös and Straus; twin Pythagorean triples, Diophantine equations

Abstract. In this paper we present our recent progress about the conjecture of Erdös-Straus by using Diophantine equations and the irreducible twin Pythagorean triples of the first type.

Introduction:

The life of Paul Erdős (March 26, 1913 in Budapest, Hungary; September 20, 1996 in Warsaw, Poland) was entirely devoted to his research. Living in destitution. He had no wife, no job, not even a house he lived with an old suitcase and a bag of orange plastic supermarket. The only possession that mattered to him was his little book [3]. He was a prolific researcher in any discipline, with more than 1,500 research articles published. In particular, many of these articles was to study his favorite fields (graph theory, number theory, combinatorics) from different angles, and to continually improve the elegance of the demonstrations.

One of the favorite maxims of Erdős was: "Sometimes you have to complicate a problem to simplify the solution."

Another famous quote often attributed incorrectly to Erdős, but in reality from the Alfred Renyi [1]: "A mathematician is a machine that turns coffee into theorems."

The Erdős-Straus conjecture implies that any rational number $4 / n$, with n an integer greater than or equal to two, can be written as a sum of three unit fractions, that is to say that there are three natural numbers not zero x, y and z such that:

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

We propose a partial demonstration of this conjecture using the twin Pythagorean triples of the first type.

Consider equality $a^2+b^2=c^2$

Indeed, a triple is said twin triple if two numbers are the consecutive component. A twin triple is necessarily irreducible. Two consecutive numbers cannot have a common divisor ... There are two categories of irreducible twin triples be called twin triples of the first type for which b and c are consecutive and twin triples of the second type those for which a and b are consecutive. Recall that a and c cannot be consequential since they are both odd ... [4,5]

Statistically, there are many more than the first type of second ...

Twin triples of the first type [4,5]

They are simple to manufacture. Indeed, if the triple (a, b, c), with $a = u^2 - v^2$, $b = 2uv$ and $c = u^2 + v^2$ (u and v of different parities and coprime), b and c are consecutive if and only if c - b = 1; $(u - v)^2 = 1$. This is true if and only if u and v consecutive: we then have u = v + 1 (since u> v).

We deduce: $a = u^2 - v^2 = 2v + 1$, $b = 2uv = 2v (v + 1)$;$c = u^2 + v^2 = 2v (v+1) + 1$.

This method thus generates all twin triples of the first type.

The first are: v = 1: (3; 4;5)

when v = 2: (5; 12; 13)

when v = 3: (7; 24; 25)

for v = 4: (9; 40; 41)

for v = 5: (11; 60; 61) ...
They are infinite

Twin triples of the second type
They are more difficult to determine ...
a and b are consecutive if and only if | a - b | = 1 or | (u - v)² - 2v² | = 1: it is recognized then the equation Pell-Fermat x² - 2y² = ± 1, by asking x = (u - v) and y = v.
There is indeed the triplets of the second type are much less common than the first type, such as before a = 1 ,000,000, there are 8 of the second type, against 499,999 the first type.
Theorem 1:
It is said that three numbers a, b and c integers form a Pythagorean triple if they satisfy the relation: a² + b² = c². [4,5]
Theorem 2:
(a, b, c) is a Pythagorean triple if and only if for any nonzero integer n, (na; nb; nc) is also a Pythagorean triple. [4,5]
Theorem 3:
If two of the three numbers of a Pythagorean triple have a common divisor d, then d also divides the third number.
So any Pythagorean triple can be reduced to an irreducible Pythagorean triple, where a, b and c are coprime two by two. [4,5]
Theorem 4:
Let a, b, c be integers. (a, b, c) is a Pythagorean triple irreducible if and only if there are two numbers u and v (u> v), of different parities and coprime, such as
a = u² - v²; b = 2uv c = u² + v². [4,5]
Conjecture 1: Conjecture of Erdös-Straus
Every rational number 4 / n, with n an integer greater than or equal to two, can be written as a sum of three unit fractions (also called Egyptian fractions) , that is to say that there are three nonzero integers x, y and z such that:

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

For a recent survey about this conjecture, the Professor Michel MINOZY gave a conference [6].
Theorem 5:
For any decomposition of the fraction 4/p sum of three Egyptian fractions, corresponds an irreducible Pythagorean triple .
Theorem 6:
For any number p different of 1 or 17 modulo 24, these two first twin Pythagorean triples (3, 4, 5) and (5, 12, 13) provide a decomposition of Straus Erdös 4/p.
Proof:
Consider a decomposition of the fraction 4/p a sum of three Egyptian fractions 4/p = 1/x + 1/y + 1/z, where p, x, y and z are positive integers, with p> 1; we can always assume that z is strictly greater than x. Let z = x + a. This gives us the equation:

$$eq := \frac{4}{p} = \frac{1}{x} + \frac{1}{y} + \frac{1}{x+a}$$

Solve this equation in x, the two roots are:

$$\frac{4ya - 2yp - ap + \sqrt{16a^2y^2 - 8a^2yp + 4p^2y^2 + a^2p^2}}{2(-4y+p)},$$

$$-\frac{-4ya + 2yp + ap + \sqrt{16a^2y^2 - 8a^2yp + 4p^2y^2 + a^2p^2}}{2(-4y+p)}$$

The discriminant must be a perfect square for x can be an integer. But this discriminant.

$$\Delta := 16\,a^2 y^2 - 8\,a^2 y\,p + 4\,p^2 y^2 + a^2 p^2$$

is the sum of the squares of two numbers (4ay – ap) and 2py because:

$$(4\,a\,y - a\,p)^2 + 4\,p^2 y^2 = \Delta$$

Thus the triplet ($4\,a\,y - a\,p$, $2\,p\,y$, $\sqrt{\Delta}$) is a Pythagorean triple
Let us write down from an irreducible triple:

$$4\,a\,y - a\,p = \alpha\,k$$

$$2\,p\,y = \beta\,k$$

$$\sqrt{\Delta} = \gamma\,k$$

By eliminating k using the first two equations we get there:

$$y = \frac{a\,\beta\,p}{4\,a\,\beta - 2\,\alpha\,p}$$

Then express the two possible values of x and take the one that is positive:

$$x = \frac{(\beta - \alpha + \sqrt{\beta^2 + \alpha^2})\,a}{2\,\alpha}$$

Finally z = x + a is:

$$z = \frac{a\,(\beta + \alpha + \sqrt{\beta^2 + \alpha^2})}{2\,\alpha}$$

The initial decomposition of 4/p is then written using an irreducible Pythagorean triple and the parameter initially introduced:

$$\frac{4}{p} = \frac{2\,\alpha}{(\beta - \alpha + \gamma)\,a} + \frac{4\,a\,\beta - 2\,\alpha\,p}{a\,\beta\,p} + \frac{2\,\alpha}{(\beta + \alpha + \gamma)\,a}$$

The parameter k of the Pythagorean triple ($\alpha\,k$, $\beta\,k$, $\gamma\,k$) is expressed as:

$$k = \frac{p^2\,a}{2\,a\,\beta - \alpha\,p}$$

In other words, the fraction 4/p is also written as:

$$\frac{4}{p} = \frac{2\,(-p^2 + 2\,\beta\,k)}{(\beta - \alpha + \gamma)\,k\,p} + \frac{2\,p}{\beta\,k} + \frac{2\,(-p^2 + 2\,\beta\,k)}{(\beta + \alpha + \gamma)\,k\,p}$$

We have thus proved the:
Theorem 5: For any decomposition of the fraction 4/p sum of three Egyptian fractions, corresponds a Pythagorean triple.
Conversely there is the problem whether for every integer p> 1 we can find a Pythagorean triple which gives rise to the decomposition of 4/p as a sum of three Egyptian fractions, if it is the case the conjecture of Erdös-Straus will be demonstrated.
Consider the simplest triplet (3, 4, 5) and establish a list of families of numbers that admit a decomposition from the irreducible triple. We have two options available to us, related to the choice α=3 or α=4 which we denote by a triplet of fractions.

For (α=3, β=4, γ=5) we have : $\dfrac{4}{p} = \left[\dfrac{1}{a},\ \dfrac{8\,a - 3\,p}{2\,a\,p},\ \dfrac{1}{2\,a}\right]$

For (α=4, β=3, γ=5) we have : $\dfrac{4}{p} = \left[\dfrac{2}{a}, \dfrac{4\,(3\,a - 2p)}{3\,a\,p}, \dfrac{2}{3\,a} \right]$

For p = 2 k, take the first identity α=3, a = k we have the classical decomposition $\left[\dfrac{1}{k}, \dfrac{1}{2\,k}, \dfrac{1}{2\,k} \right]$

For p = 3 k, consider the second identity α=4, we have the decomposition $\dfrac{4}{3\,k} = \left[\dfrac{1}{4\,k}, \dfrac{1}{k}, \dfrac{1}{12\,k} \right]$

For k = 5, take the first identity α=3, with k = 5 we have the decomposition
$\dfrac{4}{5\,k} = \left[\dfrac{1}{5\,k}, \dfrac{1}{2\,k}, \dfrac{1}{10\,k} \right]$

Similarly for p = 7 k and 13 k, the first identity α=3, we have the decompositions :
$\dfrac{4}{7\,k} = \left[\dfrac{1}{21\,k}, \dfrac{1}{2\,k}, \dfrac{1}{42\,k} \right]$ and $\dfrac{4}{13\,k} = \left[\dfrac{1}{26\,k}, \dfrac{1}{4\,k}, \dfrac{1}{52\,k} \right]$ with respectively a = 21k and a = 26k.

Similarly for p = 11 k and 23 k, the first identity α=4 we have the decompositions:
$\dfrac{4}{11\,k} = \left[\dfrac{1}{4\,k}, \dfrac{1}{33\,k}, \dfrac{1}{12\,k} \right]$ and $\dfrac{4}{23\,k} = \left[\dfrac{1}{8\,k}, \dfrac{1}{138\,k}, \dfrac{1}{24\,k} \right]$ with respectively a = 8k and a = 16k.

By cons for multiples of 17 and 19 we use the second triplet cousin (5, 12, 13) for simple decompositions:
With the triplet (12, 5, 13) we have the following decompositions
$\dfrac{4}{17\,k} = \left[\dfrac{1}{34\,k}, \dfrac{1}{5\,k}, \dfrac{1}{170\,k} \right]$ and $\dfrac{4}{19\,k} = \left[\dfrac{1}{6\,k}, \dfrac{1}{95\,k}, \dfrac{1}{30\,k} \right]$ with respectively a = 136k and a = 24k.

Using these two first cousins Pythagorean triples (3, 4, 5) and (5, 12, 13) then we can simply show the result.

<u>Theorem 6:</u> For any number p different of 1 or 17 modulo 24, these two first cousins Pythagorean triples (3, 4, 5) and (5, 12, 13) provide a decomposition of Straus Erdös 4/p.

For proof, using the fact that every prime greater than 3 equals 5, 7, 11, 13, 17, 19 or 23 modulo 24, there are only four decompositions to establish using the first two triplets cousins.

With α=3 and a=2 +3k we have

$$\dfrac{4}{5 + 8\,k} = \left[\dfrac{1}{2 + 3\,k}, \dfrac{1}{2\,(2 + 3\,k)\,(5 + 8\,k)}, \dfrac{1}{2\,(2 + 3\,k)} \right]$$

With α=3 and a=3 +3k we have $\dfrac{4}{7 + 8\,k} = \left[\dfrac{1}{3\,(1 + k)}, \dfrac{1}{2\,(1 + k)\,(7 + 8\,k)}, \dfrac{1}{6\,(1 + k)} \right]$

With α=4 and a= 8 + 8k we have

$$\dfrac{4}{11 + 12\,k} = \left[\dfrac{1}{4\,(1 + k)}, \dfrac{1}{3\,(1 + k)\,(11 + 12\,k)}, \dfrac{1}{12\,(1 + k)} \right]$$

With α=5 and a= 4 + 5k we have

$$\dfrac{4}{19 + 24\,k} = \left[\dfrac{1}{2\,(4 + 5\,k)}, \dfrac{1}{6\,(4 + 5\,k)\,(19 + 24\,k)}, \dfrac{1}{3\,(4 + 5\,k)} \right]$$

We thus find classical results in a simple manner by this method according to the doctrine of Paul Erdös.

Note: If p admits a decomposition 4/p=1/x + 1/y + 1/(x+a) associated with a twin Pythagorician triple of first type (2n+1, 2n(n+1), 2n(n+1)+1) then any multiple kp of p admits a decomposition

associated with the same twin Pythagorician triple, by writing x1=kx, y1=ky,a1=ka. Thus it suffices to prove the conjecture for primes equal to 1 or 17 mod 24.

Note that the study of progressions 2+3k, 1+24k and 17+24k requires the use of all Pythagorean triples cousins type 1, those of the form (2n+1, 2n(n+1), 2n(n+1)+1), ie the two identities

$$\frac{4}{p} = \left[\frac{1}{a\,n}, \frac{4\,a\,n^2 + 4\,a\,n - 2\,p\,n - p}{a\,n\,(n+1)\,p}, \frac{1}{a\,(n+1)} \right]$$ and

$$\frac{4}{p} = \left[\frac{2\,n}{a}, \frac{4\,(2\,a\,n + a - p\,n^2 - p\,n)}{a\,(2\,n+1)\,p}, \frac{2\,n}{a\,(2\,n+1)} \right].$$ Indeed we have the decomposition for

$p=17 + 24k$ with $n=2+3k$ and $a=4(1+k)(2+3k)(17+24k)$:

$$\frac{4}{17+24\,k} = \left[\frac{1}{2\,(17+24\,k)\,(1+k)}, \frac{1}{5+6\,k}, \frac{1}{2\,(1+k)\,(5+6\,k)\,(17+24\,k)} \right]$$

Verification of this new conjecture (for each p>2 there exist a twin Pythagorian triple of the first type which gives an Erdös-Straus decomposition of 4/p) have been made in few minutes using a simple computer for all numbers less than 100 millions.

Hence the very probably truth of the conjecture of Erdös-Straus."

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