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A Mixed Quadrature Rule by Blending Clenshaw-Curtis and Gauss-Legendre Quadrature Rules for Approximate Evaluation of Real Definite Integrals

[1]Anasuya Pati, [2]Rajani B. Dash, [3]Pritikanta Patra

[1&3] Department of Mathematics, Ravenshaw University, Cuttack-753003, Odisha, India
[2] Department of Mathematics, S.C.S.(A)College, Puri752001, Odisha, India

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ABSTRACT. A mixed quadrature rule blending Clenshaw-Curtis five point rule and Gauss-Legendre three point rule is formed. The mixed rule has been tested and found to be more effective than that of its constituent Clenshaw-Curtis five point rule.

I. INTRODUCTION

Real definite integrals of the type

$$I(f)=\int_a^b f(t)dt \tag{1.1}$$

have been successfully approximated by several authors by applying the mixed quadrature rule. The method involves construction of a symmetric quadrature rule of higher precision as a linear combination of two other rules of equal lower precision.

If we consider Gauss-Legendre rule and a Clenshaw-Curtis rule having same precision, Clenshaw-Curtis rule is better than Gauss-Legendre rule. An n-point Gauss rule is of precision 2n-1, while the precision of an n-point Clenshaw Curtis rule is n. In general Gauss type rule is of higher precision than that of Clenshaw-Curtis type rule when same abscissae are used.

In this paper, taking the advantage of the fact that Gauss-Legendre 3-point rule and Clenshaw-Curtis 5-point rule are of same precision (i.e. precision 5), we formed a mixed quadrature rule of higher precision (i.e. precision 7) taking linear combination of these rules. The mixed rule so formed has been tested on different integrals giving better results than Clenshaw-Curtis quadrature rule.

II. THE CLENSHAW-CURTIS QUADRATURE RULE

The Clenshaw-Curtis method [4] essentially approximates a function f(t) over any interval [α – h, α + h] using the Chebyshev polynomials T_r(x) of degree n.

$$f(t)=F(x)=\sum_{r=0}^{n}{}' a_r T_r(x) \qquad (-1 \le x \le 1) \tag{2.1}$$

where a_r are the expansion co-efficient and Σ' denotes a finite sum whose first term is to be halved before beginning to sum. That is,

$$F(x)=\frac{1}{2}a_0 T_0(x)+a_1 T_1(x)+\ldots+ a_n T_n(x) \tag{2.2}$$

Collocating with f (α + hx) at the n + 1 points.

$$x_i=\mathrm{Cos}\left(\frac{i\pi}{n}\right) \qquad (i=0,1,\ldots,n) \tag{2.3}$$

One can evaluate the expansion co-efficient a_r.

The Chebyshew Polynomials $T_r(x_i)$ can be expressed as

$$T_r(x_i) = \mathrm{Cos}(r\,\mathrm{Cos}^{-1}(x_i)) \qquad r \geq 0$$

$$= \mathrm{Cos}\left(\frac{ri\pi}{n}\right) \tag{2.4}$$

$$\sum_{i=0}^{n}{}'' f(\alpha + hx_i) T_r(x_i) = \sum_{i=0}^{n}{}'' \sum_{k=0}^{n}{}' a_k T_k(x_i) T_r(x_i)$$

Then

$$= \sum_{k=0}^{n}{}' \sum_{i=0}^{n}{}'' \mathrm{Cos}\left(\frac{ki\pi}{n}\right) \mathrm{Cos}\left(\frac{ri\pi}{n}\right) \tag{2.5}$$

The notation Σ" means that the first and the last terms are to be halved before summation begins.

The orthogonality of the cosine function [5] with respect to the points $x_i = \mathrm{Cos}\left(\frac{i\pi}{n}\right)$ is expressed by

$$\sum_{i=0}^{n}{}'' \mathrm{Cos}\left(\frac{ki\pi}{n}\right) \mathrm{Cos}\left(\frac{ri\pi}{n}\right) = \begin{cases} n, r = k = 0 \, or \, n \\ \frac{n}{2}, 0 < r = k < n \\ 0, r \neq k \end{cases} \tag{2.6}$$

Substituting equation (2.6) in equation (2.5) we get

$$\sum_{i=0}^{n}{}'' f(\alpha + hx_i) T_r(x_i) = \begin{cases} \frac{n}{2}.a_r, 0 \leq r = k < n \\ n.a_r, r = k = n \\ 0, r \neq k \end{cases}$$

Hence

$$a_r = \begin{cases} \frac{2}{n} \sum_{i=0}^{n}{}'' f(\alpha + hx_i) T_r(x_i), (r = 0, 1, \ldots, n-1) \\ \frac{1}{n} \sum_{i=0}^{n}{}'' f(\alpha + hx_i) T_r(x_i), (r = n) \end{cases} \tag{2.7}$$

Denoting the integral of $f(t)$ over the interval $[\alpha - h, \alpha + h]$ by I and replacing t by $(\alpha + hx)$, we get

$$I = h \int_{-1}^{1} f(\alpha + hx) dx$$

Assuming $I \cong I_n$, we write

$$I_n = h \int_{-1}^{1} \sum_{r=0}^{n}{}' a_r T_r(x) dx$$

$$= h \sum_{r=0}^{n}{}' a_r \int_{-1}^{1} T_r(x) dx$$

Substituting the values of a_r (as given in eqn. 2.7), we get

$$I_n = h \sum_{r=0}^{n}{}'' \frac{2}{n} \sum_{i=0}^{n}{}'' f(\alpha + hx_i) T_r(x_i) \int_{-1}^{1} T_r(x) dx$$

Since $\int_{-1}^{1} T_r(x)dx = \frac{-2}{r^2-1}\ (r=\text{even})$ (2.8)

we get $I_n = h\sum_{i=0}^{n}{}'' W_i f(\alpha + hx_i)$ (2.9a)

where $W_i = -\frac{4}{n}\sum_{r=0}^{n}{}'' \frac{1}{r^2-1}T_r(x_i)\quad (i = 0,\ 1, \dots,\ n)$ (2.9b)

with n = 4

$$I_4 = \frac{h}{15}[f(\alpha+h) + 8f(\alpha+\frac{h}{\sqrt{2}}) + 12f(\alpha) + 8f(\alpha-\frac{1}{\sqrt{2}}) + f(\alpha-h)] \tag{2.10}$$

III. CONSTRUCTION OF THE MIXED QUADRATURE RULE OF PRECISION SEVEN

We choose the Clenshaw-Curtis five point rule

$$I(f)=\int_{-1}^{1} f(x)dx \cong R_{cc5}(f)=\frac{1}{6}[f(-1)+8f(\frac{-1}{\sqrt{2}})+12f(0)+8f(\frac{1}{\sqrt{2}})+f(1)] \tag{3.1}$$

and the Gauss-Legendre three point rule

$$I(f)=\int_{-1}^{1} f(x)dx \cong R_{GL3}(f)= \frac{1}{9}[5f(-\sqrt{\frac{3}{5}})+8f(0)+5f(\sqrt{\frac{3}{5}})] \tag{3.2}$$

Each of the rules (3.1) and (3.2) is of precision five. Let $E_{cc5}(f)$ and $E_{GL3}(f)$ denote the errors in approximating the integral I(f) by the rules (3.1) and (3.2) respectively. Using Maclaurin's expansion of functions in Eqn. (3.1) and (3.2) we get

$$I(f)=R_{cc5}(f)+E_{cc5}(f) \tag{3.3}$$

and

$$I(f)= R_{GL3}(f)+ E_{GL3}(f) \tag{3.4}$$

where

$$E_{cc5}(f)=\frac{1}{315\times 5!}f^{vi}(0)+\frac{1}{360\times 7!}f^{viii}(0)+\dots$$

$$E_{GL3}(f)=\frac{1}{175\times 90}f^{vi}(0)+\frac{11}{1125\times 7!}f^{viii}(0)+\dots$$

Now multiplying the equations (3.3) and (3.4) by $\frac{1}{5}$ and $\frac{-1}{12}$ respectively, and then adding the resulting equations we obtain

$$I(f)=\frac{1}{7}[12R_{cc5}(f)-5R_{GL3}(f)]+\frac{1}{7}[12E_{cc5}(f)-5E_{GL3}(f)]$$

or

$$I(f)=R_{cc5GL3}(f)+E_{cc5GL3}(f) \tag{3.5}$$

where

$$R_{cc5GL3}(f)=\frac{1}{7}[12R_{cc5}(f)-5R_{GL3}(f)] \tag{3.6}$$

This is the desired mixed quadrature rule of precision seven for approximate evaluation of I(f). The truncation error generated in this approximation is given by

$$E_{cc5GL3}(f)=\frac{1}{7}[12E_{cc5}(f)-5E_{GL3}(f)]$$
$$=\frac{-1}{450\times 7!}f^{viii}(0)+\dots \tag{3.7}$$

The rule (3.6) may be called as a mixed type rule as it is constructed from two different types of rules of the same precision (i.e. precession 5).

IV. ERROR ANALYSIS

An asymptotic error estimate and an error bound of the rule (3.6) are given in theorems 4.1(a) and 4.1(b) respectively.

Theorem – 4.1(a):

Let f(x) be a sufficiently differentiable function in the closed interval [– 1, 1]. Then the error $E_{cc5GL3}(f)$ associated with the rule $R_{cc5GL3}(f)$ is given by

$$|E_{cc5GL3}(f)| \cong \frac{1}{450\times 7!}\,|f^{viii}(0)|$$

Proof: From Eqn. (3.5)

$$I(f)=R_{cc5GL3}(f)+E_{cc5GL3}(f)$$

where $$R_{cc5GL3}(f)=\frac{1}{7}[12\,R_{cc5}(f)-5\,R_{GL3}(f)]$$

$$E_{cc5GL3}(f)=\frac{1}{7}[12\,E_{cc5}(f)-5E_{GL3}(f)]$$

Hence $$E_{cc5GL3}(f)=\frac{-1}{450\times 7!}\,f^{viii}(0)+\ldots$$

So $$|E_{cc5GL3}(f)| \cong \frac{1}{450\times 7!}\,|f^{viii}(0)|$$

Theorem – 4.2

The bound for the truncation error $E_{cc5GL3}(f)=I(f)-R_{cc5GL3}(f)$ is given by

$$|E_{cc5GL3}(f)| \le \frac{M}{22050}|\eta_2-\eta_1| \quad , \eta_1,\eta_2 \in [-1,1]$$

where $$M=\max_{-1\le x\le 1}|f^{vii}(x)|$$

Proof:

We have

$$E_{cc5}(f) \cong \frac{1}{315\times 5!}\,f^{vi}(\eta_2) \qquad , \eta_2\in[-1,1]$$

$$E_{GL3}(f) \cong \frac{1}{175\times 90}\,f^{vi}(\eta_1) \qquad , \eta_1\in[-1,1]$$

Hence $$E_{cc5GL3}(f)=\frac{1}{7}[12E_{cc5}(f)-5E_{GL3}(f)]$$

$$=\frac{1}{22050}[f^{vi}(\eta_2)-f^{vi}(\eta_1)]$$

$$=\frac{1}{22050}\int_{\eta_1}^{\eta_2} f^{vii}(x)dx \qquad (\text{assuming } \eta_1<\eta_2)$$

From this we obtain

$$|E_{cc5GI3}(f)|=\left|\frac{1}{22050}\int_{\eta_1}^{\eta_2} f^{vii}(x)dx\right| \le \frac{1}{22050}\int_{\eta_1}^{\eta_2}|f^{vii}(x)|dx$$

So $$|E_{cc5GL3}(f)| \le \frac{M}{22050}|\eta_2-\eta_1|$$

which gives only a theoretical error bound as η_1, η_2 are unknown points in [– 1, 1]. It shows that the error in the approximation will be less if the points η_1, η_2 are closed to each other.

Corollary

The error bound for the truncation error E_{cc5GL3}(f) is given by

$$\left|E_{cc5GL3}(f)\right| \leq \frac{M}{11025}$$

Proof: We know from the theorem 4.1(b) that

$$\left|E_{cc5GL3}(f)\right| \leq \frac{M}{22050}\left|\eta_2 - \eta_1\right|$$

where $\quad M = \max_{-1 \leq x \leq 1} \left|f^{vii}(x)\right|$

Choosing $\left|\eta_1 - \eta_2\right| \leq 2$ we have

$$\left|E_{cc5GL3}(f)\right| \leq \frac{M}{11025}$$

The effectiveness of the method is verified applying the method on different integrals as given in the following table.

V. NUMERICAL VERIFICATION

Comparison of the mixed Quadrature rule with Clenshaw-Curtis 5-point rule in approximation of some real definite integrals

Integrals	Exact Value	Approximate Value			Error Approximated		
		Clenshaw-Curtis 5-point Quadrature rule $R_{cc_5}(f)$	Gauss-Legendre 3-point evaluation Rule $R_{cc_5}(f)$	Mixed Quadrature rule $[R_{CC_5GL_3}(f)]$	Clenshaw-Curtis 5-point Quadrature rule $E_{GL_5}(f)$	Gauss-Legendre 3-point Quadrature Rule $ELob_4(f)$	Mixed Quadrature rule $E_{GL_5}(f)$
$\int_0^1 \frac{1}{1+25x^2}dx$	0.2746801534	0.2772591236	0.2855636402	0.271327326	0.0025789702	0.0108834968	0.0033528274
$\int_0^1 Cos^3 x dx$	0.6666666667	0.666110409	0.6654018648	0.666616512	0.00055625766677	0.001264801867	0.0000501547
$\int_0^{\pi/4} \frac{1}{1+Sinx}dx$	0.5857864376	0.5857845882	0.5857818675	0.5857865316	0.0000018494	0.0000045701	0.000000094
$\int_0^1 \frac{1}{1+x}dx$	0.6931471806	0.6931372549	0.6931216932	0.6931483704	0.0000099257	0.0000254874	0.0000011898
$\int_0^1 \frac{1}{1-0.5x^4}dx$	1.143667254	1.136084102	1.13911098	1.133922046	0.007583152	0.004556274	0.009745208
$\int_0^1 \frac{1}{1+100x^2}dx$	0.1471127674	0.1374584589	0.1429380932	0.1335444344	0.0096533085	0.0041736742	0.013568333
$\int_0^2 \frac{\ln x}{x}dx$	0.240226507	0.2402471446	0.2402788877	0.2402236852	0.0000206376	0.0000534807	0.0000028218
$\int_1^2 \frac{1}{e^x-1}dx$	0.3132616875	0.3132517625	0.3132362016	0.3132628774	0.000009925	0.0000254859	0.0000011899

VI. CONCLUSION

Above examples give a clear picture about the effectiveness of imposing mixed quadrature rule. The quadrature rule R*cc5GL3*(f) reduces the number of steps required to approximate an integral in its constituent Clenshaw-Curtis quadrature rule.

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