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JORDAN *u*-GENERALIZED REVERSE DERIVATIONS ON SEMIPRIME RINGS

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ABSTRACT: In this paper, we prove that if G is a Jordan u-generalized reverse derivation of a semi prime ring R of char.$\neq 2$, then G is a u-generalized reverse derivation. Similarly, we show that if G is a Jordan $u-*$ generalized reverse derivation of a semi prime ring R of char.$\neq 2$, then G is a $u-*$generalized reverse derivation. We also prove that the commutativity of R if $G([x,y])=0$.

INTRODUCTION: M.Bresar [2] proved that for a semiprime ring R, if G is a function from R *to* R and $D{:}R\to R$ is an additive mapping such that $G(xy)=G(x)y+xD(y)$, for all $x,y\in R$,then D is uniquely determined by G and moreover G must be a derivation. In [1] Ashraf and Rehman proved that if R is a ring of char.$\neq 2$ such that R has a commutator which is not a zero divisor, then every Jordan generalized derivation on R *i*s a generalized derivation.We know that an additive mapping $G{:}R\to R$ is a Jordan generalized reverse derivation if there exists a derivation D from R to R such that $G(x^2)=G(x)x+xD(x)$, for all x in R. An additive mapping $D$from R to itself is a u-reverse derivation if $D(xy)=D(y)u(x)+yD(x)$ hold where u is a homomorphism of R, for all x,yin R. An additive mapping $G{:}R\to R$ is a u-generalized reverse derivation if there exists a derivation D from R to R such that $G(xy)=G(y)u(x)+yD(x)$,for all $x,y\in R$ and an additive mapping $G{:}R\to R$ is a Jordan u-generalized reverse derivation if there exists a derivation $D{:}R\to R$ such that $G(x^2)=G(x)u(x)+xD(x)$,for all $x\in R$. An additive mapping D from R to itself is a $u-*$ reverse derivation if $D(xy)=u(x)D(y)+D(x)y$, where u is an anti - homomorphism in R, for all $x,y\in R$. An additive mapping $G{:}R\to R$ is a $u-*$ generalized reverse derivation if there exists a derivation D from R to R such that $G(xy)=u(x)G(y)+D(x)y$, where u is an anti-homomorphism in R, for all $x,y\in R$and an additive mapping $G{:}R\to R$ is a Jordan $u-*$generalized reverse derivation if there exists a derivation from R to R such that $G(x^2)=u(x)G(x)+D(x)x$, where u is an anti-homomorphism in R, for all $x\in R$. Clearly, every u-generalized derivation on a ring is a Jordan u-generalized derivation. But the converse statement does not hold in general. Throughout this paper R will be a semiprime ring and Z its center.

First we prove the following Lemmas:

Lemma1: Let R be a semiprime ring. If $a,b\in R$ are such that $axb=0$,for all $x\in R$, then $ab=ba=0$.

Proof : We take any $x\in R$,

$(ab)x(ab)=a(bxa)b=0$,

$(ba)x(ba)=b(axb)a=0$

By the semiprimeness of R, it follows that $ab=ba=0$. ■

Lemma 2: Let R be a semi prime ring and $\theta,\phi{:}R\times R\to R$ bi additive mappings. If $\theta(x,y)w\,\phi(x,y)=0$, for all $x,y,w\in R$ then $\theta(x,y)w\,\phi(s,t)=0$, forall $x,y,s,t,w\in R$.

Proof: We replace x by $x+s$, then

$\theta(x+s,y)w\phi(x+s,y)=0$, $\quad\theta(x,y)w\phi(s,y)$

$=-\theta(s,y)w\phi(x,y)$

By using the biadditivity of θ and ϕ , we get,
$\big(\theta(x,y)w\phi(s,y)\big)z\big(\theta(x,y)w\phi(s,y)\big) = -\theta(s,y)w\phi(s,y)z\theta(x,y)w\phi(x,y) = 0$
Hence $\theta(x,y)w\phi(s,y) = 0$ by semiprimeness of R. Now we replace y by $y+t$ and obtain the assertion of lemma with a similar approach as above. ■

Lemma 3: Let R be a semi prime ring and $a\in R$ some fixed element. If $a[x,y]= 0$, for all x, $y\in R$, then there exists an ideal U of R such that $a\in U\subset Z$ holds.

Proof: We have $[z,a]x[z,a]=[za-az]x[z,a]$

$$= zax[z,a]-azx[z,a]$$

$$= za[z,xa]- za[z,x]a-a[z,zxa]+ a[z,zx]a= 0$$

Hence $a\in Z$. Since $zaw[x,y]= 0$, for all $z,w,x,y\in R$,we can repeat the above argument with zaw instead of a to obtain $RaR\subset Z$ and now it is obvious that the ideal generated by a is central. ■

Now we prove the following results.

Theorem 1: If G is a Jordan u-generalized reverse derivation of a semiprime ring R of char.$\neq 2$, then G is a u-generalized reverse derivation.

Proof: Since $G(x^2) = G(x)u(x) + xD(x)$, for all $x\epsilon R$ (1)

We replace x by $x+ y$ in equ.(1), then
$G(xy + yx) = G(x)u(y) + G(y)u(x) + xD(y) + yD(x)$, for all $x, y\epsilon R$ (2)

Again replacing y by $xy+ yx$ in equ.(2), and using equ.(2), we obtain,
$G(x^2y + yx^2) + 2G(xyx) = G(x)u(xy) + G(x)u(yx) + G(y)u(x^2) + yD(x)u(x) +$
$G(x)u(yx) + xD(y)u(x) + xD(xy + yx) + (xy + yx)D(x)$, for all $x, y\epsilon R$ (3)

On the other hand, we replace x by x^2 in equ.(2) and adding $2G(xyx)$ on both sides, we get,

$G(x^2y + yx^2) + 2G(xyx) = G(x)u(xy) + xD(x)u(y) + G(y)u(x^2) + x^2D(y) + yD(x)u(x) +$
$yxD(x) + 2G(xyx)$, for all $x, y\epsilon R$ (4)

By comparing equ.'s (3) and (4), we obtain,

$G(x)u(yx) + G(yxx) + 2xD(xy) = xD(yx) + 2G(xyx)$, for all $x, y\epsilon R$ (5)

We put $x= x+ z$ in equ. (5), then
$G(xyz + zyx) = G(x)u(zy) + G(z)u(xy) + xD(zy) + zD(xy)$, for all $x, y, z\ \epsilon R$ (6)

Let $f = G(yxzxy + xyzyx)$, we shall compute it in two different ways. By using equ.(5), we have,
$f = G(y)u(yxzx) + yD(yxzx) + G(x)u(xyzy) + xD(xyzy)$, for all $x, y, z\epsilon R$ (7)

By using equ.(6),we have,
$f = G(yx)u(xyz) + G(xy)u(yxz) + yxD(xyz) + xyD(yxz)$, for all $x, y, z\epsilon R$ (8)

By comparing equ.'s (7) and (8), we get,
$\theta(y,x)u(xyz) + \theta(x,y)u(yxz) = 0$, for all $x, y, z\epsilon R$. (9)

Where $\theta(x,y)$ stands for $G(xy)-G(y)u(x)- yD(x)$. In the concept of the definition of θ , equ.(2) can be rewritten as,$\theta(x,y)= -\theta(y,x)$

By using this notation in equ.(9), we get,$\theta(x,y)u(z)[u(x),u(y)]=0$, for all $x,y,z\in R$ (10)

By using Lemma 2, we get, $\theta(x,y)u(z)[u(s),u(t)] =0$, for all $x,y,s,t\in R$ (11)

By using Lemma 1, we obtain, $\theta(x, y)[u(s),u(t)]= 0$, for all $x,y,s,t\in R$ (12)

Now, we fix $x,y\in R$ and write θ instead of $\theta(x, y)$ to simplify further writing.

By using Lemma 3, we get the existence of an ideal U such that $\theta \in U\subset Z$ holds. In particular, $b\theta, \theta b\epsilon Z$ for all $b\in R$. This gives us,
$x.\theta^2y=\theta^2y.x = y\theta^2.x = y.\theta^2x$

This gives us $4G(x.\theta^2y) = 4.G(y.\theta^2x)$

Now we will compute each side of this equality by using equ.(2), and the above notation
$4G(x.\theta^2y) = 2G(x\theta^2y + \theta^2yx)$
$= 2G(\theta^2y)u(x) + 2\theta^2y.D(x) + 2G(x)u(\theta^2y) + 2xD(\theta^2y)$

$= G(\theta^2 y + y\theta^2)u(x) + 2xD(\theta^2 y) + 2\theta^2 y.D(x) + 2G(x)u(\theta^2 y)$
$= 2G(x)u(\theta^2 y) + G(y)u(\theta^2)u(x) + yD(\theta^2)u(x) + G(\theta)u(y\theta)u(x) + \theta.D(y\theta)u(x) + 2xD(\theta^2 y) + 2\theta^2 yD(x)$

$= 2G(x)u(\theta^2 y) + G(\theta)u(y\theta x) + \theta D(\theta)u(y)u(x) + \theta\theta D(y)u(x) + 2xD(\theta^2 y) + 2\theta^2 yD(x) + G(y)u(\theta^2)u(x) + yD(\theta^2)u(x)$

$\Rightarrow 4G(x.\theta^2 y) = 2G(x)u(\theta^2 y) + G(\theta)u(y\theta x) + \theta.D(\theta)u(yx) + G(y)u(\theta^2 x) + \theta^2 D(y)u(x) + yD(\theta^2)u(x) + 2xD(\theta^2 y) + 2\theta^2 yD(x)$, for all $x, y\epsilon R$ (13)

Moreover

$4G(y.\theta^2 x) = 2G(y\theta^2 x + \theta^2 xy)$

$= 2G(\theta^2 x)u(y) + 2\theta^2 xD(y) + 2G(y)u(\theta^2 x) + 2yD(\theta^2 x)$

$= G(\theta^2 x + x\theta^2)u(y) + 2yD(\theta^2 x) + 2\theta^2 xD(y) + 2G(y)u(\theta^2 x)$

$= 2G(y)u(\theta^2 x) + G(x)u(\theta^2)u(y) + xD(\theta^2)u(y) + G(\theta)u(x\theta)u(y) + \theta D(x\theta)u(y) + 2yD(\theta^2 x) + 2\theta^2 xD(y)$

$= 2G(y)u(\theta^2 x) + G(x)u(\theta^2 y) + xD(\theta^2)u(y) + G(\theta)u(x\theta y) + \theta D(\theta)u(x)u(y) + \theta.\theta.D(x)u(y) + 2yD(\theta^2 x) + 2\theta^2 xD(y)$

$\Rightarrow 4G(y.\theta^2 x) =$
$2G(y)u(\theta^2 x) + G(\theta)u(x\theta y) + \theta D(\theta)u(xy) + G(x)u(\theta^2 y) + \theta^2 D(x)u(y) + xD(\theta^2)u(y) + 2yD(\theta^2 x) + 2\theta^2 xD(y)$, for all $x, y\epsilon R$. (14)

By comparing equations (13) & (14) and using the following notations.

$u(\theta yx) = u(\theta y).u(x) = u(x).u(\theta y) = u(x\theta y) = u(\theta xy)$,

$\theta D(\theta)u(yx) = D(\theta)\theta u(yx) = D(\theta)\theta u(xy) = \theta D(\theta)u(xy)$,

$x\theta D(\theta)u(y) = D(\theta)x\theta u(y) = D(\theta)\theta xu(y) = D(\theta)\theta u(y)x = \theta u(y)D(\theta)x = u(y)\theta D(\theta)x$

We obtain,
$G(x)u(\theta^2 y) + xD(\theta^2 y) = G(y)u(\theta^2 x) + yD(\theta^2 x)$, which gives,
$\phi(x, y)u(\theta^2) = \phi(y, x)u(\theta^2)$
(15)
Where $\phi(x, y)$ stands for $G(y)u(x) + yD(x)$
On the other hand, we also have,
$4G(xy\theta^2) = 4G(x\theta.y\theta)$
We will compute each side of this equality by using equation (2) and the properties of θ, so, we get

$4G(xy\theta^2) = 2G(xy\theta^2 + \theta^2 xy)$

$= 2G(\theta^2)u(xy) + 2\theta^2 D(xy) + 2G(xy)u(\theta^2) + 2xyD(\theta^2)$

Which gives,

$4G(xy\theta^2) = 2G(\theta^2)u(xy) + 2\theta^2 D(xy) + 2G(xy)u(\theta^2) + 2xyD(\theta^2)$, for all $x, y\epsilon R$. (16)

More over,

$4G(x\theta.y\theta) = 2G(x\theta y\theta + y\theta x\theta)$

$= 2G(y\theta).u(x\theta) + 2y\theta.D(x\theta) + 2G(x\theta)u(y\theta) + 2x\theta.D(y\theta)$

$= G(y\theta + \theta y)u(x\theta) + 2y\theta.D(x\theta) + G(x\theta + \theta x)u(y\theta) + 2x\theta.D(\theta y)$

$= G(\theta)u(y)u(x\theta) + \theta.D(y)u(x\theta) + G(y)u(\theta)u(x\theta) + yD(\theta)u(x\theta) + 2y\theta D(x\theta) +$
$G(\theta)u(x)u(y\theta) + \theta.D(x)u(y\theta) + G(x)u(\theta)u(y\theta) + x.D(\theta)u(y\theta) + 2x\theta.D(\theta y)$
$\Rightarrow 4G(x\theta.y\theta) = G(x)u(y\theta^2) + G(\theta)u(xy\theta) + xD(\theta)u(y\theta) + \theta D(x)u(y\theta) + 2x\theta.D(\theta y) +$
$G(y)u(x\theta^2) + G(\theta)u(yx\theta) + yD(\theta)u(x\theta) + 2y\theta D(x\theta) + \theta.D(y)u(x\theta)$, for all$x, y \epsilon R$ (17)
Comparing equation (16) & (17), we obtain,
$G(xy)u(\theta^2) = \phi(x,y)u(\theta^2)$, for all $x, y \epsilon R$. (18)
But $\theta(x,y) = G(xy) - G(y)u(x) - yD(x)$
$\phi(x,y) = G(y)u(x) + yD(x)$
$\theta(x,y) = G(xy) - \phi(x,y)$
And this means $\theta^3 = 0$ show that, $\theta^2 R\theta^2 = \theta^4 R = (0)$
$\theta R\theta = \theta^2 R = (0)$,
Which implies $\theta = 0$, and the proof is complete. ■

Theorem 2: If G is a Jordan u–*generalized reverse derivation of a semiprime ring R of char $\neq 2$, then G is an u–*generalized reverse derivation.
Proof: Since $G(x^2)=u(x)G(x)+ D(x)x$
We replace x by $x+ y$, then
$G(xy+ yx)=u(x)G(y)+u(y)G(x)+D(x)y+D(y)x$, for all $x,y\in R$ (19)
Consider $W=G(x(xy+ yx)+(xy+ yx)x)$
$=u(x)G(xy+ yx)+D(x)(xy+ yx)+u(xy+ yx)G(x)+D(xy+ yx)x$
$=u(x)G(xy)+u(x)G(yx)+ D(x)xy+ D(x)yx+u(xy)G(x)+ u(yx)G(x)+ D(xy)x+ D(yx)x$
$=u(x)u(x)G(y)+u(x)D(x)y+u(x)u(y)G(x)+ u(x)D(y)x+ D(x)xy+ D(x)yx+u(y)u(x)G(x)+$
$u(x)u(y)G(x)+u(x)D(y)x+ D(x)yx+u(y)D(x)x+ D(y)x^2$
$= u(x)u(x)G(y)+u(x)D(x)y+ 2u(x)u(y)G(x)+ 2u(x)D(y)x+ D(x)xy+ 2D(x)yx+u(y)u(x)G(x)+u(y)D(x)x+ D(y)x^2$ (20)

On the other hand,
$W=G(x^2y+ xyx+ xyx+ yx^2)$
$=G(x^2y+ yx^2+2xyx)$
$=u(x^2)G(y)+ D(x^2)y+u(yx)G(x)+ D(yx)x+2G(xyx)$
$=u(x)u(x)G(y)+u(x)D(x)y+ D(x)xy+u(x)u(y)G(x)+u(y)D(x)x+ D(y)x^2+2G(xyx)$ (21)
By comparing the equ.'s (20) and (21) for W, we obtain,
$2u(x)u(y)G(x)+2u(x)D(y)x+2D(x)yx=2G(xyx)$
$\Rightarrow 2[u(x)u(y)G(x)+u(x)D(y)x+D(x)yx]=2G(xyx)$
$\Rightarrow G(xyx)=u(x)u(y)G(x)+u(x)D(y)x+D(x)yx$
$\Rightarrow G(xyx)=u(yx)G(x)+ D(xy)x$
We put $z= xy$, then $G(zx)=u(z)G(x)+D(z)x$,which implies that G is a u–* generalized reverse derivation. ■
It is clear that if we assume that the reverse derivation D to be the zero reverse derivation in the above Theorems 1and 2, then we get the following result.
Corollary 1: Let R be a semi prime ring of char.$\neq 2$ and T:$R\rightarrow R$an additive mapping which satisfies $T(x^2)=T(x)u(x)$, for all $x\in R$. Then it is a left (right) centralizer.
Theorem 3: Let R be a semiprime ring of char.$\neq 2$ and G be a non-zero Jordan u-generalized reverse derivation of R. If $G([x,y])=0$,then R is commutative.

Proof : We substitute xy for y in $G([x,y]) = 0$

Then we obtain,

$0=G([x,xy])$

$\Rightarrow 0=G[x,y]u(x)+[x,y]D(x)$

$\Rightarrow 0=[x,y]D(x)$, for all $x,y\in R$

We replace y by ry, then

$0= [x,ry]D(x)$, for all $x,y\in R$

$\Rightarrow r[x,y]D(x)+[x,r]yD(x)= 0$

$\Rightarrow [x,r]yD(x)= 0$

$\Rightarrow [x,r]RD(x)$, for all $y\in R$

Now we choose a family $P=\{p_\alpha/\alpha\in\Lambda\}$ of prime rings R such that $\cap P_\alpha=\{0\}$ and let P denote a fixed one of the P_α. From the above equation, it follows that either $[x,r]\in p$ (or) $D(x)\in p$, for all $x\in R$. Let $A=\{x\in R/x\in Z\}$ and $B=\{x\in R/D(x)=0\}$. Then A and B are two additive sub groups of $(R,+)$ such that $R= A\cup B$. However, a group cannot be the union of proper sub groups. Hence either $R= A$ or $R= B$. If $R= A$, then $R\subset Z$ and so R is commutative. If $R= B$, then $R=0$ which contradicts the hypothesis. So, we must have $r\in Z$, for all $r\in R$. Hence R is commutative. ■

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