

JORDAN u -GENERALIZED REVERSE DERIVATIONS ON SEMIPRIME RINGS

Dr. C. Jaya Subba Reddy¹, G. Venkata Bhaskara Rao² S.Mallikarjuna Rao³

cjsreddysvu@gmail.com gvbr0001@gmail.com s.mallikarjunarao123@gmail.com

^{1,2,3}Department of Mathematics, S.V.University, Tirupati-517502, AndhraPradesh,India.

KEYWORDS: Semiprime ring, Generalized Derivation, Jordan Derivation, Generalized reverse derivation, Jordan generalized derivation, Jordan generalized reverse derivation.

ABSTRACT: In this paper, we prove that if G is a Jordan u -generalized reverse derivation of a semi prime ring R of char. $\neq 2$, then G is a u -generalized reverse derivation. Similarly, we show that if G is a Jordan u -* generalized reverse derivation of a semi prime ring R of char. $\neq 2$, then G is a u -*generalized reverse derivation. We also prove that the commutativity of R if $G([x,y])=0$.

INTRODUCTION: M.Bresar [2] proved that for a semiprime ring R , if G is a function from R to R and $D:R \rightarrow R$ is an additive mapping such that $G(xy)=G(x)y+xD(y)$, for all $x,y \in R$, then D is uniquely determined by G and moreover G must be a derivation. In [1] Ashraf and Rehman proved that if R is a ring of char. $\neq 2$ such that R has a commutator which is not a zero divisor, then every Jordan generalized derivation on R is a generalized derivation. We know that an additive mapping $G:R \rightarrow R$ is a Jordan generalized reverse derivation if there exists a derivation D from R to R such that $G(x^2)=G(x)x+xD(x)$, for all x in R . An additive mapping D from R to itself is a u -reverse derivation if $D(xy)=D(y)u(x)+yD(x)$ hold where u is a homomorphism of R , for all x,y in R . An additive mapping $G:R \rightarrow R$ is a u -generalized reverse derivation if there exists a derivation D from R to R such that $G(xy)=G(y)u(x)+yD(x)$, for all $x,y \in R$ and an additive mapping $G:R \rightarrow R$ is a Jordan u -generalized reverse derivation if there exists a derivation $D:R \rightarrow R$ such that $G(x^2)=G(x)u(x)+xD(x)$, for all $x \in R$. An additive mapping D from R to itself is a u -* reverse derivation if $D(xy)=u(x)D(y)+D(x)y$, where u is an anti-homomorphism in R , for all $x,y \in R$. An additive mapping $G:R \rightarrow R$ is a u -* generalized reverse derivation if there exists a derivation D from R to R such that $G(xy)=u(x)G(y)+D(x)y$, where u is an anti-homomorphism in R , for all $x,y \in R$ and an additive mapping $G:R \rightarrow R$ is a Jordan u -*generalized reverse derivation if there exists a derivation from R to R such that $G(x^2)=u(x)G(x)+D(x)x$, where u is an anti-homomorphism in R , for all $x \in R$. Clearly, every u -generalized derivation on a ring is a Jordan u -generalized derivation. But the converse statement does not hold in general. Throughout this paper R will be a semiprime ring and Z its center.

First we prove the following Lemmas:

Lemma1: Let R be a semiprime ring. If $a,b \in R$ are such that $axb=0$, for all $x \in R$, then $ab=ba=0$.

Proof : We take any $x \in R$,

$$(ab)x(ab)=a(bxa)b=0,$$

$$(ba)x(ba)=b(axb)a=0$$

By the semiprimeness of R , it follows that $ab=ba=0$. ■

Lemma 2: Let R be a semi prime ring and $\theta, \phi: R \times R \rightarrow R$ bi additive mappings. If $\theta(x,y)w\phi(x,y)=0$, for all $x,y,w \in R$ then $\theta(x,y)w\phi(s,t)=0$, for all $x,y,s,t,w \in R$.

Proof: We replace x by $x+s$, then

$$\theta(x+s,y)w\phi(x+s,y)=0,$$

$$\theta(x,y)w\phi(s,y)$$

$$= -\theta(s,y)w\phi(x,y)$$

By using the biadditivity of θ and ϕ , we get,

$$(\theta(x, y)w\phi(s, y))z(\theta(x, y)w\phi(s, y)) = -\theta(s, y)w\phi(s, y)z\theta(x, y)w\phi(x, y) = 0$$

Hence $\theta(x, y)w\phi(s, y) = 0$ by semiprimeness of R . Now we replace y by $y+t$ and obtain the assertion of lemma with a similar approach as above. ■

Lemma 3: Let R be a semi prime ring and $a \in R$ some fixed element. If $a[x, y] = 0$, for all $x, y \in R$, then there exists an ideal U of R such that $a \in U \subset Z$ holds.

$$\begin{aligned} \text{Proof: We have } [z, a]x[z, a] &= [za - az]x[z, a] \\ &= zax[z, a] - azx[z, a] \\ &= za[z, xa] - za[z, x]a - a[z, zxa] + a[z, zx]a = 0 \end{aligned}$$

Hence $a \in Z$. Since $zaw[x, y] = 0$, for all $z, w, x, y \in R$, we can repeat the above argument with zaw instead of a to obtain $RaR \subset Z$ and now it is obvious that the ideal generated by a is central. ■

Now we prove the following results.

Theorem 1: If G is a Jordan u -generalized reverse derivation of a semiprime ring R of char. $\neq 2$, then G is a u -generalized reverse derivation.

$$\text{Proof: Since } G(x^2) = G(x)u(x) + xD(x), \text{ for all } x \in R \quad (1)$$

We replace x by $x+y$ in equ.(1), then

$$G(xy + yx) = G(x)u(y) + G(y)u(x) + xD(y) + yD(x), \text{ for all } x, y \in R \quad (2)$$

Again replacing y by $xy + yx$ in equ.(2), and using equ.(2), we obtain,

$$G(x^2y + yx^2) + 2G(xyx) = G(x)u(xy) + G(x)u(yx) + G(y)u(x^2) + yD(x)u(x) + G(x)u(yx) + xD(y)u(x) + xD(xy + yx) + (xy + yx)D(x), \text{ for all } x, y \in R \quad (3)$$

On the other hand, we replace x by x^2 in equ.(2) and adding $2G(xyx)$ on both sides, we get,

$$G(x^2y + yx^2) + 2G(xyx) = G(x)u(xy) + xD(x)u(y) + G(y)u(x^2) + x^2D(y) + yD(x)u(x) + yxD(x) + 2G(xyx), \text{ for all } x, y \in R \quad (4)$$

By comparing equ.'s (3) and (4), we obtain,

$$G(x)u(yx) + G(yxx) + 2xD(xy) = xD(yx) + 2G(xyx), \text{ for all } x, y \in R \quad (5)$$

We put $x = x+z$ in equ. (5), then

$$G(xyz + zyx) = G(x)u(zx) + G(z)u(xy) + xD(zx) + zD(xy), \text{ for all } x, y, z \in R \quad (6)$$

Let $f = G(yxzxy + xyzyx)$, we shall compute it in two different ways. By using equ.(5), we have,

$$f = G(y)u(yxzx) + yD(yxzx) + G(x)u(xyzy) + xD(xyzy), \text{ for all } x, y, z \in R \quad (7)$$

By using equ.(6), we have,

$$f = G(yx)u(xyz) + G(xy)u(yxz) + yxD(xyz) + xyD(yxz), \text{ for all } x, y, z \in R \quad (8)$$

By comparing equ.'s (7) and (8), we get,

$$\theta(y, x)u(xyz) + \theta(x, y)u(yxz) = 0, \text{ for all } x, y, z \in R. \quad (9)$$

Where $\theta(x, y)$ stands for $G(xy) - G(y)u(x) - yD(x)$. In the concept of the definition of θ , equ.(2) can be rewritten as, $\theta(x, y) = -\theta(y, x)$

$$\text{By using this notation in equ.(9), we get, } \theta(x, y)u(z)[u(x), u(y)] = 0, \text{ for all } x, y, z \in R \quad (10)$$

$$\text{By using Lemma 2, we get, } \theta(x, y)u(z)[u(s), u(t)] = 0, \text{ for all } x, y, s, t \in R \quad (11)$$

$$\text{By using Lemma 1, we obtain, } \theta(x, y)[u(s), u(t)] = 0, \text{ for all } x, y, s, t \in R \quad (12)$$

Now, we fix $x, y \in R$ and write θ instead of $\theta(x, y)$ to simplify further writing.

By using Lemma 3, we get the existence of an ideal U such that $\theta \in U \subset Z$ holds. In particular, $b\theta, \theta b \in Z$ for all $b \in R$. This gives us,

$$x \cdot \theta^2 y = \theta^2 y \cdot x = y \theta^2 \cdot x = y \cdot \theta^2 x$$

$$\text{This gives us } 4G(x \cdot \theta^2 y) = 4 \cdot G(y \cdot \theta^2 x)$$

Now we will compute each side of this equality by using equ.(2), and the above notation

$$4G(x \cdot \theta^2 y) = 2G(x\theta^2 y + \theta^2 yx)$$

$$= 2G(\theta^2 y)u(x) + 2\theta^2 y \cdot D(x) + 2G(x)u(\theta^2 y) + 2xD(\theta^2 y)$$

$$\begin{aligned}
 &= G(\theta^2y + y\theta^2)u(x) + 2xD(\theta^2y) + 2\theta^2y.D(x) + 2G(x)u(\theta^2y) \\
 &= 2G(x)u(\theta^2y) + G(y)u(\theta^2)u(x) + yD(\theta^2)u(x) + G(\theta)u(y\theta)u(x) + \theta.D(y\theta)u(x) + \\
 &2xD(\theta^2y) + 2\theta^2yD(x) \\
 &= 2G(x)u(\theta^2y) + G(\theta)u(y\theta x) + \theta D(\theta)u(y)u(x) + \theta\theta D(y)u(x) + 2xD(\theta^2y) + 2\theta^2yD(x) + \\
 &G(y)u(\theta^2)u(x) + yD(\theta^2)u(x) \\
 &\Rightarrow 4G(x.\theta^2y) = 2G(x)u(\theta^2y) + G(\theta)u(y\theta x) + \theta.D(\theta)u(yx) + G(y)u(\theta^2x) + \theta^2D(y)u(x) + \\
 &yD(\theta^2)u(x) + 2xD(\theta^2y) + 2\theta^2yD(x), \text{ for all } x, y \in R \tag{13}
 \end{aligned}$$

Moreover

$$\begin{aligned}
 4G(y.\theta^2x) &= 2G(y\theta^2x + \theta^2xy) \\
 &= 2G(\theta^2x)u(y) + 2\theta^2xD(y) + 2G(y)u(\theta^2x) + 2yD(\theta^2x) \\
 &= G(\theta^2x + x\theta^2)u(y) + 2yD(\theta^2x) + 2\theta^2xD(y) + 2G(y)u(\theta^2x) \\
 &= 2G(y)u(\theta^2x) + G(x)u(\theta^2)u(y) + xD(\theta^2)u(y) + G(\theta)u(x\theta)u(y) + \theta D(x\theta)u(y) + \\
 &2yD(\theta^2x) + 2\theta^2xD(y) \\
 &= 2G(y)u(\theta^2x) + G(x)u(\theta^2y) + xD(\theta^2)u(y) + G(\theta)u(x\theta y) + \theta D(\theta)u(x)u(y) + \\
 &\theta.\theta.D(x)u(y) + 2yD(\theta^2x) + 2\theta^2xD(y) \\
 &\Rightarrow 4G(y.\theta^2x) = \\
 &2G(y)u(\theta^2x) + G(\theta)u(x\theta y) + \theta D(\theta)u(xy) + G(x)u(\theta^2y) + \theta^2D(x)u(y) + xD(\theta^2)u(y) + \\
 &2yD(\theta^2x) + 2\theta^2xD(y), \text{ for all } x, y \in R. \tag{14}
 \end{aligned}$$

By comparing equations (13) & (14) and using the following notations.

$$u(\theta yx) = u(\theta y).u(x) = u(x).u(\theta y) = u(x\theta y) = u(\theta xy),$$

$$\theta D(\theta)u(yx) = D(\theta)\theta u(yx) = D(\theta)\theta u(xy) = \theta D(\theta)u(xy),$$

$$x\theta D(\theta)u(y) = D(\theta)x\theta u(y) = D(\theta)\theta x u(y) = D(\theta)\theta u(y)x = \theta u(y)D(\theta)x = u(y)\theta D(\theta)x$$

We obtain,

$$G(x)u(\theta^2y) + xD(\theta^2y) = G(y)u(\theta^2x) + yD(\theta^2x), \text{ which gives,}$$

$$\phi(x, y)u(\theta^2) = \phi(y, x)u(\theta^2)$$

(15)

Where $\phi(x, y)$ stands for $G(y)u(x) + yD(x)$

On the other hand, we also have,

$$4G(xy\theta^2) = 4G(x\theta.y\theta)$$

We will compute each side of this equality by using equation (2) and the properties of θ , so, we get

$$4G(xy\theta^2) = 2G(xy\theta^2 + \theta^2xy)$$

$$= 2G(\theta^2)u(xy) + 2\theta^2D(xy) + 2G(xy)u(\theta^2) + 2xyD(\theta^2)$$

Which gives,

$$4G(xy\theta^2) = 2G(\theta^2)u(xy) + 2\theta^2D(xy) + 2G(xy)u(\theta^2) + 2xyD(\theta^2), \text{ for all } x, y \in R. \tag{16}$$

More over,

$$4G(x\theta.y\theta) = 2G(x\theta y\theta + y\theta x\theta)$$

$$= 2G(y\theta).u(x\theta) + 2y\theta.D(x\theta) + 2G(x\theta)u(y\theta) + 2x\theta.D(y\theta)$$

$$= G(y\theta + \theta y)u(x\theta) + 2y\theta.D(x\theta) + G(x\theta + \theta x)u(y\theta) + 2x\theta.D(\theta y)$$

$$= G(\theta)u(y)u(x\theta) + \theta.D(y)u(x\theta) + G(y)u(\theta)u(x\theta) + yD(\theta)u(x\theta) + 2y\theta D(x\theta) +$$

$$G(\theta)u(x)u(y\theta) + \theta.D(x)u(y\theta) + G(x)u(\theta)u(y\theta) + x.D(\theta)u(y\theta) + 2x\theta.D(\theta y)$$

$$\Rightarrow 4G(x\theta.y\theta) = G(x)u(y\theta^2) + G(\theta)u(xy\theta) + xD(\theta)u(y\theta) + \theta D(x)u(y\theta) + 2x\theta.D(\theta y) + G(y)u(x\theta^2) + G(\theta)u(yx\theta) + yD(\theta)u(x\theta) + 2y\theta D(x\theta) + \theta.D(y)u(x\theta), \text{ for all } x, y \in R \quad (17)$$

Comparing equation (16) & (17), we obtain,

$$G(xy)u(\theta^2) = \phi(x, y)u(\theta^2), \text{ for all } x, y \in R. \quad (18)$$

$$\text{But } \theta(x, y) = G(xy) - G(y)u(x) - yD(x)$$

$$\phi(x, y) = G(y)u(x) + yD(x)$$

$$\theta(x, y) = G(xy) - \phi(x, y)$$

$$\text{And this means } \theta^3 = 0 \text{ show that, } \theta^2 R \theta^2 = \theta^4 R = (0)$$

$$\theta R \theta = \theta^2 R = (0),$$

Which implies $\theta = 0$, and the proof is complete. \blacksquare

Theorem 2: If G is a Jordan u -*generalized reverse derivation of a semiprime ring R of char $\neq 2$, then G is an u -*generalized reverse derivation.

Proof: Since $G(x^2) = u(x)G(x) + D(x)x$

We replace x by $x + y$, then

$$G(xy + yx) = u(x)G(y) + u(y)G(x) + D(x)y + D(y)x, \text{ for all } x, y \in R \quad (19)$$

Consider $W = G(x(xy + yx)) + (xy + yx)x$

$$= u(x)G(xy + yx) + D(x)(xy + yx) + u(xy + yx)G(x) + D(xy + yx)x$$

$$= u(x)G(xy) + u(x)G(yx) + D(x)xy + D(x)yx + u(xy)G(x) + u(yx)G(x) + D(xy)x + D(yx)x$$

$$= u(x)u(x)G(y) + u(x)D(x)y + u(x)u(y)G(x) + u(x)D(y)x + D(x)xy + D(x)yx + u(y)u(x)G(x) +$$

$$u(x)u(y)G(x) + u(x)D(y)x + D(x)yx + u(y)D(x)x + D(y)x^2$$

$$= u(x)u(x)G(y) + u(x)D(x)y + 2u(x)u(y)G(x) + 2u(x)D(y)x + D(x)xy + 2D(x)yx + u(y)u(x)G(x) + u(y)D(x)x + D(y)x^2 \quad (20)$$

On the other hand,

$$W = G(x^2 y + xyx + xyx + yx^2)$$

$$= G(x^2 y + yx^2 + 2xyx)$$

$$= u(x^2)G(y) + D(x^2)y + u(yx)G(x) + D(yx)x + 2G(xy x)$$

$$= u(x)u(x)G(y) + u(x)D(x)y + D(x)xy + u(x)u(y)G(x) + u(y)D(x)x + D(y)x^2 + 2G(xy x) \quad (21)$$

By comparing the equ.'s (20) and (21) for W , we obtain,

$$2u(x)u(y)G(x) + 2u(x)D(y)x + 2D(x)yx = 2G(xy x)$$

$$\Rightarrow 2[u(x)u(y)G(x) + u(x)D(y)x + D(x)yx] = 2G(xy x)$$

$$\Rightarrow G(xy x) = u(x)u(y)G(x) + u(x)D(y)x + D(x)yx$$

$$\Rightarrow G(xy x) = u(yx)G(x) + D(xy)x$$

We put $z = xy$, then $G(zx) = u(z)G(x) + D(z)x$, which implies that G is a u -* generalized reverse derivation. \blacksquare

It is clear that if we assume that the reverse derivation D to be the zero reverse derivation in the above Theorems 1 and 2, then we get the following result.

Corollary 1: Let R be a semi prime ring of char. $\neq 2$ and $T: R \rightarrow R$ an additive mapping which satisfies $T(x^2) = T(x)u(x)$, for all $x \in R$. Then it is a left (right) centralizer.

Theorem 3: Let R be a semiprime ring of char. $\neq 2$ and G be a non-zero Jordan u -generalized reverse derivation of R . If $G([x, y]) = 0$, then R is commutative.

Proof : We substitute xy for y in $G([x,y]) = 0$

Then we obtain,

$$0 = G([x,xy])$$

$$\Rightarrow 0 = G[x,y]u(x) + [x,y]D(x)$$

$$\Rightarrow 0 = [x,y]D(x), \text{ for all } x, y \in R$$

We replace y by ry , then

$$0 = [x,ry]D(x), \text{ for all } x, y \in R$$

$$\Rightarrow r[x,y]D(x) + [x,r]yD(x) = 0$$

$$\Rightarrow [x,r]yD(x) = 0$$

$$\Rightarrow [x,r]RD(x), \text{ for all } y \in R$$

Now we choose a family $P = \{p_\alpha / \alpha \in \Lambda\}$ of prime rings R such that $\cap P_\alpha = \{0\}$ and let P denote a fixed one of the P_α . From the above equation, it follows that either $[x,r] \in p$ (or) $D(x) \in p$, for all $x \in R$. Let $A = \{x \in R / x \in Z\}$ and $B = \{x \in R / D(x) = 0\}$. Then A and B are two additive sub groups of $(R, +)$ such that $R = A \cup B$. However, a group cannot be the union of proper sub groups. Hence either $R = A$ or $R = B$. If $R = A$, then $R \subset Z$ and so R is commutative. If $R = B$, then $R = 0$ which contradicts the hypothesis. So, we must have $r \in Z$, for all $r \in R$. Hence R is commutative. ■

References:

[1] Ashraf. M. and Rehman N.U., On Jordan generalized derivations in rings, Math. J. Okayama Univ. 42(2000), 7-9.

[2] Bresar. M., On the distance of the composition of two derivations to the generalized derivations, Glasgow Math. J. 33 (1991), 89-93.