Slant and hemislant submanifolds of a 3—dimensional indefinite trans-Sasakian manifold

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Keywords: Slant submanifold, indefinite trans-Sasakian manifold, hemislant submanifold.

Abstract. In this paper we would like to establish some of the properties of slant and hemislant submanifolds of an indefinite trans-Sasakian manifold. We have four sections in this paper. Section (1) is introductory. In Section (2) we recall some necessary details of an indefinite trans-Sasakian manifold. In Section (3) we have obtained some interesting properties on a totally umbilical slant submanifolds of an indefinite trans-Sasakian manifold. Finally, in Section (4), some results on integrability conditions of the distributions of hemislant submanifolds of an indefinite trans-Sasakian manifold have been obtained.

1. Introduction

The study of slant submanifolds in complex spaces was initiated by B.Y.Chen as a natural generalization of both holomorphic and totally real submanifolds in ([1],[2]). After him, A.Lotta in 1996 extended the notion to the setting of almost contact metric manifolds [3]. Further modifications regarding semislant submanifolds were introduced by N.Papaghiuc [4]. These submanifolds are a generalized version of CR-submanifolds. J.L.Cabrerizo et.al. ([5],[6]) extended the study of semislant submanifolds of Kaehler manifold to the setting of Sasakian manifolds. In particular, totally umbilical proper slant submanifold of a Kaehler manifold has also been discussed in [7]. Recently, Khan et.al. [8] carried some investigation on these submanifolds in the setting of Lorentzian paracontact manifolds.

The idea of hemislant submanifold was introduced by Carriazo as a particular class of bislant submanifolds, and he called them antislant submanifolds in [9]. Later on, in 2011 Siraj Uddin et.al. studied totally umbilical proper slant and hemislant submanifolds of an LP-cosymplectic manifold. In the present note, our aim is to extend the study of slant and hemislant submanifolds of an indefinite trans-Sasakian manifold.

2. Preliminaries

Let $\tilde{M}$ be an $(2n + 1)$-dimensional indefinite almost contact metric manifold with indefinite almost contact metric structure $(\phi, \xi, \eta, \tilde{g})$, where $\phi$ is a tensor of type $(1, 1)$ having rank $2n$, $\xi$ is a vector field, $\eta$ is a 1-form and $\tilde{g}$ is Riemannian metric, satisfying following properties :

\begin{align}
(2.1) & \quad \phi^2 X = -X + \eta(X)\xi, \quad \eta \circ \phi = 0, \quad \phi \xi = 0, \quad \eta(\xi) = 1, \\
(2.2) & \quad \tilde{g}(\phi X, \phi Y) = \tilde{g}(X, Y) - \epsilon \eta(X)\eta(Y), \\
(2.3) & \quad \tilde{g}(X, \xi) = \epsilon \eta(X),
\end{align}

for all vector fields $X, Y$ on $\tilde{M}$. It is easy to see that $\tilde{g}(\xi, \xi) = \epsilon = \pm 1$. 

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An indefinite almost contact metric structure \((\phi, \xi, \eta, \bar{g})\) is called an indefinite trans-Sasakian structure if

\[
(2.4) \quad (\bar{\nabla}_X \phi)Y = \alpha[g(X,Y)\xi - \epsilon\eta(Y)X] + \beta[g(\phi X,Y)\xi - \epsilon\eta(Y)\phi X],
\]

for functions \(\alpha\) and \(\beta\) on \(\tilde{M}\) of type \((\alpha, \beta)\), where \(\bar{\nabla}\) is the Levi-Civita connection on \(\tilde{M}\). On indefinite trans-Sasakian manifold we have,

\[
(2.5) \quad \bar{\nabla}_X \xi = -\alpha \epsilon \phi X + \beta \epsilon [X - \eta(X)]\xi,
\]

for any \(X \in T\tilde{M}\) where \(T\tilde{M}\) is the Lie algebra of vector fields on \(\tilde{M}\).

**Definition (2.1):** An \(n\)-dimensional Riemannian submanifold \(M\) of an indefinite trans-Sasakian manifold \(\tilde{M}\) is called a contact CR-submanifold if

i) \(\xi\) is tangent to \(M\),

ii) there exists on \(M\) a differentiable distribution \(D : x \rightarrow D_x \subset T_x(M)\), such that \(D_x\) is invariant under \(\phi\); i.e., \(\phi D_x \subset D_x\), for each \(x \in M\) and the orthogonal complementary distribution \(D^\perp : x \rightarrow D_x^\perp \subset T_x^\perp(M)\) of the distribution \(D\) on \(M\) is totally real; i.e., \(\phi D_x^\perp \subset T_x^\perp(M)\), where \(T_x(M)\) and \(T_x^\perp(M)\) are the tangent space and the normal space of \(M\) at \(x\) respectively.

Let \(M\) be a submanifold of an indefinite trans-Sasakian manifold \(\tilde{M}\) with induced metric \(g\) and let \(\nabla\) is the induced connection on the tangent bundle \(TM\) and \(\nabla^\perp\) is the induced connection on the normal bundle \(T^\perp M\) of \(M\). Let \(F(M)\) be the algebra of smooth functions on \(M\) and \(F(TM)\) be the \(F(M)\)-module of smooth functions of a vector bundle \(TM\) over \(M\).

The Gauss and Weingarten formulae are characterized by

\[
(2.6) \quad \bar{\nabla}_X Y = \nabla_X Y + h(X,Y),
\]

\[
(2.7) \quad \bar{\nabla}_X N = -A_N X + \nabla^\perp_X N,
\]

for any \(X, Y \in TM\), \(N \in T^\perp M\), \(h\) is the second fundamental form and \(A_N\) is the Weingarten map associated with \(N\) via

\[
(2.8) \quad g(A_N X, Y) = g(h(X,Y), N).
\]

For any \(X \in \Gamma(TM)\) we can write,

\[
(2.9) \quad \phi X = PX + FX,
\]

where \(PX\) is the tangential component and \(FX\) is the normal component of \(\phi X\). Similarly for any \(N \in \Gamma(T^\perp M)\) we can put

\[
(2.10) \quad \phi N = BN + CN,
\]

where \(BN\) denote the tangential component and \(CN\) denote the normal component of \(\phi N\).

The covariant derivatives of the tensor fields \(\phi\), \(P\) and \(F\) are defined as
(2.11) \((\tilde{\nabla}_X \phi) Y = \tilde{\nabla}_X \phi Y - \phi \tilde{\nabla}_X Y\) \quad \forall \ X, Y \in T\tilde{M},

(2.12) \((\tilde{\nabla}_X P) Y = \nabla_X PY - P \nabla_X Y\) \quad \forall \ X, Y \in TM,

(2.13) \((\tilde{\nabla}_X F) Y = \nabla_X FY - F \nabla_X Y\) \quad \forall \ X, Y \in TM. A submanifold \( M \) is said to be invariant if \( F \) is identically zero, i.e., \( \phi X \in \Gamma(TM) \) for any \( X \in \Gamma(TM) \). On the other hand, \( M \) is said to be anti-invariant if \( P \) is identically zero, i.e., \( \phi X \in \Gamma(T^\perp M) \) for any \( X \in \Gamma(TM) \).

A submanifold \( M \) of an indefinite trans-Sasakian manifold \( \tilde{M} \) is called totally umbilical if

\[(2.14) \ h(X, Y) = g(X, Y) H,\]

for any \( X, Y \in \Gamma(TM) \). The mean curvature vector \( H \) is denoted by

\[(2.15) \ H = \sum_{i=1}^{k} h(e_i, e_i),\]

where \( k \) is the dimension of \( M \) and \((e_1, e_2, e_3, \ldots, e_k)\) is the local orthonormal frame on \( M \).

A submanifold \( M \) is said to be totally geodesic if \( h(X, Y) = 0 \) for each \( X, Y \in \Gamma(TM) \) and is minimal if \( H = 0 \) on \( M \).

3. Slant submanifold

A submanifold \( M \) of a 3–dimensional indefinite trans-Sasakian manifold \( \tilde{M} \) is said to be slant submanifold if for any \( X \in M \) and \( X \in TM - < \xi > \), the angle between \( \phi X \) and \( TM \) is constant. The constant angle \( \theta \in [0, \pi/2] \) is then called slant angle of the slant submanifold \( M \). Obviously if \( \theta = 0 \), \( M \) is invariant and if \( \theta = \pi/2 \), \( M \) is an anti-invariant submanifold. A slant submanifold is called proper slant if it is neither invariant nor anti-invariant submanifold. If \( M \) is a slant submanifold of a 3–dimensional indefinite trans-Sasakian manifold, then we can decompose tangent bundle \( TM \) of \( M \) as

\[(3.1) \ TM = D \bigoplus < \xi >,\]

where the orthogonal complementary distribution \( D \) of \( < \xi > \) is known as the slant distribution on \( M \). If \( \mu \) is \( \phi \)-invariant of the normal bundle \( T^\perp M \), then

\[(3.2) \ T^\perp M = FTM \bigoplus < \mu >.\]

Defining the endomorphism \( P : TM \rightarrow TM \), whose square, \( P^2 \) will be denoted by \( Q \). Then the tensor fields on \( M \) of type \((1,1)\) determined by these endomorphism will be denoted by the same letters, respectively \( P \) and \( Q \).

We are already having the following result for a slant submanifold.

**Theorem(3.1)** : Let \( M \) be a submanifold of a 3–dimensional indefinite trans-Sasakian manifold \( \tilde{M} \), such that \( \xi \in TM \). Then by [6], \( M \) is slant if and only if \( \exists \) a constant \( \lambda \in [0, 1] \) such that
\[ P^2 = \lambda(-I + \eta \bigotimes \xi). \]

Again, if \( \theta \) is slant angle of \( M \), then \( \lambda = \cos^2 \theta \).

We can easily draw the following consequences for a 3–dimensional indefinite trans-Sasakian manifold \( \tilde{M} \) from [6]

\[
\begin{align*}
(3.4) & \quad g(PX, PY) = \cos^2 \theta [g(X, Y) - \epsilon \eta(X) \eta(Y)], \\
(3.5) & \quad g(FX, FY) = \sin^2 \theta [g(X, Y) - \epsilon \eta(X) \eta(Y)],
\end{align*}
\]

for any \( X, Y \) tangent to \( M \).

Now we prove some interesting results on slant submanifold \( M \) of a 3–dimensional indefinite trans-Sasakian manifold \( \tilde{M} \).

**Theorem (3.2) :** If \( M \) is a totally umbilical slant submanifold of a 3–dimensional indefinite trans-Sasakian manifold \( \tilde{M} \), and if \( H \in \Gamma(\mu) \) then

\( \text{(i)} \) \( M \) is invariant.

\( \text{(ii)} \) \( M \) is not a proper slant submanifold of \( \tilde{M} \).

**Proof :** From (2.14) we have

\[ h(PX, PY) = g(PX, PY)H. \]

Using (2.6) and (3.4) we can write

\[
\begin{align*}
(3.7) & \quad \tilde{\nabla}_PXPY - \nabla_PXPY = h(PX, PY) \\
& \quad \quad = g(PX, PY)H \\
& \quad \quad = \cos^2 \theta [g(X, Y) - \epsilon \eta(X) \eta(Y)]H.
\end{align*}
\]

Replacing \( Y \) with \( X \) in (3.7) we have

\[ \tilde{\nabla}_PX - \nabla_PX = \cos^2 \theta [g(X, X) - \epsilon \eta(X) \eta(X)]H. \]

Using (2.9) we can have

\[ \tilde{\nabla}_PX\phi X - \tilde{\nabla}_PXFX - \nabla_PXPX = \cos^2 \theta[||X||^2 - \epsilon \eta^2(X)]H. \]

From (2.11) we get

\[ \phi \tilde{\nabla}_PX X - (\tilde{\nabla}_PX \phi) X - \tilde{\nabla}_PXFX - \nabla_PXPX = \cos^2 \theta[||X||^2 - \epsilon \eta^2(X)]H. \]
By using equation (2.4), (2.9), (2.14) and Gauss and Weingarten formulae we calculate

\begin{equation}
(3.11) \quad P \nabla_P X + F \nabla_P X + g(PX, X)\phi H - \alpha g(PX, X)\xi + \alpha \epsilon \eta(X)PX \\
- \beta g(\phi PX, X)\xi + \beta \epsilon \eta(X)\phi PX + AFX PX - \nabla_P FX - \nabla_P PX \\
= \cos^2 \theta [ ||X||^2 - \epsilon \eta^2(X) ] H.
\end{equation}

Equating the normal components we get

\begin{equation}
(3.12) \quad F \nabla_P X - \nabla_P^FX = \cos^2 \theta [ ||X||^2 - \epsilon \eta^2(X) ] H.
\end{equation}

On the other hand from (3.5) we infer

\begin{equation}
(3.13) \quad g(FX, FX) = \sin^2 \theta [g(X, X) - \epsilon \eta^2(X)],
\end{equation}

for any \( X \in \Gamma(TM) \). Taking the covariant derivative of the above equation w.r.t \( PX \), we obtain

\begin{equation}
(3.14) \quad 2g(\nabla_P FX, FX) = 2\sin^2 \theta g(\nabla_P X, X) - 2\sin^2 \theta \epsilon \eta(X)g(\nabla_P X, \xi) \\
- 2\sin^2 \theta \epsilon \eta(X)g(X, \nabla_P X).
\end{equation}

Using the property of metric connection and using (2.12) and (2.13) we have

\begin{equation}
(3.15) \quad g(\nabla_P^FX, FX) = \sin^2 \theta g(\nabla_P X, X).
\end{equation}

Now taking the inner product in (3.12) with \( FX \), for any \( X \in \Gamma(TM) \), then

\begin{equation}
(3.16) \quad g(F \nabla_P X, FX) - g(\nabla_P^FX, FX) = \cos^2 \theta [ ||X||^2 - \epsilon \eta^2(X) ] g(H, FX).
\end{equation}

After using (2.6), (3.5) and (3.15) and having some brief calculation we derive

\begin{equation}
(3.17) \quad - \epsilon \sin^2 \theta \eta(X)g(\nabla_P X, \xi) = \cos^2 \theta [ ||X||^2 - \epsilon \eta^2(X) ] g(H, FX).
\end{equation}

Since \( \nabla \) is the metric connection then the above equation can be written as

\begin{equation}
(3.18) \quad \epsilon \sin^2 \theta \eta(X)g(X, \nabla_P X) = \cos^2 \theta [ ||X||^2 - \epsilon \eta^2(X) ] g(H, FX).
\end{equation}

As \( \tilde{M} \) is a 3-dimensional indefinite trans-Sasakian manifold, then using the fact that

\begin{equation}
(3.19) \quad \nabla_P X = -\alpha \epsilon \phi PX + \beta \epsilon [PX - \eta(PX)\xi],
\end{equation}

we can easily conclude from (3.18) and (3.19)

\begin{equation}
(3.20) \quad \tan^2 \theta = \frac{||X||^2 - \epsilon \eta^2(X) }{ \eta(X)[g(X, -\alpha \epsilon \phi PX + \beta \epsilon [PX - \beta \epsilon \eta(PX)\xi])] },
\end{equation}

if \( H \in \Gamma(\mu) \) then, \( \tan \theta = 0 = \tan(n\pi) \). Since \( \theta \in [0, \pi/2] \) hence \( \theta = 0 \).
Therefore $M$ is invariant i.e. $M$ is not a proper slant manifold.

**Theorem (3.3)**: Every totally umbilical proper slant submanifold $M$ of a 3-dimensional indefinite trans-Sasakian manifold $\tilde{M}$ is totally geodesic, provided $\nabla^+ X H \in \Gamma(\mu)$ for any $X \in TM$ and $H \in \Gamma(\mu)$.

**Proof**: As $\tilde{M}$ is a 3-dimensional indefinite trans-Sasakian manifold we have

\[(\tilde{\nabla}_X \phi)Y = \tilde{\nabla}_X \phi Y - \phi \tilde{\nabla}_X Y.\]

From the fact that $\phi Y = PY + FY$ and $\tilde{M}$ is a 3-dimensional indefinite trans-Sasakian manifold we infer

\[\tilde{\nabla}_X PY + \tilde{\nabla}_X FY = P\nabla_X Y + F\nabla_X Y + \phi h(X, Y) + \alpha [g(X, Y) \xi - \epsilon(Y)X]
+ \beta [g(\phi X, Y) \xi - \epsilon(Y)\phi X].\]

Using (2.12), (2.13) and (2.14) we obtain

\[\nabla_X PY + h(X, PY) - A_{FY} X + \nabla^+ X FY = P\nabla_X Y + F\nabla_X Y + g(X, Y) \phi H
+ \alpha [g(X, Y) \xi - \epsilon(Y)X] + \beta [g(\phi X, Y) \xi - \epsilon(Y)\phi X].\]

Taking inner product with $\phi H$ and using the fact that $H \in \Gamma(\mu)$, from (2.2) and (2.14) we get

\[g(X, PY)g(H, \phi H) + g(\nabla^+ X FY, \phi H) = g(X, Y)||H||^2 + \alpha g(X, Y)g(\phi H, \xi)
- \alpha \epsilon(Y)g(X, \phi H) + \beta g(\phi X, Y)g(\phi H, \xi) - \beta \epsilon(Y)g(\phi X, \phi H).\]

Now we consider

\[\tilde{\nabla}_X \phi H = \phi \tilde{\nabla}_X H + (\tilde{\nabla}_X \phi)H.\]

Using equation (2.6), (2.7), (2.9), (2.10) and (3.25) we calculate

\[-A_{\phi H} X + \nabla^+ X \phi H = -PA_H X - FA_H X + B\nabla^+ X H + C\nabla^+ X H
+ \alpha g(X, H) \xi - \alpha \epsilon(H)X + \beta g(\phi H, H) \xi - \beta \epsilon(H)\phi X.\]

Taking inner product with $FY$, for any $Y \in \Gamma(TM)$ we have

\[g(\nabla^+ X \phi H, FY) = -g(FA_H X, FY) + g(C\nabla^+ X H, FY) + \alpha g(X, H) g(FY, \xi)
- \alpha \epsilon(H)g(X, FY) + \beta g(\phi H, H) g(FY, \xi)
- \beta \epsilon(H)g(\phi X, FY).\]

Since $C\nabla^+ X H \in \Gamma(\mu)$, then by (3.5) the above equation takes the form
Using (2.7), (2.8) and (2.14) and having some brief calculations we obtain

\begin{equation}
(3.29) \quad g(\nabla_X \phi H, FY) = -\sin^2 \theta [g(A_H X, Y) - \epsilon \eta(A_H X) \eta(Y)] - \alpha \eta(H) g(X, FY) - \beta \eta(H) g(\phi X, FY).
\end{equation}

The above equation can be written as

\begin{equation}
(3.30) \quad g(\nabla_X FY, \phi H) = \sin^2 \theta [g(X, Y) - \epsilon \eta(X) \eta(Y)] ||H||^2 - \alpha \eta(H) g(X, FY) - \beta \eta(H) g(\phi X, FY).
\end{equation}

Again using the fact that \( H \in \Gamma(\mu) \) then by (2.7) we have

\begin{equation}
(3.31) \quad g(\nabla_X FY, \phi H) = \sin^2 \theta [g(X, Y) - \epsilon \eta(X) \eta(Y)] ||H||^2 - \alpha \eta(H) g(X, FY) - \beta \eta(H) g(\phi X, FY).
\end{equation}

From (3.24) and (3.31) we get

\begin{equation}
(3.32) \quad g(X, Y)||H||^2 - \alpha \epsilon \eta(Y) g(X, \phi H) + g(X, PY) g(\phi H, H) - \beta \epsilon \eta(Y) g(\phi X, \phi H) = \sin^2 \theta g(X, Y)||H||^2 - \epsilon \sin^2 \theta \eta(Y) \eta(X)||H||^2 - \alpha \epsilon \eta(Y) g(X, \phi H) - \beta \epsilon \eta(Y) g(\phi X, FY) - \alpha \epsilon \eta(H) g(X, FY) - \beta \epsilon \eta(H) g(\phi X, FY) = 0.
\end{equation}

The equation (3.33) has a solution if \( H = 0 \). Hence \( M \) is totally geodesic in \( \tilde{M} \). Therefore the proof is done.

**Theorem (3.4):** Let \( M \) be a slant submanifold of a 3–dimensional indefinite trans-Sasakian manifold \( \tilde{M} \). Then \( Q \) is parallel if and only if \( M \) is anti-invariant.

**Proof:** Let \( Q \) be the slant angle of \( M \) in \( \tilde{M} \), then for any \( X, Y \) in \( TM \), using equation (3.3) we infer

\begin{equation}
(3.34) \quad P^2 Y = QY = \cos^2 \theta (-Y + \eta(Y) \xi).
\end{equation}

Putting \( Y = \nabla_X Y \) we have

\begin{equation}
(3.35) \quad Q \nabla_X Y = \cos^2 \theta (-\nabla_X Y + \eta(\nabla_X Y) \xi).
\end{equation}
By taking the covariant derivative of (3.34) w.r.t \( X \in TM \) we get

\[
\nabla_X Q Y = \cos^2 \theta (-\nabla_X Y + \eta(\nabla_X Y)\xi) + g(Y, \nabla_X \xi) + \eta(Y)\nabla_X \xi.
\]

Again using (3.35) and (3.36) we obtain

\[
(\tilde{\nabla}_X Q) Y = \cos^2 \theta g(Y, \nabla_X \xi) - \cos^2 \theta \eta(Y)\nabla_X \xi.
\]

Using (2.5) in (3.37) we can easily observe that if \( \theta = \pi/2 \) then \((\tilde{\nabla}_X Q) Y = 0\) i.e. \( Q \) is parallel and thus assertion is proved.

4. Hemislant submanifold

We assume that \( M \) is a hemislant submanifold of a 3–dimensional indefinite trans-Sasakian manifold such that the structure vector field \( \xi \) is tangent to \( M \). At first, we define a hemislant submanifold and then we derive integrability condition of the involved distributions \( D_1 \) and \( D_2 \).

**Definition(4.1)**: A submanifold \( M \) of a 3–dimensional indefinite trans-Sasakian manifold is said to be a hemislant submanifold if there exist two orthogonal complementary distributions \( D_1 \) and \( D_2 \) satisfying the following properties:

(i) \( TM = D_1 \oplus D_2 \oplus \langle \xi \rangle \),

(ii) \( D_1 \) is a slant distribution with slant angle \( \theta \neq \pi/2 \).

(iii) \( D_2 \) is totally real that is \( \phi D_2 \subseteq T^\perp M \).

Further if \( \mu \) is \( \phi \)-invariant subspace of the normal bundle \( T^\perp M \), then for hemislant submanifold, the normal bundle \( T^\perp M \) can be decomposed as,

\[
T^\perp M = FD_1 \oplus FD_2 \oplus \langle \mu \rangle.
\]

Now, our task is to obtain the integrability conditions of the involved distributions.

**Proposition(4.1)**: Let \( M \) be a hemislant submanifold of a 3–dimensional indefinite trans-Sasakian manifold \( \tilde{M} \), then the anti-invariant distribution \( D_2 \) is integrable iff

\[
A_{FZ} W = A_{FW} Z,
\]

for any \( Z, W \in D_2 \).

**Proof**: For any \( Z, W \in D_2 \), we know

\[
\phi[Z, W] = \phi \tilde{\nabla}_Z W - \phi \tilde{\nabla}_W Z.
\]

After using (2.7) and (2.11) we get

\[
\phi[Z, W] = -A_{FW} Z + \nabla^\perp_Z FW + A_{FZ} W - \nabla^\perp_W FZ + (\tilde{\nabla}_W \phi)Z - (\tilde{\nabla}_Z \phi)W.
\]
From the fact that $\tilde{M}$ is a 3–dimensional indefinite trans-Sasakian manifold we infer

\[(4.5) \phi[Z,W] = -A_W Z + \nabla^\bot_Z F W + A_Z W - \nabla^\bot_W F Z - \alpha \epsilon(Z) W + \beta [g(\phi W, Z) \xi - \epsilon(Z) \phi W] - \alpha \epsilon(W) Z - \beta [g(\phi Z, W) \xi - \epsilon(W) \phi Z].\]

Therefore we have

\[(4.6) \phi[Z,W] = -A_W Z + \nabla^\bot_Z F W + A_Z W - \nabla^\bot_W F Z.\]

As $D_2$ is an anti-invariant distribution, the tangential part of above equation should be identically zero, hence we obtain the required result.

**Proposition (4.2) :** Let $M$ be an hemislant submanifold of a 3–dimensional indefinite trans-Sasakian manifold $\tilde{M}$, then the invariant distribution $D_1 \ominus <\xi>$ is integrable iff

\[(4.7) g(h(Z, PW) + \nabla^\bot_Z F W - h(W, PZ) + \nabla^\bot_W F Z, FX) = 0.\]

for any $Z, W \in D_1 \ominus <\xi>$ and $X \in D_2$.

**Proof :** For any $Z, W \in D_1 \ominus <\xi>$, we have

\[(4.8) \phi[Z,W] = \phi \tilde{\nabla}_Z W - \phi \tilde{\nabla}_W Z.\]

Since $\tilde{M}$ is a 3–dimensional indefinite trans-Sasakian manifold and using (2.11) we get

\[(4.9) \phi[Z,W] = \tilde{\nabla}_Z \phi W - (\tilde{\nabla}_Z \phi) W - \tilde{\nabla}_W \phi Z + (\tilde{\nabla}_W \phi) Z.\]

Again using (2.4) and (2.9) we obtain

\[(4.10) \phi[Z,W] = \tilde{\nabla}_Z PW + \tilde{\nabla}_Z FW - \tilde{\nabla}_W PZ - \tilde{\nabla}_W FZ - \alpha \epsilon(Z) W + \beta g(\phi W, Z) \xi - \beta \epsilon(Z) P W - \beta \epsilon(Z) F W + \alpha \epsilon(W) Z - \beta g(\phi Z, W) \xi + \beta \epsilon(W) P Z + \beta \epsilon(W) F Z.\]

Taking inner product with $FX$ for any $X \in D_2$ and using Gauss and Weingarten formulae we calculate

\[(4.11) g(\phi[Z,W], FX) = 0,\]

if

\[g(h(Z, PW) + \nabla^\bot_Z F W - h(W, PZ) + \nabla^\bot_W F Z, FX) = 0.\]

Since $\xi$ is tangential to $D_1$, we obtain the required integrability condition.

**Theorem (4.1) :** Let $M$ be a totally umbilical hemislant submanifold $M$ of a 3–dimensional indefinite trans-Sasakian manifold $\tilde{M}$, then at least one of the following statement is true:
(i) the dimension of anti-invariant distribution is one, i.e. \( \dim D^\perp = 1 \).
(ii) The mean curvature vector \( H \in \mu \).
(iii) \( M \) is proper slant submanifold of \( \tilde{M} \).

**Proof:** Now we consider \( M \) as a totally umbilical hemislant submanifold of a 3—dimensional indefinite trans-Sasakian manifold \( \tilde{M} \). For any \( Z, W \in TM \), one has

\[
(4.12) \quad \nabla_Z \phi W = (\nabla_Z \phi) W + \phi \nabla_Z W.
\]

Taking vectors \( Z, W \in D_2 \), then from Gauss and Weingarten formulae we have

\[
(4.13) \quad -A_{FW} Z + \nabla_\frac{1}{2} FW = (\nabla_Z \phi) W + \phi (\nabla_Z W + h(Z, W)).
\]

Using (2.4), (2.9) and (2.10) we obtain

\[
(4.14) \quad -A_{FW} Z + \nabla_\frac{1}{2} FW = P \nabla_Z W + F \nabla_Z W + Bh(Z, W) + Ch(Z, W)
\]
\[
+ \alpha g(Z, W) \xi - \alpha \eta(Z) W + \beta [g(PZ, W) \xi - \eta(W) PZ] - \beta \eta(W) FZ
\]
\[
+ \beta g(FZ, W) \xi.
\]

Equating the tangential components we calculate

\[
(4.15) \quad -A_{FW} Z = P \nabla_Z W + Bh(Z, W) + \beta g(PZ, W) \xi - \beta \eta(W) PZ
\]
\[
+ \alpha g(Z, W) \xi - \alpha \eta(W) Z.
\]

\[
(4.16) \quad P \nabla_Z W = A_{FW} Z - Bh(Z, W) - \beta g(PZ, W) \xi + \beta \eta(W) PZ
\]
\[
- \alpha g(Z, W) \xi - \alpha \eta(W) Z.
\]

Taking the inner product with \( V \in D_2 \), we get

\[
(4.17) \quad g(P \nabla_Z W, V) = g(A_{FW} Z, V) - g(Bh(Z, W), V) + \beta \eta(W) g(PZ, V)
\]
\[
- \alpha \eta(W) g(Z, V).
\]

From equation (2.8) and using the fact that \( W \in D_2 \), the above equation takes the form

\[
(4.18) \quad 0 = g(h(Z, V), FW) + g(Bh(Z, W), V).
\]

As \( M \) is totally umbilical, we infer

\[
(4.19) \quad 0 = g(Z, V) g(H, FW) + g(Z, W) g(BH, V).
\]

Thus the above equation has a solution if either \( Z = W = V = \xi \) i.e. \( \dim D_2 = 1 \) or \( H \in \mu \) or \( D_2 = 0 \).
Now we give an example of a 3–dimensional indefinite trans-Sasakian manifold:

**Example** : Consider a 3-dimensional submanifold of $R^5$ with its usual structure defined as:

The $(1, 1)$ tensor $\phi$ is defined as:

$$
\begin{pmatrix}
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$\eta = dt \quad \xi = \frac{\partial}{\partial t}$ and $g = -dx_1^2 - dx_2^2 - dy_1^2 - dy_2^2 + \eta \otimes \eta$.

Now for any $\theta \in [0, \Pi/2]$,

$$x(u, v, t) = 2(u \cos \theta, u \sin \theta, v, 0, t).$$

If we denote by $M$ a slant submanifold, then its tangent space $TM$ span by the vectors:

$$e_1 = \frac{\partial}{\partial v} + 2 \cos \theta v \frac{\partial}{\partial t} = \cos \theta (2(\frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial t})) + \sin \theta (2(\frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial t}))$$
$$e_2 = \frac{\partial}{\partial v} = 2 \frac{\partial}{\partial y_1}, e_3 = \frac{\partial}{\partial t} = \xi.$$

Moreover, the vector fields:

$$e_1^* = -\sin \theta (2(\frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial t})) + \cos \theta (2(\frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial t}))$$
$$e_2^* = 2 \frac{\partial}{\partial y_2}$$

form a basis of $T^1M$. Furthermore, using Gauss formula, we get $\tilde{\nabla}_{e_i} e_i = 0$, for $i = 1, 2$. Thus, we have:

$$h(e_1, e_1) = 0, \quad h(e_2, e_2) = 0, \quad h(e_1, e_2) = 0$$

and hence, we can conclude that $M$ is totally geodesic.

**References**


