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Slant and hemislant submanifolds of a $3-$dimensional indefinite trans-Sasakian manifold

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Abstract. In this paper we would like to establish some of the properties of slant and hemislant submanifolds of an indefinite trans-Sasakian manifold. We have four sections in this paper. Section (1) is introductory. In Section (2) we recall some necessary details of an indefinite trans-Sasakian manifold. In Section (3) we have obtained some interesting properties on a totally umbilical slant submanifolds of an indefinite trans-Sasakian manifold. Finally, in Section (4), some results on integrability conditions of the distributions of hemislant submanifolds of an indefinite trans-Sasakian manifold have been obtained.

1. Introduction

The study of slant submanifolds in complex spaces was initiated by B.Y.Chen as a natural generalization of both holomorphic and totally real submanifolds in ([1],[2]). After him, A.Lotta in 1996 extended the notion to the setting of almost contact metric manifolds [3]. Further modifications regarding semislant submanifolds were introduced by N.Papaghiuc [4]. These submanifolds are a generalized version of CR-submanifolds. J.L.Cabrerizo et.al. ([5],[6]) extended the study of semislant submanifolds of Kaehler manifold to the setting of Sasakian manifolds. In particular, totally umbilical proper slant submanifold of a Kaehler manifold has also been discussed in [7]. Recently, Khan et.al. [8] carried some investigation on these submanifolds in the setting of Lorentzian paracontact manifolds.

The idea of hemislant submanifold was introduced by Carriazo as a particular class of bislant submanifolds, and he called them antislant submanifolds in [9]. Later on, in 2011 Siraj Uddin et.al. studied totally umbilical proper slant and hemislant submanifolds of an LP-cosymplectic manifold.

In the present note, our aim is to extend the study of slant and hemislant submanifolds of an indefinite trans-Sasakian manifold.

2. Preliminaries

Let $\tilde{M}$ be an $(2n+1)$-dimensional indefinite almost contact metric manifold with indefinite almost contact metric structure $(\phi, \xi, \eta, \tilde{g})$, where ϕ is a tensor of type $(1,1)$ having rank $2n$, ξ is a vector field, η is a 1-form and $\tilde{g}$ is Riemannian metric, satisfying following properties :

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta \circ \phi = 0, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \tag{2.1}$$

$$\tilde{g}(\phi X, \phi Y) = \tilde{g}(X, Y) - \epsilon\eta(X)\eta(Y), \tag{2.2}$$

$$\tilde{g}(X, \xi) = \epsilon\eta(X), \tag{2.3}$$

for all vector fields X, Y on $\tilde{M}$. It is easy to see that $g(\xi, \xi) = \epsilon = \pm 1$.

An indefinite almost contact metric structure $(\phi, \xi, \eta, \tilde{g})$ is called an indefinite trans-Sasakian structure if

$$(\tilde{\nabla}_X \phi)Y = \alpha[g(X,Y)\xi - \epsilon\eta(Y)X] + \beta[g(\phi X, Y)\xi - \epsilon\eta(Y)\phi X], \tag{2.4}$$

for functions α and β on $\tilde{M}$ of type (α, β), where $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$. On indefinite trans-Sasakian manifold we have,

$$\tilde{\nabla}_X \xi = -\alpha\epsilon\phi X + \beta\epsilon[X - \eta(X)\xi], \tag{2.5}$$

for any $X \in T\tilde{M}$ where $T\tilde{M}$ is the Lie algebra of vector fields on $\tilde{M}$.

Definition(2.1) : An n-dimensional Riemannian submanifold M of an indefinite trans-Sasakian manifold $\tilde{M}$ is called a contact CR-submanifold if
i) ξ is tangent to M,
ii) there exists on M a differentiable distribution $D : x \longrightarrow D_x \subset T_x(M)$, such that D_x is invariant under ϕ; i.e., $\phi D_x \subset D_x$, for each $x \in M$ and the orthogonal complementary distribution $D^\perp : x \longrightarrow D_x^\perp \subset T_x{}^\perp(M)$ of the distribution D on M is totally real; i.e., $\phi D_x^\perp \subset T_x^\perp(M)$, where $T_x(M)$ and $T_x^\perp(M)$ are the tangent space and the normal space of M at x respectively.

Let M be a submanifold of an indefinite trans-Sasakian manifold $\tilde{M}$ with induced metric g and let ∇ is the induced connection on the tangent bundle TM and $\nabla^\perp$ is the induced connection on the normal bundle $T^\perp M$ of M. Let $F(M)$ be the algebra of smooth functions on M and $\Gamma(TM)$ be the $F(M)$-module of smooth functions of a vector bundle TM over M.
The Gauss and Weingarten formulae are characterized by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.6}$$

$$\tilde{\nabla}_X N = -A_N X + \nabla_X^\perp N, \tag{2.7}$$

for any $X, Y \in TM$, $N \in T^\perp M$, h is the second fundamental form and A_N is the Weingarten map associated with N via

$$g(A_N X, Y) = g(h(X,Y), N). \tag{2.8}$$

For any $X \in \Gamma(TM)$ we can write,

$$\phi X = PX + FX, \tag{2.9}$$

where PX is the tangential component and FX is the normal component of ϕX. Similarly for any $N \in \Gamma(T^\perp M)$ we can put

$$\phi N = BN + CN, \tag{2.10}$$

where BN denote the tangential component and CN denote the normal component of ϕN.

The covariant derivatives of the tensor fields ϕ, P and F are defined as

(2.11) $$(\tilde{\nabla}_X\phi)Y = \tilde{\nabla}_X\phi Y - \phi\tilde{\nabla}_X Y \qquad \forall\ X, Y \in T\tilde{M},$$

(2.12) $$(\tilde{\nabla}_X P)Y = \nabla_X PY - P\nabla_X Y \qquad \forall\ X, Y \in TM,$$

(2.13) $(\tilde{\nabla}_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y \qquad \forall\ X, Y \in TM$. A submanifold M is said to be invariant if F is identically zero, i.e., $\phi X \in \Gamma(TM)$ for any $X \in \Gamma(TM)$. On the other hand, M is said to be anti-invariant if P is identically zero, i.e., $\phi X \in \Gamma(T^{\perp}M)$ for any $X \in \Gamma(TM)$.
A submanifold M of an indefinite trans-Sasakian manifold $\tilde{M}$ is called totally umbilical if

(2.14) $$h(X, Y) = g(X, Y)H,$$

for any $X, Y \in \Gamma(TM)$. The mean curvature vector H is denoted by

(2.15) $$H = \sum_{i=1}^{k} h(e_i, e_i),$$

where k is the dimension of M and $(e_1, e_2, e_3, ..., e_k)$ is the local orthonormal frame on M.
A submanifold M is said to be totally geodesic if $h(X, Y) = 0$ for each $X, Y \in \Gamma(TM)$ and is minimal if $H = 0$ on M.

3. Slant submanifold

A submanifold M of a $3-$dimensional indefinite trans-Sasakian manifold $\tilde{M}$ is said to be slant submanifold if for any $X \in M$ and $X \in TM - <\xi>$, the angle between ϕX and TM is constant. The constant angle $\theta \in [0, \pi/2]$ is then called slant angle of the slant submanifold M. Obviously if $\theta = 0$, M is invariant and if $\theta = \pi/2$, M is an anti-invariant submanifold. A slant submanifold is called proper slant if it is neither invariant nor anti-invariant submanifold. If M is a slant submanifold of a $3-$dimensional indefinite trans-Sasakian manifold, then we can decompose tangent bundle TM of M as

(3.1) $$TM = D \bigoplus <\xi>,$$

where the orthogonal complementary distribution D of $<\xi>$ is known as the slant distribution on M. If μ is ϕ-invariant of the normal bundle $T^{\perp}M$, then

(3.2) $$T^{\perp}M = FTM \bigoplus <\mu>.$$

Defining the endomorphism $P : TM \longrightarrow TM$, whose square, P^2 will be denoted by Q. Then the tensor fields on M of type $(1, 1)$ determined by these endomorphism will be denoted by the same letters, respectively P and Q.

We are already having the following result for a slant submanifold.

Theorem(3.1) : Let M be a submanifold of a $3-$dimensional indefinite trans-Sasakian manifold $\tilde{M}$, such that $\xi \in TM$. Then by [6], M is slant if and only if $\exists$ a constant $\lambda \in [0, 1]$ such that

(3.3) $$P^2 = \lambda(-I + \eta \bigotimes \xi).$$

Again, if θ is slant angle of M, then $\lambda = cos^2\theta$.

We can easily draw the following consequences for a $3-$dimensional indefinite trans-Sasakian manifold $\tilde{M}$ from [6]

(3.4) $$g(PX, PY) = cos^2\theta[g(X, Y) - \epsilon\eta(X)\eta(Y)],$$

(3.5) $$g(FX, FY) = sin^2\theta[g(X, Y) - \epsilon\eta(X)\eta(Y)],$$

for any X, Y tangent to M.

Now we prove some interesting results on slant submanifold M of a $3-$dimensional indefinite trans-Sasakian manifold $\tilde{M}$.

Theorem(3.2) : If M is a totally umbilical slant submanifold of a $3-$dimensional indefinite trans-Sasakian manifold $\tilde{M}$, and if $H \in \Gamma(\mu)$ then
(i) M is invariant.
(ii) M is not a proper slant submanifold of $\tilde{M}$.

Proof : From (2.14) we have

(3.6) $$h(PX, PY) = g(PX, PY)H.$$

Using (2.6) and (3.4) we can write

(3.7) $$\tilde{\nabla}_{PX}PY - \nabla_{PX}PY = h(PX, PY)$$

$$= g(PX, PY)H$$

$$= cos^2\theta[g(X, Y) - \epsilon\eta(X)\eta(Y)]H.$$

Replacing Y with X in (3.7) we have

(3.8) $$\tilde{\nabla}_{PX}PX - \nabla_{PX}PX = cos^2\theta[g(X, X) - \epsilon\eta(X)\eta(X)]H.$$

Using (2.9) we can have

(3.9) $$\tilde{\nabla}_{PX}\phi X - \tilde{\nabla}_{PX}FX - \nabla_{PX}PX = cos^2\theta[\ ||X||^2 - \epsilon\eta^2(X)\ ]H.$$

From (2.11) we get

(3.10) $$\phi\tilde{\nabla}_{PX}X - (\tilde{\nabla}_{PX}\phi)X - \tilde{\nabla}_{PX}FX - \nabla_{PX}PX = cos^2\theta[\ ||X||^2 - \epsilon\eta^2(X)\ ]H.$$

By using equation (2.4), (2.9), (2.14) and Gauss and Weingarten formulae we calculate

$$(3.11)\quad P\nabla_{PX}X + F\nabla_{PX}X + g(PX,X)\phi H - \alpha g(PX,X)\xi + \alpha\epsilon\eta(X)PX$$

$$-\beta g(\phi PX,X)\xi + \beta\epsilon\eta(X)\phi PX + A_{FX}PX - \nabla^{\perp}_{PX}FX - \nabla_{PX}PX$$

$$= cos^2\theta[\,||X||^2 - \epsilon\eta^2(X)\,]H.$$

Equating the normal components we get

$$(3.12)\qquad F\nabla_{PX}X - \nabla^{\perp}_{PX}FX = cos^2\theta[\,||X||^2 - \epsilon\eta^2(X)\,]H.$$

On the other hand from (3.5) we infer

$$(3.13)\qquad g(FX,FX) = sin^2\theta[g(X,X) - \epsilon\eta^2(X)],$$

for any $X \in \Gamma(TM)$. Taking the covariant derivative of the above equation w.r.t PX, we obtain

$$(3.14)\quad 2g(\tilde{\nabla}_{PX}FX,FX) = 2sin^2\theta g(\tilde{\nabla}_{PX}X,X) - 2sin^2\theta\epsilon^2\eta(X)g(\tilde{\nabla}_{PX}X,\xi)$$

$$-2sin^2\theta\epsilon^2\eta(X)g(X,\tilde{\nabla}_{PX}\xi).$$

Using the property of metric connection and using (2.12) and (2.13) we have

$$(3.15)\qquad g(\nabla^{\perp}_{PX}FX,FX) = sin^2\theta g(\nabla_{PX}X,X).$$

Now taking the inner product in (3.12) with FX, for any $X \in \Gamma(TM)$, then

$$(3.16)\quad g(F\nabla_{PX}X,FX) - g(\nabla^{\perp}_{PX}FX,FX) = cos^2\theta[\,||X||^2 - \epsilon\eta^2(X)\,]g(H,FX).$$

After using (2.6), (3.5) and (3.15) and having some brief calculation we derive

$$(3.17)\quad -\epsilon sin^2\theta\eta(X)g(\tilde{\nabla}_{PX}X,\xi) = cos^2\theta[\,||X||^2 - \epsilon\eta^2(X)\,]g(H,FX).$$

Since $\tilde{\nabla}$ is the metric connection then the above equation can be written as

$$(3.18)\quad \epsilon sin^2\theta\eta(X)g(X,\tilde{\nabla}_{PX}\xi) = cos^2\theta[\,||X||^2 - \epsilon\eta^2(X)\,]g(H,FX).$$

As $\tilde{M}$ is a $3-$dimensional indefinite trans-Sasakian manifold, then using the fact that

$$(3.19)\qquad \tilde{\nabla}_{PX}\xi = -\alpha\epsilon\phi PX + \beta\epsilon[PX - \eta(PX)\xi],$$

we can easily conclude from (3.18) and (3.19)

$$(3.20)\qquad tan^2\theta = \frac{[\,||X||^2 - \epsilon\eta^2(X)\,]g(H,FX)}{\epsilon\eta(X)g(X,-\alpha\epsilon\phi PX + \beta\epsilon PX - \beta\epsilon\xi\eta(PX)\xi)}\quad ,$$

if $H \in \Gamma(\mu)$ then, tan $\theta = 0$ =tan$(n\pi)$. Since $\theta \in [0,\pi/2]$ hence $\theta = 0$.

Therefore M is invariant i.e. M is not a proper slant manifold.

Theorem(3.3) : Every totally umbilical proper slant submanifold M of a $3-$dimensional indefinite trans-Sasakian manifold $\tilde{M}$ is totally geodesic, provided $\nabla^{\perp}_{X}H \in \Gamma(\mu)$ for any $X \in TM$ and $H \in \Gamma(\mu)$.

Proof : As $\tilde{M}$ is a $3-$dimensional indefinite trans-Sasakian manifold we have

$$(\tilde{\nabla}_X\phi)Y = \tilde{\nabla}_X\phi Y - \phi\tilde{\nabla}_X Y. \tag{3.21}$$

From the fact that $\phi Y = PY + FY$ and $\tilde{M}$ is a $3-$dimensional indefinite trans-Sasakian manifold we infer

$$\tilde{\nabla}_X PY + \tilde{\nabla}_X FY = P\nabla_X Y + F\nabla_X Y + \phi h(X,Y) + \alpha[g(X,Y)\xi - \epsilon\eta(Y)X] \tag{3.22}$$

$$+ \beta[g(\phi X,Y)\xi - \epsilon\eta(Y)\phi X].$$

Using (2.12), (2.13) and (2.14) we obtain

$$\nabla_X PY + h(X,PY) - A_{FY}X + \nabla^{\perp}_X FY = P\nabla_X Y + F\nabla_X Y + g(X,Y)\phi H \tag{3.23}$$

$$+ \alpha[g(X,Y)\xi - \epsilon\eta(Y)X] + \beta[g(\phi X,Y)\xi - \epsilon\eta(Y)\phi X].$$

Taking inner product with ϕH and using the fact that $H \in \Gamma(\mu)$, from (2.2) and (2.14) we get

$$g(X,PY)g(H,\phi H) + g(\nabla^{\perp}_X FY,\phi H) = g(X,Y)||H||^2 + \alpha g(X,Y)g(\phi H,\xi) \tag{3.24}$$

$$- \alpha\epsilon\eta(Y)g(X,\phi H) + \beta g(\phi X,Y)g(\phi H,\xi) - \beta\epsilon\eta(Y)g(\phi X,\phi H).$$

Now we consider

$$\tilde{\nabla}_X\phi H = \phi\tilde{\nabla}_X H + (\tilde{\nabla}_X\phi)H. \tag{3.25}$$

Using equation (2.6), (2.7), (2.9), (2.10) and (3.25) we calculate

$$- A_{\phi H}X + \nabla^{\perp}_X\phi H = -PA_H X - FA_H X + B\nabla^{\perp}_X H + C\nabla^{\perp}_X H \tag{3.26}$$

$$+ \alpha g(X,H)\xi - \alpha\epsilon\eta(H)X + \beta g(\phi H,H)\xi - \beta\epsilon\eta(H)\phi X.$$

Taking inner product with FY, for any $Y \in \Gamma(TM)$ we have

$$g(\nabla^{\perp}_X\phi H, FY) = -g(FA_H X, FY) + g(C\nabla^{\perp}_X H, FY) + \alpha g(X,H)g(FY,\xi) \tag{3.27}$$

$$- \alpha\epsilon\eta(H)g(X,FY) + \beta g(\phi H,H)g(FY,\xi)$$
$$- \beta\epsilon\eta(H)g(\phi X,FY).$$

Since $C\nabla^{\perp}_X H \in \Gamma(\mu)$, then by (3.5) the above equation takes the form

(3.28)$g(\nabla^{\perp}_{X}\phi H, FY) = -sin^2\theta\ [g(A_H X, Y) - \epsilon\eta(A_H X)\eta(Y)] - \alpha\epsilon\eta(H)g(X, FY)$
$- \beta\epsilon\eta(H)g(\phi X, FY).$

Using (2.7), (2.8) and (2.14) and having some brief calculations we obtain

(3.29) $g(\tilde{\nabla}_X\phi H, FY) = -sin^2\theta[g(X, Y) - \epsilon\eta(X)\eta(Y)]\ ||H||^2 - \alpha\epsilon\eta(H)g(X, FY)$
$- \beta\epsilon\eta(H)g(\phi X, FY).$

The above equation can be written as

(3.30) $g(\tilde{\nabla}_X FY, \phi H) = sin^2\theta[g(X, Y) - \epsilon\eta(X)\eta(Y)]\ ||H||^2 - \alpha\epsilon\eta(H)g(X, FY)$
$- \beta\epsilon\eta(H)g(\phi X, FY).$

Again using the fact that $H \in \Gamma(\mu)$ then by (2.7) we have

(3.31) $g(\nabla^{\perp}_X FY, \phi H) = sin^2\theta[g(X, Y) - \epsilon\eta(X)\eta(Y)]||H||^2 - \alpha\epsilon\eta(H)g(X, FY)$
$- \beta\epsilon\eta(H)g(\phi X, FY).$

From (3.24) and (3.31) we get

(3.32) $g(X, Y)||H||^2 - \alpha\epsilon\eta(Y)g(X, \phi H) + g(X, PY)g(\phi H, H) - \beta\epsilon\eta(Y)g(\phi X, \phi H)$

$= sin^2\theta g(X, Y)||H||^2 - \epsilon sin^2\theta\eta(Y)\eta(X)||H||^2$

$- \alpha\epsilon\eta(H)g(X, FY) - \beta\epsilon\eta(H)g(\phi X, FY).$

After having some calculations we infer

(3.33) $g(X, Y)||H||^2 cos^2\theta + \epsilon sin^2\theta\eta(X)\eta(Y)||H||^2 - \alpha\epsilon\eta(Y)g(X, \phi H) + g(X, PY)g(\phi H, H)$

$- \beta\epsilon\eta(Y)g(\phi X, \phi H) + \alpha\epsilon\eta(H)g(X, FY) + \beta\epsilon\eta(H)g(\phi X, FY) = 0.$

The equation (3.33) has a solution if $H = 0$. Hence M is totally geodesic in $\tilde{M}$. Therefore the proof is done.

Theorem(3.4) : Let M be a slant submanifold of a $3-$dimensional indefinite trans-Sasakian manifold $\tilde{M}$. Then Q is parallel if and only if M is anti-invariant.

Proof : Let Q be the slant angle of M in $\tilde{M}$, then for any X, Y in TM, using equation (3.3) we infer

(3.34) $P^2 Y = QY = cos^2\theta(-Y + \eta(Y)\xi).$

Putting $Y = \nabla_X Y$ we have

(3.35) $Q\nabla_X Y = cos^2\theta(-\nabla_X Y + \eta(\nabla_X Y)\xi).$

By taking the covariant derivative of (3.34) w.r.t $X \in TM$ we get

$$\nabla_X QY = cos^2\theta(-\nabla_X Y + \eta(\nabla_X Y)\xi) + g(Y, \nabla_X \xi)\xi + \eta(Y)\nabla_X \xi). \tag{3.36}$$

Again using (3.35) and (3.36) we obtain

$$(\tilde{\nabla}_X Q)Y = cos^2\theta \; g(Y, \nabla_X \xi)\xi - cos^2\theta \; \eta(Y)\nabla_X \xi. \tag{3.37}$$

Using (2.5) in (3.37) we can easily observe that if $\theta = \pi/2$ then $(\tilde{\nabla}_X Q)Y = 0$ i.e. Q is parallel and thus assertion is proved.

4. Hemislant submanifold

We assume that M is a hemislant submanifold of a $3-$dimensional indefinite trans-Sasakian manifold such that the structure vector field ξ is tangent to M. At first, we define a hemislant submanifold and then we derive integrability condition of the involved distributions D_1 and D_2.

Definition(4.1) : A submanifold M of a $3-$dimensional indefinite trans-Sasakian manifold is said to be a hemislant submanifold if there exist two orthogonal complementary distributions D_1 and D_2 satisfying the following properties :

(i) $TM = D_1 \bigoplus D_2 \bigoplus < \xi >$,
(ii)D_1 is a slant distribution with slant angle $\theta \neq \pi/2$.
(iii) D_2 is totally real that is $\phi D_2 \subseteq T^{\perp}M$.
Further if μ is ϕ-invariant subspace of the normal bundle $T^{\perp}M$, then for hemislant submanifold, the normal bundle $T^{\perp}M$ can be decomposed as,

$$T^{\perp}M = FD_1 \bigoplus FD_2 \bigoplus < \mu > . \tag{4.1}$$

Now, our task is to obtain the integrability conditions of the involved distributions.

Proposition(4.1) : Let M be a hemislant submanifold of a $3-$dimensional indefinite trans-Sasakian manifold $\tilde{M}$, then the anti-invariant distribution D_2 is integrable iff

$$A_{FZ}W = A_{FW}Z, \tag{4.2}$$

for any $Z, W \in D_2$.

Proof : For any $Z, W \in D_2$, we know

$$\phi[Z, W] = \phi\tilde{\nabla}_Z W - \phi\tilde{\nabla}_W Z. \tag{4.3}$$

After using (2.7) and (2.11) we get

$$\phi[Z, W] = -A_{FW}Z + \nabla^{\perp}_Z FW + A_{FZ}W - \nabla^{\perp}_W FZ + (\tilde{\nabla}_W \phi)Z - (\tilde{\nabla}_Z \phi)W. \tag{4.4}$$

From the fact that $\tilde{M}$ is a $3-$dimensional indefinite trans-Sasakian manifold we infer

$$(4.5)\quad \phi[Z,W] = -A_{FW}Z + \nabla^{\perp}_{Z}FW + A_{FZ}W - \nabla^{\perp}_{W}FZ - \alpha\epsilon\eta(Z)W$$

$$+ \beta[g(\phi W, Z)\xi - \epsilon\eta(Z)\phi W] - \alpha\epsilon\eta(W)Z$$

$$- \beta[g(\phi Z, W)\xi - \epsilon\eta(W)\phi Z].$$

Therefore we have

$$(4.6)\qquad \phi[Z,W] = -A_{FW}Z + \nabla^{\perp}_{Z}FW + A_{FZ}W - \nabla^{\perp}_{W}FZ.$$

As D_2 is an anti-invariant distribution, the tangential part of above equation should be identically zero, hence we obtain the required result.

Proposition(4.2) : Let M be an hemislant submanifold of a $3-$dimensional indefinite trans-Sasakian manifold $\tilde{M}$, then the invariant distribution $D_1 \bigoplus < \xi >$ is integrable iff

$$(4.7)\qquad g(h(Z,PW) + \nabla^{\perp}_{Z}FW - h(W,PZ) + \nabla^{\perp}_{W}FZ, FX) = 0.$$

for any $Z, W \in D_1 \bigoplus < \xi >$ and $X \in D_2$.

Proof : For any $Z, W \in D_1 \bigoplus < \xi >$, we have

$$(4.8)\qquad \phi[Z,W] = \phi\tilde{\nabla}_Z W - \phi\tilde{\nabla}_W Z.$$

Since $\tilde{M}$ is a $3-$dimensional indefinite trans-Sasakian manifold and using (2.11) we get

$$(4.9)\qquad \phi[Z,W] = \tilde{\nabla}_Z\phi W - (\tilde{\nabla}_Z\phi)W - \tilde{\nabla}_W\phi Z + (\tilde{\nabla}_W\phi)Z$$

Again using (2.4) and (2.9) we obtain

$$(4.10)\quad \phi[Z,W] = \tilde{\nabla}_Z PW + \tilde{\nabla}_Z FW - \tilde{\nabla}_W PZ - \tilde{\nabla}_W FZ - \alpha\epsilon\eta(Z)W$$

$$+ \beta g(\phi W, Z)\xi - \beta\epsilon\eta(Z)PW - \beta\epsilon\eta(Z)FW + \alpha\epsilon\eta(W)Z$$

$$- \beta g(\phi Z, W)\xi + \beta\epsilon\eta(W)PZ + \beta\epsilon\eta(W)FZ.$$

Taking inner product with FX for any $X \in D_2$ and using Gauss and Weingarten formulae we calculate

$$(4.11)\qquad g(\phi[Z,W], FX) = 0,$$

if

$g(h(Z,PW) + \nabla^{\perp}_{Z}FW - h(W,PZ) + \nabla^{\perp}_{W}FZ, FX) = 0.$

Since ξ is tangential to D_1, we obtain the required integrability condition.

Theorem(4.1) : Let M be a totally umbilical hemislant submanifold M of a $3-$dimensional indefinite trans-Sasakian manifold $\tilde{M}$, then at least one of the following statement is true :

(i) the dimension of anti-invariant distribution is one, i.e. dim $D^{\perp} = 1$.
(ii) The mean curvature vector $H \in \mu$.
(iii) M is proper slant submanifold of $\tilde{M}$.

Proof : Now we consider M as a totally umbilical hemislant submanifold of a $3-$dimensional indefinite trans-Sasakian manifold $\tilde{M}$. For any $Z, W \in TM$, one has

$$\tilde{\nabla}_Z \phi W = (\tilde{\nabla}_Z \phi) W + \phi \tilde{\nabla}_Z W. \tag{4.12}$$

Taking vectors $Z, W \in D_2$, then from Gauss and Weingarten formulae we have

$$-A_{FW} Z + \nabla^{\perp}_Z FW = (\tilde{\nabla}_Z \phi) W + \phi(\nabla_Z W + h(Z, W)). \tag{4.13}$$

Using (2.4), (2.9) and (2.10) we obtain

$$\begin{aligned} -A_{FW} Z + \nabla^{\perp}_Z FW &= P\nabla_Z W + F\nabla_Z W + Bh(Z, W) + Ch(Z, W) \\ &+ \alpha g(Z, W)\xi - \alpha\epsilon\eta(Z)W + \beta[g(PZ, W)\xi - \epsilon\eta(W)PZ] - \beta\epsilon\eta(W)FZ \\ &+ \beta g(FZ, W)\xi. \end{aligned} \tag{4.14}$$

Equating the tangential components we calculate

$$\begin{aligned} -A_{FW} Z &= P\nabla_Z W + Bh(Z, W) + \beta g(PZ, W)\xi - \beta\epsilon\eta(W)PZ \\ &+ \alpha g(Z, W)\xi - \alpha\epsilon\eta(W)Z. \end{aligned} \tag{4.15}$$

$$\begin{aligned} P\nabla_Z W &= A_{FW} Z - Bh(Z, W) - \beta g(PZ, W)\xi + \beta\epsilon\eta(W)PZ \\ &- \alpha g(Z, W)\xi - \alpha\epsilon\eta(W)Z. \end{aligned} \tag{4.16}$$

Taking the inner product with $V \in D_2$, we get

$$\begin{aligned} g(P\nabla_Z W, V) &= g(A_{FW} Z, V) - g(Bh(Z, W), V) + \beta\epsilon\eta(W)g(PZ, V) \\ &- \alpha\epsilon\eta(W)g(Z, V). \end{aligned} \tag{4.17}$$

From equation (2.8) and using the fact that $W \in D_2$, the above equation takes the form

$$0 = g(h(Z, V), FW) + g(Bh(Z, W), V). \tag{4.18}$$

As M is totally umbilical, we infer

$$0 = g(Z, V)g(H, FW) + g(Z, W)g(BH, V). \tag{4.19}$$

Thus the above equation has a solution if either $Z = W = V = \xi$ i.e. dim$D_2 = 1$ or $H \in \mu$ or $D_2 = 0$.

Now we give an example of a $3-$dimensional indefinite trans-Sasakian manifold:

Example : Consider a 3-dimensional submanifold of R^5 with its usual structure defined as :

The $(1,1)$ tensor ϕ is defined as:

$$\begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\eta = dt \quad \xi = \frac{\partial}{\partial t} \quad and \ g = -dx_1^2 - dx_2^2 - dy_1^2 - dy_2^2 + \eta \bigotimes \eta.$

Now for any $\theta \in [0, \Pi/2]$,

$x(u,v,t) = 2(ucos\theta, usin\theta, v, 0, t).$

If we denote by M a slant submanifold, then its tangent space TM span by the vectors:

$e_1 = \frac{\partial}{\partial u} + 2cos\theta v\frac{\partial}{\partial t} = cos\theta(2(\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial t})) + sin\theta(2(\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial t})$
$e_2 = \frac{\partial}{\partial v} = 2\frac{\partial}{\partial y_1}, e_3 = \frac{\partial}{\partial t} = \xi.$

Moreover, the vector fields:

$e_1^* = -sin\theta(2(\frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial t})) + cos\theta(2(\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial t})$
$e_2^* = 2\frac{\partial}{\partial y_2}$

form a basis of $T^{\perp}M$. Furthermore, using Gauss formula, we get $\tilde{\nabla}_{e_i}e_i = 0$, for $i = 1, 2$.Thus, we have:

$h(e_1, e_1) = 0, \ \ h(e_2, e_2) = 0, \ \ h(e_1, e_2) = 0$

and hence, we can conclude that M is totally geodesic.

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