

Slant and hemislant submanifolds of a 3–dimensional indefinite trans-Sasakian manifold

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Abstract. In this paper we would like to establish some of the properties of slant and hemislant submanifolds of an indefinite trans-Sasakian manifold. We have four sections in this paper. Section (1) is introductory. In Section (2) we recall some necessary details of an indefinite trans-Sasakian manifold. In Section (3) we have obtained some interesting properties on a totally umbilical slant submanifolds of an indefinite trans-Sasakian manifold. Finally, in Section (4), some results on integrability conditions of the distributions of hemislant submanifolds of an indefinite trans-Sasakian manifold have been obtained.

1. Introduction

The study of slant submanifolds in complex spaces was initiated by B.Y.Chen as a natural generalization of both holomorphic and totally real submanifolds in ([1],[2]). After him, A.Lotta in 1996 extended the notion to the setting of almost contact metric manifolds [3]. Further modifications regarding semislant submanifolds were introduced by N.Papaghiuc [4]. These submanifolds are a generalized version of CR-submanifolds. J.L.Cabrerizo et.al. ([5],[6]) extended the study of semislant submanifolds of Kaehler manifold to the setting of Sasakian manifolds. In particular, totally umbilical proper slant submanifold of a Kaehler manifold has also been discussed in [7]. Recently, Khan et.al. [8] carried some investigation on these submanifolds in the setting of Lorentzian paracontact manifolds.

The idea of hemislant submanifold was introduced by Carriazo as a particular class of bislant submanifolds, and he called them antislant submanifolds in [9]. Later on, in 2011 Siraj Uddin et.al. studied totally umbilical proper slant and hemislant submanifolds of an LP-cosymplectic manifold.

In the present note, our aim is to extend the study of slant and hemislant submanifolds of an indefinite trans-Sasakian manifold.

2. Preliminaries

Let \tilde{M} be an $(2n + 1)$ -dimensional indefinite almost contact metric manifold with indefinite almost contact metric structure $(\phi, \xi, \eta, \tilde{g})$, where ϕ is a tensor of type $(1, 1)$ having rank $2n$, ξ is a vector field, η is a 1-form and \tilde{g} is Riemannian metric, satisfying following properties :

$$(2.1) \quad \phi^2 X = -X + \eta(X)\xi, \quad \eta \circ \phi = 0, \quad \phi\xi = 0, \quad \eta(\xi) = 1,$$

$$(2.2) \quad \tilde{g}(\phi X, \phi Y) = \tilde{g}(X, Y) - \epsilon\eta(X)\eta(Y),$$

$$(2.3) \quad \tilde{g}(X, \xi) = \epsilon\eta(X),$$

for all vector fields X, Y on \tilde{M} . It is easy to see that $\tilde{g}(\xi, \xi) = \epsilon = \pm 1$.

An indefinite almost contact metric structure $(\phi, \xi, \eta, \tilde{g})$ is called an indefinite trans-Sasakian structure if

$$(2.4) \quad (\tilde{\nabla}_X \phi)Y = \alpha[g(X, Y)\xi - \epsilon\eta(Y)X] + \beta[g(\phi X, Y)\xi - \epsilon\eta(Y)\phi X],$$

for functions α and β on \tilde{M} of type (α, β) , where $\tilde{\nabla}$ is the Levi-Civita connection on \tilde{M} . On indefinite trans-Sasakian manifold we have,

$$(2.5) \quad \tilde{\nabla}_X \xi = -\alpha\epsilon\phi X + \beta\epsilon[X - \eta(X)\xi],$$

for any $X \in T\tilde{M}$ where $T\tilde{M}$ is the Lie algebra of vector fields on \tilde{M} .

Definition(2.1) : An n -dimensional Riemannian submanifold M of an indefinite trans-Sasakian manifold \tilde{M} is called a contact CR-submanifold if

- i) ξ is tangent to M ,
- ii) there exists on M a differentiable distribution $D : x \rightarrow D_x \subset T_x(M)$, such that D_x is invariant under ϕ ; i.e., $\phi D_x \subset D_x$, for each $x \in M$ and the orthogonal complementary distribution $D^\perp : x \rightarrow D_x^\perp \subset T_x^\perp(M)$ of the distribution D on M is totally real; i.e., $\phi D_x^\perp \subset T_x^\perp(M)$, where $T_x(M)$ and $T_x^\perp(M)$ are the tangent space and the normal space of M at x respectively.

Let M be a submanifold of an indefinite trans-Sasakian manifold \tilde{M} with induced metric g and let ∇ is the induced connection on the tangent bundle TM and ∇^\perp is the induced connection on the normal bundle $T^\perp M$ of M . Let $F(M)$ be the algebra of smooth functions on M and $\Gamma(TM)$ be the $F(M)$ -module of smooth functions of a vector bundle TM over M .

The Gauss and Weingarten formulae are characterized by

$$(2.6) \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$

$$(2.7) \quad \tilde{\nabla}_X N = -A_N X + \nabla_X^\perp N,$$

for any $X, Y \in TM, N \in T^\perp M$, h is the second fundamental form and A_N is the Weingarten map associated with N via

$$(2.8) \quad g(A_N X, Y) = g(h(X, Y), N).$$

For any $X \in \Gamma(TM)$ we can write,

$$(2.9) \quad \phi X = PX + FX,$$

where PX is the tangential component and FX is the normal component of ϕX . Similarly for any $N \in \Gamma(T^\perp M)$ we can put

$$(2.10) \quad \phi N = BN + CN,$$

where BN denote the tangential component and CN denote the normal component of ϕN .

The covariant derivatives of the tensor fields ϕ, P and F are defined as

$$(2.11) \quad (\tilde{\nabla}_X \phi)Y = \tilde{\nabla}_X \phi Y - \phi \tilde{\nabla}_X Y \quad \forall X, Y \in T\tilde{M},$$

$$(2.12) \quad (\tilde{\nabla}_X P)Y = \nabla_X PY - P\nabla_X Y \quad \forall X, Y \in TM,$$

(2.13) $(\tilde{\nabla}_X F)Y = \nabla_X^\perp FY - F\nabla_X Y \quad \forall X, Y \in TM$. A submanifold M is said to be invariant if F is identically zero, i.e., $\phi X \in \Gamma(TM)$ for any $X \in \Gamma(TM)$. On the other hand, M is said to be anti-invariant if P is identically zero, i.e., $\phi X \in \Gamma(T^\perp M)$ for any $X \in \Gamma(TM)$. A submanifold M of an indefinite trans-Sasakian manifold \tilde{M} is called totally umbilical if

$$(2.14) \quad h(X, Y) = g(X, Y)H,$$

for any $X, Y \in \Gamma(TM)$. The mean curvature vector H is denoted by

$$(2.15) \quad H = \sum_{i=1}^k h(e_i, e_i),$$

where k is the dimension of M and $(e_1, e_2, e_3, \dots, e_k)$ is the local orthonormal frame on M .

A submanifold M is said to be totally geodesic if $h(X, Y) = 0$ for each $X, Y \in \Gamma(TM)$ and is minimal if $H = 0$ on M .

3. Slant submanifold

A submanifold M of a 3-dimensional indefinite trans-Sasakian manifold \tilde{M} is said to be slant submanifold if for any $X \in M$ and $X \in TM - \langle \xi \rangle$, the angle between ϕX and TM is constant. The constant angle $\theta \in [0, \pi/2]$ is then called slant angle of the slant submanifold M . Obviously if $\theta = 0$, M is invariant and if $\theta = \pi/2$, M is an anti-invariant submanifold. A slant submanifold is called proper slant if it is neither invariant nor anti-invariant submanifold. If M is a slant submanifold of a 3-dimensional indefinite trans-Sasakian manifold, then we can decompose tangent bundle TM of M as

$$(3.1) \quad TM = D \oplus \langle \xi \rangle,$$

where the orthogonal complementary distribution D of $\langle \xi \rangle$ is known as the slant distribution on M . If μ is ϕ -invariant of the normal bundle $T^\perp M$, then

$$(3.2) \quad T^\perp M = FTM \oplus \langle \mu \rangle.$$

Defining the endomorphism $P : TM \rightarrow TM$, whose square, P^2 will be denoted by Q . Then the tensor fields on M of type $(1, 1)$ determined by these endomorphism will be denoted by the same letters, respectively P and Q .

We are already having the following result for a slant submanifold.

Theorem(3.1) : Let M be a submanifold of a 3-dimensional indefinite trans-Sasakian manifold \tilde{M} , such that $\xi \in TM$. Then by [6], M is slant if and only if \exists a constant $\lambda \in [0, 1]$ such that

$$(3.3) \quad P^2 = \lambda(-I + \eta \otimes \xi).$$

Again, if θ is slant angle of M , then $\lambda = \cos^2\theta$.

We can easily draw the following consequences for a 3–dimensional indefinite trans-Sasakian manifold \tilde{M} from [6]

$$(3.4) \quad g(PX, PY) = \cos^2\theta[g(X, Y) - \epsilon\eta(X)\eta(Y)],$$

$$(3.5) \quad g(FX, FY) = \sin^2\theta[g(X, Y) - \epsilon\eta(X)\eta(Y)],$$

for any X, Y tangent to M .

Now we prove some interesting results on slant submanifold M of a 3–dimensional indefinite trans-Sasakian manifold \tilde{M} .

Theorem(3.2) : If M is a totally umbilical slant submanifold of a 3–dimensional indefinite trans-Sasakian manifold \tilde{M} , and if $H \in \Gamma(\mu)$ then

- (i) M is invariant.
- (ii) M is not a proper slant submanifold of \tilde{M} .

Proof : From (2.14) we have

$$(3.6) \quad h(PX, PY) = g(PX, PY)H.$$

Using (2.6) and (3.4) we can write

$$(3.7) \quad \begin{aligned} \tilde{\nabla}_{PX}PY - \nabla_{PX}PY &= h(PX, PY) \\ &= g(PX, PY)H \\ &= \cos^2\theta[g(X, Y) - \epsilon\eta(X)\eta(Y)]H. \end{aligned}$$

Replacing Y with X in (3.7) we have

$$(3.8) \quad \tilde{\nabla}_{PX}PX - \nabla_{PX}PX = \cos^2\theta[g(X, X) - \epsilon\eta(X)\eta(X)]H.$$

Using (2.9) we can have

$$(3.9) \quad \tilde{\nabla}_{PX}\phi X - \tilde{\nabla}_{PX}FX - \nabla_{PX}PX = \cos^2\theta[||X||^2 - \epsilon\eta^2(X)]H.$$

From (2.11) we get

$$(3.10) \quad \phi\tilde{\nabla}_{PX}X - (\tilde{\nabla}_{PX}\phi)X - \tilde{\nabla}_{PX}FX - \nabla_{PX}PX = \cos^2\theta[||X||^2 - \epsilon\eta^2(X)]H.$$

By using equation (2.4), (2.9), (2.14) and Gauss and Weingarten formulae we calculate

$$(3.11) \quad P\nabla_{PX}X + F\nabla_{PX}X + g(PX, X)\phi H - \alpha g(PX, X)\xi + \alpha\epsilon\eta(X)PX \\ - \beta g(\phi PX, X)\xi + \beta\epsilon\eta(X)\phi PX + A_{FX}PX - \nabla_{PX}^\perp FX - \nabla_{PX}PX \\ = \cos^2\theta[\|X\|^2 - \epsilon\eta^2(X)]H.$$

Equating the normal components we get

$$(3.12) \quad F\nabla_{PX}X - \nabla_{PX}^\perp FX = \cos^2\theta[\|X\|^2 - \epsilon\eta^2(X)]H.$$

On the other hand from (3.5) we infer

$$(3.13) \quad g(FX, FX) = \sin^2\theta[g(X, X) - \epsilon\eta^2(X)],$$

for any $X \in \Gamma(TM)$. Taking the covariant derivative of the above equation w.r.t PX , we obtain

$$(3.14) \quad 2g(\tilde{\nabla}_{PX}FX, FX) = 2\sin^2\theta g(\tilde{\nabla}_{PX}X, X) - 2\sin^2\theta\epsilon^2\eta(X)g(\tilde{\nabla}_{PX}X, \xi) \\ - 2\sin^2\theta\epsilon^2\eta(X)g(X, \tilde{\nabla}_{PX}\xi).$$

Using the property of metric connection and using (2.12) and (2.13) we have

$$(3.15) \quad g(\nabla_{PX}^\perp FX, FX) = \sin^2\theta g(\nabla_{PX}X, X).$$

Now taking the inner product in (3.12) with FX , for any $X \in \Gamma(TM)$, then

$$(3.16) \quad g(F\nabla_{PX}X, FX) - g(\nabla_{PX}^\perp FX, FX) = \cos^2\theta[\|X\|^2 - \epsilon\eta^2(X)]g(H, FX).$$

After using (2.6), (3.5) and (3.15) and having some brief calculation we derive

$$(3.17) \quad -\epsilon\sin^2\theta\eta(X)g(\tilde{\nabla}_{PX}X, \xi) = \cos^2\theta[\|X\|^2 - \epsilon\eta^2(X)]g(H, FX).$$

Since $\tilde{\nabla}$ is the metric connection then the above equation can be written as

$$(3.18) \quad \epsilon\sin^2\theta\eta(X)g(X, \tilde{\nabla}_{PX}\xi) = \cos^2\theta[\|X\|^2 - \epsilon\eta^2(X)]g(H, FX).$$

As \tilde{M} is a 3-dimensional indefinite trans-Sasakian manifold, then using the fact that

$$(3.19) \quad \tilde{\nabla}_{PX}\xi = -\alpha\epsilon\phi PX + \beta\epsilon[PX - \eta(PX)\xi],$$

we can easily conclude from (3.18) and (3.19)

$$(3.20) \quad \tan^2\theta = \frac{[\|X\|^2 - \epsilon\eta^2(X)]g(H, FX)}{\epsilon\eta(X)g(X, -\alpha\epsilon\phi PX + \beta\epsilon PX - \beta\epsilon\xi\eta(PX)\xi)},$$

if $H \in \Gamma(\mu)$ then, $\tan \theta = 0 = \tan(n\pi)$. Since $\theta \in [0, \pi/2]$ hence $\theta = 0$.

Therefore M is invariant i.e. M is not a proper slant manifold.

Theorem(3.3) : Every totally umbilical proper slant submanifold M of a 3–dimensional indefinite trans-Sasakian manifold \tilde{M} is totally geodesic, provided $\nabla_X^\perp H \in \Gamma(\mu)$ for any $X \in TM$ and $H \in \Gamma(\mu)$.

Proof : As \tilde{M} is a 3–dimensional indefinite trans-Sasakian manifold we have

$$(3.21) \quad (\tilde{\nabla}_X \phi)Y = \tilde{\nabla}_X \phi Y - \phi \tilde{\nabla}_X Y.$$

From the fact that $\phi Y = PY + FY$ and \tilde{M} is a 3–dimensional indefinite trans-Sasakian manifold we infer

$$(3.22) \quad \tilde{\nabla}_X PY + \tilde{\nabla}_X FY = P\nabla_X Y + F\nabla_X Y + \phi h(X, Y) + \alpha[g(X, Y)\xi - \epsilon\eta(Y)X] \\ + \beta[g(\phi X, Y)\xi - \epsilon\eta(Y)\phi X].$$

Using (2.12), (2.13) and (2.14) we obtain

$$(3.23) \quad \nabla_X PY + h(X, PY) - A_{FY}X + \nabla_X^\perp FY = P\nabla_X Y + F\nabla_X Y + g(X, Y)\phi H \\ + \alpha[g(X, Y)\xi - \epsilon\eta(Y)X] + \beta[g(\phi X, Y)\xi - \epsilon\eta(Y)\phi X].$$

Taking inner product with ϕH and using the fact that $H \in \Gamma(\mu)$, from (2.2) and (2.14) we get

$$(3.24) \quad g(X, PY)g(H, \phi H) + g(\nabla_X^\perp FY, \phi H) = g(X, Y)\|H\|^2 + \alpha g(X, Y)g(\phi H, \xi) \\ - \alpha\epsilon\eta(Y)g(X, \phi H) + \beta g(\phi X, Y)g(\phi H, \xi) - \beta\epsilon\eta(Y)g(\phi X, \phi H).$$

Now we consider

$$(3.25) \quad \tilde{\nabla}_X \phi H = \phi \tilde{\nabla}_X H + (\tilde{\nabla}_X \phi)H.$$

Using equation (2.6), (2.7), (2.9), (2.10) and (3.25) we calculate

$$(3.26) \quad -A_{\phi H}X + \nabla_X^\perp \phi H = -PA_H X - FA_H X + B\nabla_X^\perp H + C\nabla_X^\perp H \\ + \alpha g(X, H)\xi - \alpha\epsilon\eta(H)X + \beta g(\phi H, H)\xi - \beta\epsilon\eta(H)\phi X.$$

Taking inner product with FY , for any $Y \in \Gamma(TM)$ we have

$$(3.27) \quad g(\nabla_X^\perp \phi H, FY) = -g(FA_H X, FY) + g(C\nabla_X^\perp H, FY) + \alpha g(X, H)g(FY, \xi) \\ - \alpha\epsilon\eta(H)g(X, FY) + \beta g(\phi H, H)g(FY, \xi) \\ - \beta\epsilon\eta(H)g(\phi X, FY).$$

Since $C\nabla_X^\perp H \in \Gamma(\mu)$, then by (3.5) the above equation takes the form

$$(3.28) \quad g(\nabla_X^\perp \phi H, FY) = -\sin^2\theta [g(A_H X, Y) - \epsilon\eta(A_H X)\eta(Y)] - \alpha\epsilon\eta(H)g(X, FY) - \beta\epsilon\eta(H)g(\phi X, FY).$$

Using (2.7), (2.8) and (2.14) and having some brief calculations we obtain

$$(3.29) \quad g(\tilde{\nabla}_X \phi H, FY) = -\sin^2\theta [g(X, Y) - \epsilon\eta(X)\eta(Y)] \|H\|^2 - \alpha\epsilon\eta(H)g(X, FY) - \beta\epsilon\eta(H)g(\phi X, FY).$$

The above equation can be written as

$$(3.30) \quad g(\tilde{\nabla}_X FY, \phi H) = \sin^2\theta [g(X, Y) - \epsilon\eta(X)\eta(Y)] \|H\|^2 - \alpha\epsilon\eta(H)g(X, FY) - \beta\epsilon\eta(H)g(\phi X, FY).$$

Again using the fact that $H \in \Gamma(\mu)$ then by (2.7) we have

$$(3.31) \quad g(\nabla_X^\perp FY, \phi H) = \sin^2\theta [g(X, Y) - \epsilon\eta(X)\eta(Y)] \|H\|^2 - \alpha\epsilon\eta(H)g(X, FY) - \beta\epsilon\eta(H)g(\phi X, FY).$$

From (3.24) and (3.31) we get

$$(3.32) \quad g(X, Y) \|H\|^2 - \alpha\epsilon\eta(Y)g(X, \phi H) + g(X, PY)g(\phi H, H) - \beta\epsilon\eta(Y)g(\phi X, \phi H) \\ = \sin^2\theta g(X, Y) \|H\|^2 - \epsilon\sin^2\theta \eta(Y)\eta(X) \|H\|^2 \\ - \alpha\epsilon\eta(H)g(X, FY) - \beta\epsilon\eta(H)g(\phi X, FY).$$

After having some calculations we infer

$$(3.33) \quad g(X, Y) \|H\|^2 \cos^2\theta + \epsilon\sin^2\theta \eta(X)\eta(Y) \|H\|^2 - \alpha\epsilon\eta(Y)g(X, \phi H) + g(X, PY)g(\phi H, H) \\ - \beta\epsilon\eta(Y)g(\phi X, \phi H) + \alpha\epsilon\eta(H)g(X, FY) + \beta\epsilon\eta(H)g(\phi X, FY) = 0.$$

The equation (3.33) has a solution if $H = 0$. Hence M is totally geodesic in \tilde{M} . Therefore the proof is done.

Theorem(3.4) : Let M be a slant submanifold of a 3-dimensional indefinite trans-Sasakian manifold \tilde{M} . Then Q is parallel if and only if M is anti-invariant.

Proof : Let Q be the slant angle of M in \tilde{M} , then for any X, Y in TM , using equation (3.3) we infer

$$(3.34) \quad P^2Y = QY = \cos^2\theta(-Y + \eta(Y)\xi).$$

Putting $Y = \nabla_X Y$ we have

$$(3.35) \quad Q\nabla_X Y = \cos^2\theta(-\nabla_X Y + \eta(\nabla_X Y)\xi).$$

By taking the covariant derivative of (3.34) w.r.t $X \in TM$ we get

$$(3.36) \quad \nabla_X QY = \cos^2\theta(-\nabla_X Y + \eta(\nabla_X Y)\xi) + g(Y, \nabla_X \xi)\xi + \eta(Y)\nabla_X \xi.$$

Again using (3.35) and (3.36) we obtain

$$(3.37) \quad (\tilde{\nabla}_X Q)Y = \cos^2\theta g(Y, \nabla_X \xi)\xi - \cos^2\theta \eta(Y)\nabla_X \xi.$$

Using (2.5) in (3.37) we can easily observe that if $\theta = \pi/2$ then $(\tilde{\nabla}_X Q)Y = 0$ i.e. Q is parallel and thus assertion is proved.

4. Hemislant submanifold

We assume that M is a hemislant submanifold of a 3–dimensional indefinite trans-Sasakian manifold such that the structure vector field ξ is tangent to M . At first, we define a hemislant submanifold and then we derive integrability condition of the involved distributions D_1 and D_2 .

Definition(4.1) : A submanifold M of a 3–dimensional indefinite trans-Sasakian manifold is said to be a hemislant submanifold if there exist two orthogonal complementary distributions D_1 and D_2 satisfying the following properties :

- (i) $TM = D_1 \oplus D_2 \oplus \langle \xi \rangle$,
- (ii) D_1 is a slant distribution with slant angle $\theta \neq \pi/2$.
- (iii) D_2 is totally real that is $\phi D_2 \subseteq T^\perp M$.

Further if μ is ϕ -invariant subspace of the normal bundle $T^\perp M$, then for hemislant submanifold, the normal bundle $T^\perp M$ can be decomposed as,

$$(4.1) \quad T^\perp M = FD_1 \oplus FD_2 \oplus \langle \mu \rangle .$$

Now, our task is to obtain the integrability conditions of the involved distributions.

Proposition(4.1) : Let M be a hemislant submanifold of a 3–dimensional indefinite trans-Sasakian manifold \tilde{M} , then the anti-invariant distribution D_2 is integrable iff

$$(4.2) \quad A_{FZ}W = A_{FW}Z,$$

for any $Z, W \in D_2$.

Proof : For any $Z, W \in D_2$, we know

$$(4.3) \quad \phi[Z, W] = \phi\tilde{\nabla}_Z W - \phi\tilde{\nabla}_W Z.$$

After using (2.7) and (2.11) we get

$$(4.4) \quad \phi[Z, W] = -A_{FW}Z + \nabla_Z^\perp FW + A_{FZ}W - \nabla_W^\perp FZ + (\tilde{\nabla}_W \phi)Z - (\tilde{\nabla}_Z \phi)W.$$

From the fact that \tilde{M} is a 3–dimensional indefinite trans-Sasakian manifold we infer

$$(4.5) \quad \begin{aligned} \phi[Z, W] &= -A_{FW}Z + \nabla_Z^\perp FW + A_{FZ}W - \nabla_W^\perp FZ - \alpha\epsilon\eta(Z)W \\ &+ \beta[g(\phi W, Z)\xi - \epsilon\eta(Z)\phi W] - \alpha\epsilon\eta(W)Z \\ &- \beta[g(\phi Z, W)\xi - \epsilon\eta(W)\phi Z]. \end{aligned}$$

Therefore we have

$$(4.6) \quad \phi[Z, W] = -A_{FW}Z + \nabla_Z^\perp FW + A_{FZ}W - \nabla_W^\perp FZ.$$

As D_2 is an anti-invariant distribution, the tangential part of above equation should be identically zero, hence we obtain the required result.

Proposition(4.2) : Let M be an hemislant submanifold of a 3–dimensional indefinite trans-Sasakian manifold \tilde{M} , then the invariant distribution $D_1 \oplus \langle \xi \rangle$ is integrable iff

$$(4.7) \quad g(h(Z, PW) + \nabla_Z^\perp FW - h(W, PZ) + \nabla_W^\perp FZ, FX) = 0.$$

for any $Z, W \in D_1 \oplus \langle \xi \rangle$ and $X \in D_2$.

Proof : For any $Z, W \in D_1 \oplus \langle \xi \rangle$, we have

$$(4.8) \quad \phi[Z, W] = \phi\tilde{\nabla}_Z W - \phi\tilde{\nabla}_W Z.$$

Since \tilde{M} is a 3–dimensional indefinite trans-Sasakian manifold and using (2.11) we get

$$(4.9) \quad \phi[Z, W] = \tilde{\nabla}_Z \phi W - (\tilde{\nabla}_Z \phi)W - \tilde{\nabla}_W \phi Z + (\tilde{\nabla}_W \phi)Z$$

Again using (2.4) and (2.9) we obtain

$$(4.10) \quad \begin{aligned} \phi[Z, W] &= \tilde{\nabla}_Z PW + \tilde{\nabla}_Z FW - \tilde{\nabla}_W PZ - \tilde{\nabla}_W FZ - \alpha\epsilon\eta(Z)W \\ &+ \beta g(\phi W, Z)\xi - \beta\epsilon\eta(Z)PW - \beta\epsilon\eta(Z)FW + \alpha\epsilon\eta(W)Z \\ &- \beta g(\phi Z, W)\xi + \beta\epsilon\eta(W)PZ + \beta\epsilon\eta(W)FZ. \end{aligned}$$

Taking inner product with FX for any $X \in D_2$ and using Gauss and Weingarten formulae we calculate

$$(4.11) \quad g(\phi[Z, W], FX) = 0,$$

if

$$g(h(Z, PW) + \nabla_Z^\perp FW - h(W, PZ) + \nabla_W^\perp FZ, FX) = 0.$$

Since ξ is tangential to D_1 , we obtain the required integrability condition.

Theorem(4.1) : Let M be a totally umbilical hemislant submanifold M of a 3–dimensional indefinite trans-Sasakian manifold \tilde{M} , then at least one of the following statement is true :

- (i) the dimension of anti-invariant distribution is one, i.e. $\dim D^\perp = 1$.
- (ii) The mean curvature vector $H \in \mu$.
- (iii) M is proper slant submanifold of \tilde{M} .

Proof : Now we consider M as a totally umbilical hemislant submanifold of a 3–dimensional indefinite trans-Sasakian manifold \tilde{M} . For any $Z, W \in TM$, one has

$$(4.12) \quad \tilde{\nabla}_Z \phi W = (\tilde{\nabla}_Z \phi)W + \phi \tilde{\nabla}_Z W.$$

Taking vectors $Z, W \in D_2$, then from Gauss and Weingarten formulae we have

$$(4.13) \quad -A_{FW}Z + \nabla_Z^\perp FW = (\tilde{\nabla}_Z \phi)W + \phi(\nabla_Z W + h(Z, W)).$$

Using (2.4), (2.9) and (2.10) we obtain

$$(4.14) \quad -A_{FW}Z + \nabla_Z^\perp FW = P\nabla_Z W + F\nabla_Z W + Bh(Z, W) + Ch(Z, W) \\ + \alpha g(Z, W)\xi - \alpha \epsilon \eta(Z)W + \beta[g(PZ, W)\xi - \epsilon \eta(W)PZ] - \beta \epsilon \eta(W)FZ \\ + \beta g(FZ, W)\xi.$$

Equating the tangential components we calculate

$$(4.15) \quad -A_{FW}Z = P\nabla_Z W + Bh(Z, W) + \beta g(PZ, W)\xi - \beta \epsilon \eta(W)PZ \\ + \alpha g(Z, W)\xi - \alpha \epsilon \eta(W)Z.$$

$$(4.16) \quad P\nabla_Z W = A_{FW}Z - Bh(Z, W) - \beta g(PZ, W)\xi + \beta \epsilon \eta(W)PZ \\ - \alpha g(Z, W)\xi - \alpha \epsilon \eta(W)Z.$$

Taking the inner product with $V \in D_2$, we get

$$(4.17) \quad g(P\nabla_Z W, V) = g(A_{FW}Z, V) - g(Bh(Z, W), V) + \beta \epsilon \eta(W)g(PZ, V) \\ - \alpha \epsilon \eta(W)g(Z, V).$$

From equation (2.8) and using the fact that $W \in D_2$, the above equation takes the form

$$(4.18) \quad 0 = g(h(Z, V), FW) + g(Bh(Z, W), V).$$

As M is totally umbilical, we infer

$$(4.19) \quad 0 = g(Z, V)g(H, FW) + g(Z, W)g(BH, V).$$

Thus the above equation has a solution if either $Z = W = V = \xi$ i.e. $\dim D_2 = 1$ or $H \in \mu$ or $D_2 = 0$.

Now we give an example of a 3–dimensional indefinite trans-Sasakian manifold:

Example : Consider a 3-dimensional submanifold of R^5 with its usual structure defined as :

The $(1, 1)$ tensor ϕ is defined as:

$$\begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\eta = dt \quad \xi = \frac{\partial}{\partial t} \quad \text{and} \quad g = -dx_1^2 - dx_2^2 - dy_1^2 - dy_2^2 + \eta \otimes \eta.$$

Now for any $\theta \in [0, \Pi/2]$,

$$x(u, v, t) = 2(ucos\theta, usin\theta, v, 0, t).$$

If we denote by M a slant submanifold, then its tangent space TM span by the vectors:

$$\begin{aligned} e_1 &= \frac{\partial}{\partial u} + 2cos\theta v \frac{\partial}{\partial t} = cos\theta(2(\frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial t})) + sin\theta(2(\frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial t})) \\ e_2 &= \frac{\partial}{\partial v} = 2\frac{\partial}{\partial y_1}, e_3 = \frac{\partial}{\partial t} = \xi. \end{aligned}$$

Moreover, the vector fields:

$$\begin{aligned} e_1^* &= -sin\theta(2(\frac{\partial}{\partial x_1} + y_1 \frac{\partial}{\partial t})) + cos\theta(2(\frac{\partial}{\partial x_2} + y_2 \frac{\partial}{\partial t})) \\ e_2^* &= 2\frac{\partial}{\partial y_2} \end{aligned}$$

form a basis of $T^\perp M$. Furthermore, using Gauss formula, we get $\tilde{\nabla}_{e_i} e_i = 0$, for $i = 1, 2$. Thus, we have:

$$h(e_1, e_1) = 0, \quad h(e_2, e_2) = 0, \quad h(e_1, e_2) = 0$$

and hence, we can conclude that M is totally geodesic.

References

- [1] B.Y.Chen, *Slant immersions*, Bulletin of the Australian Mathematical Society, vol. 41, no.1, pp. 135-147, 1990.
- [2] B.Y.Chen, *Geometry of slant Submanifolds*, Katholieke Universiteit Leuven, Leuven, Belgium, 1990.
- [3] A.Lotta, *Slant Submanifolds in contact geometry*, Bull.Math.Soc. Roumanie, 39(1996), 183-198.
- [4] N.Papaghiuc, *Semi-slant submanifolds of Kaehlerian manifold*, An.Stiint.Univ.Al.I.CuzaIasi, Ser.Noua, mat. 40(1)(1994), 55-61.

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- [5] J.L.Cabrerizo, A.Carriazo, L.M.Fernandez, M.Fernandez, *Semi-slant submanifolds of a Sasakian manifold*, *Geom.Dedicata* 78(1999), 183-199.
- [6] J.L.Cabrerizo, A.Carriazo, L.M.Fernandez, M.Fernandez, *Slant submanifolds in Sasakian manifolds*, *Glasg. Math. J.* 42(2000).
- [7] B.Sahin, *Every totally umbilical proper slant submanifold of a Kaehler manifold is totally geodesic*, *Results in Mathematics*, vol.54, no.1-2, pp. 167-172, 2009.
- [8] M.A.Khan, K.Singh and V.A.Khan, *Slant Submanifolds of LP-contact manifolds*, *Differential Geometry Dynamical Systems*, vol.12, pp. 102-108, 2010.
- [9] A.Carriazo, *Bi-slant immersions*, in *Proceedings of the Integrated Car Rental and Accounts Management System*, pp 88-97, Kharagpur, West Bengal, India, 2000.