The Bulletin of Society for Mathematical Services and Standards *Online: 2015-10-01*
ISSN: 2277-8020, Vol. 15, pp 1-4
doi:10.18052/www.scipress.com/BSMaSS.15.1
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GENERALIZED REVERSE DERIVATIONS ON SEMIPRIME RINGS

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KEYWORDS: Semiprime ring, Derivation, Reverse derivation, Generalized reverse derivation, Center.

ABSTRACT: In this paper we extend our ideas from reverse derivation towards the Generalized reverse derivations on semiprime rings. In this Paper, we prove that if d is a non-zero reverse derivation of a semi prime ring R and f is a generalized reverse derivation, then f is a strong commutativity preserving. Using this, we prove that R is commutative.

INTRODUCTION: Bell and Martindale [3] studied centralizing mappings of semiprime rings and proved that if d is a non-zero derivation of a prime ring R such that $[d(x),x]=0$, for all x in a non-zero left ideal of R, then R is commutative. Bell and Daif [2] investigated commutativity in prime and semiprime rings admitting a derivation or an endomorphism which is strong commutativity preserving on a non-zero right ideal. Ali and Shah [1] extended some results of Bell and Martindale [3] for generalized derivations. Bresar [6] studied centralizing mappings and derivations in prime rings and proved that if U be a non-zero left ideal of a prime ring R and d and g are derivations of R satisfying $d(u)u - ug(u) \in Z$, for all $u \in U$ and $d \neq 0$ then R is commutative. Vukman [11] studied some properties of generalized derivations of semiprime rings. Bresar and Vukman [5] have studied the notion of reverse derivation and some properties of reverse derivations. M.Samman and N.Alyamani [9] have studied some properties of reverse derivations on semiprime rings and proved that a mapping d on a semiprime ring R is a reverse derivation if and only if, it is a central derivation. Also proved that if a prime ring R admits a non-zero reverse derivation, then R is commutative. K.Suvarna and D.S.Irfana [10] have studied some properties of prime and semiprime rings with generalized derivations on a non-zero left ideal of R.

PRELIMINARIES: An additive map d from a ring R to R is called a derivation if $d(xy) = d(x)y + xd(y)$) for all x, y in R. An additive map d from a ring R to R is called a reverse derivation if $d(xy) = d(y)x + yd(x)$) for all x, y in R. An additive mapping $f: R \to R$ is said to be a right generalized derivation if there exists a derivation $d: R \to R$ such that $f(xy) = f(x)y + xd(y)$ for all $x, y \in R$. An additive mapping $f: R \to R$ is said to be a left generalized derivation if there exists a derivation $d: R \to R$ such that $f(xy) = d(x)y + xf(y)$, for all x,y in R. An additive mapping $f: R \to R$ is said to be a generalized derivation if it is both right and left generalized derivation exists. We know that an additive mapping $f: R \to R$ is a right generalized reverse derivation if there exists a derivation $d: R \to R$ such that $f(xy) = f(y)x + yd(x)$, for all $x, y \in R$ and f is a left generalized reverse derivation if there exists a derivation $d: R \to R$ such that $f(xy) = d(y)x + yf(x)$, for all x,y in R. Finally, f is a generalized reverse derivation of R associated with d if it is both right and left generalized reverse derivation of R. A mapping $f: R \to R$ is called centralizing reverse derivation if $[x,f(x)] \in Z$, for all $x \in R$. If $[x, f(x)] = 0$, for all x in R, then f is called commuting reverse derivation. A mapping $f: R \to R$ is called strong commutativity preserving if $[f(x), f(y)] = [x, y]$ for all $x, y \in R$. Throughout this paper, R will denote a semi prime ring and Z its center.

Remark 1: For a non-zero element $a \in Z$, if $ab \in Z$, then $b \in Z$.

To prove the main results we require the following lemmas:

Lemma 1: If f is an additive mapping from R to R such that f is centralizing on a left ideal U of R, then $f(x) \in Z$, for all $x \in U \cap Z$.

Proof: Since f is centralizing on U, we have $[f(x+y),x+y]\in Z$, for all $x,y\in U$. This implies that

$\Rightarrow [f(x),y]+[f(y),x]\in Z$ (1)

Now if $x\in Z$, then from equation (1), we have,

$\Rightarrow [f(x),y]\in Z$

We replace y by $f(x)y$, then

$\Rightarrow f(x)[f(x),y]\in Z$

If $[f(x),y]=0$, then $f(x)\in C_R(U)$, the centralizer of U in R and by [3] belongs to Z. But on the other hand, if $[f(x),y]\neq 0$, it again follows from the Remark 1 that $f(x)\in Z$. ■

Lemma 2: Let R be a semiprime ring and U a non-zero ideal of R. If Z in R centralizes the set $[U,U]$, then Z centralizes U.

Proof: Let Z centralizes $[U,U]$. Then for all x,y in U, we have,

$\Rightarrow z[x,xy]=[x,xy]z$, which can be rewritten as $zx[x,y]=x[x,y]z$.

Hence $[z,x][x,y]=0$, for all x,y in U (2)

If we replace y by yz, then we get,

$\Rightarrow [z,x]\ U\ [z,x]=\{0\}$

Since U is an ideal, it follows that,

$\Rightarrow [z,x]\ UR\ [z,x]U=\{0\}=U[z,x]RU[z,x]$

So that, $[z,x]U=U[z,x]=\{0\}$.

Thus $[[z,x],x]=0$, for all x in U and by[4], z centralizes U. ■

Now we prove the following results:

Theorem 1: Let $d:R\to R$ be a non-zero derivation of a semiprime ring R and f be a Generalized reverse derivation on a non-zero left ideal U of R. If f acts as a homomorphism on U, then f is strong commutativity preserving on U .

Proof: We assume that f acts as a homomorphism on U and f be a generalized reverse derivation on U. Then,

$\Rightarrow f(xy)=f(x)f(y)=f(y)x+yd(x)$, for all x,y in U (3)

We replace y by yz, $z\in U$, the second equality of equation (3), we have,

$\Rightarrow f(x)f(yz)=f(yz)x+yzd(x)=f(y)f(z)x+yzd(x)$ (4)

Since f is a homomorphism, on the other hand, we have,

$\Rightarrow f(x)f(yz)=f(x)f(y)f(z)$

$=f(xy)f(z)$

$=(f(y)x+yd(x))f(z)$

$=f(y)xf(z)+yd(x)f(z)$

$=f(y)f(z)x+yd(x)f(z)$ (5)

From equations (4) and (5), we get,

$\Rightarrow yd(x)f(z)=yd(x)z$

$\Rightarrow yd(x)(f(z)-z)=0$ (6)

We replace z by $[z,y]$ in equation (6), we get,

$\Rightarrow yd(x)(f([z,y])-[z,y])=0$

$\Rightarrow d(x)y(f([z,y])-[z,y])=0$

By replacing y by $(f([z,y])-[z,y])rd(x), r\in R$, we get,

$\Rightarrow d(x)(f([z,y])-[z,y])rd(x)(f([z,y])-[z,y])=0$

$\Rightarrow d(x)(f([z,y])-[z,y])Rd(x)(f([z,y])-[z,y])=0$

Since R is semiprime, we have,

$\Rightarrow d(x)(f([z,y])-[z,y])=0$

Since $d\neq 0$, we have,
$\Rightarrow(f([z,y])-[z,y])=0$
$\Rightarrow f([z,y])=[z,y]$
$\Rightarrow[f(z),f(y)]=[z,y]$
Hence f is a strong commutativity preserving on U. ■

Theorem 2: Let R be a semiprime ring and f be a Generalized reverse derivation on a non-zero left ideal U of R . If f acts as a homomorphism on U, then f is Commuting on U.

Proof: From the above Theorem1, f is strong commutativity preserving. Then for all $x,y\in U$, we have,
$\Rightarrow[x,yx]=[f(x), f(yx)]$
$\Rightarrow[x,y]x=[x,y]f(x)$
$\Rightarrow[x,y]f(x)-[x,y]x=0$
$\Rightarrow[x,y](f(x)-x)=0$ (7)

From $[x,xy]=[f(x),f(xy)]$ we can similarly show that $(f(x)-x)[x,y]=0$, for all $x,y\in U$
We replace y by ry, then we get,
$\Rightarrow[x,r]y(f(x)-x)=0$
This implies that $[x,r]U(f(x)-x)=0$ and so, $[x,r]RU(f(x)-x)=0$

Since R is semiprime, it must contain a family $w=\{P_\alpha/\alpha\epsilon\Lambda\}$ of prime ideals such that $\cap P_\alpha=0$. If P is a member of w and $x\in U$, then from the last equation , we have, $U(f(x)-x)\subseteq P$ (or) $[x,R]\subseteq P$.
Suppose there exists $y\in U$ such that $[y,R]\not\subseteq P$. This implies that $U(f(y)-y)\subseteq P$.
Let z be any element of U such that $[y+z,R]\subseteq P$. This means that $[z,R]\not\subseteq P$ and hence $(f(z)-z)\subseteq P$. On the other hand if $[y+z,R]\subseteq P$, then $U(f(y+z)-(y+z))\subseteq P$. This implies that $U(f(z)-z)\subseteq P$.
Thus we conclude that $U(f(z)-z)\subseteq P$ for all $z\in U$ and hence $[U,U](f(z)-z)\subseteq P$ for all $z\in U$.
Since P is arbitrary and $\cap P_\alpha=0$, we have, $[U,U](f(z)-z)=\{0\}$, for all $z\in U$.
Similarly, we can show that $(f(z)-z)[U,U]=\{0\}$.
This implies that $(f(z)-z)\in C_R[U,U]$, for all $z\in U$.
By Lemma 2 and by [7], we have $(f(x)-x)\in C_R(U)$, for all $z\in U$. Thus we have $[f(x)-x,x]=0$, for all $x\in U$. This implies that $[f(x),x]=0$, for all $x\in U$.
Hence f is commuting on U. ■

Theorem 3: Let R be a semiprime ring and $d:R\rightarrow R$ be a non-zero derivation. If f *is* a Generalized reverse derivation on a left ideal U of R, then R *i*s commutative.

Proof: Since f is commuting on U, by the above Theorem 2, then we have,
$[f(x),x]=0$, for all $x\in U$ (8)
We replace x by $x+y$, in the above equation, then we get,
$\Rightarrow[f(x),y]+[f(y),x]=0$ (9)
Now by substituting $y=xy$, then we get,
$\Rightarrow[yd(x),x]=0$ (10)
We replace y by ry, then we get,
$\Rightarrow[r,x]yd(x)=0$
$\Rightarrow[r,x]Ud(x)=0$, for all $x\in U$ and $r\in R$.
Since R is semiprime ring, it must contain a family $w=\{P_\alpha/\alpha\epsilon\Lambda\}$ of prime ideals such that $\cap P_\alpha=0$.

If P is a member of w and $x \in U$, then from the last equation $[R,x] \subseteq P$ or $d(x) \subseteq P$. Since d is non-zero on R, then by [8], d is non-zero on U. Suppose $d(x) \nsubseteq P$, for some $x \in U$, then $[R,x] \subseteq P$. Suppose $z \in U$ is such that $z \notin Z$, then $d(z) \subseteq P$ and $x+z \notin Z$. This implies $d(x+z) \subseteq P$ and so $d(x) \subseteq P$, is a contradiction to our assumption that $d(x) \nsubseteq P$. So, this implies $z \in Z$, for all $z \in U$.
Thus U is commutative and hence by [8], R is commutative. ■

Theorem 4: Let U be a left ideal of a semiprime ring R suchthat $U \cap Z \neq 0$.Let d be a non-zero derivation and f be a Generalized reverse derivation on R such that f is Centralizing on U. Then R is commutative.

Proof: We assume that $Z \neq 0$ because f is commuting on U, then there is nothing to prove.
Since f is centralizing on U, we have $[f(x),x] \in Z$, for all $x \in U$. If we replace x by $(x+y)$, then $[f(x+y),x+y] \in Z$, for all $x,y \in U$.

$\Rightarrow [f(x),y]+[f(y),x] \in Z$, for all $x,y \in U$ (11)

We replace x by yz in equ.(11), then

$\Rightarrow [f(z),y]y+z[d(y),y]+[f(y),y]z \in Z$

Now, by Lemma 1, $f(z) \in Z$ and therefore, the above equation becomes,

$\Rightarrow z[d(y),y]+[f(y),y]z \in Z$

But f is centralizing on U, we have,

$\Rightarrow [f(y),y]z \in Z$ and consequently $z[d(y),y] \in Z$.

Since z is non-zero, it follows from the Remark 1 that $[d(y),y] \in Z$.
This implies that d is centralizing on U and hence by[3], we conclude that R is commutative. ■

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