GENERALIZED REVERSE DERIVATIONS ON SEMIPRIME RINGS

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ABSTRACT: In this paper we extend our ideas from reverse derivation towards the Generalized reverse derivations on semiprime rings. In this Paper, we prove that if d is a non-zero reverse derivation of a semi prime ring R and f is a generalized reverse derivation, then f is a strong commutativity preserving. Using this, we prove that R is commutative.

INTRODUCTION: Bell and Martindale [3] studied centralizing mappings of semiprime rings and proved that if d is a non-zero derivation of a prime ring R such that [d(x),x]=0, for all x in a non-zero left ideal of R, then R is commutative. Bell and Daif [2] investigated commutativity in prime and semiprime rings admitting a derivation or an endomorphism which is strong commutativity preserving on a non-zero right ideal. Ali and Shah [1] extended some results of Bell and Martindale [3] for generalized derivations. Bresar [6] studied centralizing mappings and derivations in prime rings and proved that if U be a non-zero left ideal of a prime ring R and d and g are derivations of R satisfying d(u)u − ug(u) ∈ Z, for all u∈U and d≠ 0 then R is commutative. Vukman [11] studied some properties of generalized derivations of semiprime rings. Bresar and Vukman [5] have studied the notion of reverse derivation and some properties of reverse derivations. M.Samman and N.Alyamani [9] have studied some properties of reverse derivations on semiprime rings and proved that a mapping d on a semiprime ring R is a reverse derivation if and only if, it is a central derivation. Also proved that if a prime ring R admits a non-zero reverse derivation, then R is commutative. K.Suvarna and D.S.Irfana [10] have studied some properties of prime and semiprime rings with generalized derivations on a non-zero left ideal of R.

PRELIMINARIES: An additive map d from a ring R to R is called a derivation if d(xy) = $d(x)y + xd(y)$ for all x, y in R. An additive map d from a ring R to R is called a reverse derivation if $d(xy) = d(y)x + yd(x)$ for all x, y in R. An additive mapping f : R → R is said to be a right generalised derivation if there exists a derivation $d : R → R$ such that $f(xy) = f(x)y + xd(y)$ for all x, y ∈ R. An additive mapping $f : R → R$ is said to be a left generalised derivation if there exists a derivation $d : R → R$ such that $f(xy) = f(y)x + yf(x)$, for all x, y ∈ R. An additive mapping $f : R → R$ is said to be a generalised derivation if it is both right and left generalised derivation exists. We know that an additive mapping $f : R → R$ is a right generalised reverse derivation if there exists a derivation $d : R → R$ such that $f(xy) = f(y)x + yd(x)$, for all x, y ∈ R and f is a left generalised reverse derivation if there exists a derivation $d : R → R$ such that $f(xy) = f(y)x + yf(x)$, for all x, y ∈ R. Finally, f is a generalised reverse derivation of R associated with d if it is both right and left generalised reverse derivation of R. A mapping $f : R → R$ is called centralizing reverse derivation if $[xf(x)] ∈ Z$, for all x ∈ R. If $[xf(x)] = 0$, for all x ∈ R, then f is called commuting reverse derivation. A mapping $f : R → R$ is called strong commutativity preserving if $[f(x),f(y)] = [x,y]$ for all x, y ∈ R. Throughout this paper, R will denote a semi prime ring and Z its center.

Remark 1: For a non-zero elementa ∈ Z, if ab ∈ Z, then b ∈ Z.

To prove the main results we require the following lemmas:

Lemma 1: If f is an additive mapping from R to R such that f is centralizing on a left ideal U of R, then f(x) ∈ Z, for all x ∈ U ∩ Z.
**Proof:** Since \( f \) is centralizing on \( U \), we have \([f(x+y), x+y] \in Z\), for all \( x, y \in U \). This implies that
\[
[f(x), y] + [f(y), x] = [x, y] \in Z.
\]
(1)

Now if \( x \in Z \), then from equation (1), we have,
\[
[f(x), y] \in Z.
\]
We replace \( y \) by \( f(x)y \), then
\[
[f(x), f(x)y] \in Z.
\]
If \([f(x), y] = 0\), then \( f(x) \in C_R(U) \), the centralizer of \( U \) in \( R \) and by [3] belongs to \( Z \). But on the other hand, if \([f(x), y] \neq 0\), it again follows from the Remark 1 that \( f(x) \in Z \).

**Lemma 2:** Let \( R \) be a semiprime ring and \( U \) a non-zero ideal of \( R \). If \( Z \) in \( R \) centralizes the set \([U, U]\), then \( Z \) centralizes \( U \).

**Proof:** Let \( Z \) centralizes \([U, U]\). Then for all \( x, y \) in \( U \), we have,
\[
[z, x][y] = [x, y][z],
\]
which can be rewritten as \( z[x, y] = x[y, z] \).

Hence \([z, x][y] = 0\), for all \( x, y \) in \( U \)
\[
(2)
\]
If we replace \( y \) by \( yz \), then we get,
\[
[z, x][x] = \{0\}.
\]
Since \( U \) is an ideal, it follows that,
\[
[z, x][U] = \{0\} = U[z, x]R[U[z, x]]
\]
So that, \([z, x]U = U[z, x] = \{0\}\).

Thus \([z, x][x] = 0\), for all \( x \) in \( U \) and by [4], \( z \) centralizes \( U \).

Now we prove the following results:

**Theorem 1:** Let \( d: R \rightarrow R \) be a non-zero derivation of a semiprime ring \( R \) and \( f \) be a Generalized reverse derivation on a non-zero left ideal \( U \) of \( R \). If \( f \) acts as a homomorphism on \( U \), then \( f \) is strong commutativity preserving on \( U \).

**Proof:** We assume that \( f \) acts as a homomorphism on \( U \) and \( f \) be a generalized reverse derivation on \( U \). Then,
\[
f(xy) = f(x)f(y) = f(y)x + yd(x), \text{ for all } x, y \text{ in } U
\]
(3)
We replace \( y \) by \( yz \), then we get,
\[
f(xy) = f(xyz) + yzd(x) = f(y)(f(x)z) + yd(x)
\]
(4)
Since \( f \) is a homomorphism, on the other hand, we have,
\[
f(xy) = f(x)f(y) = f(x)(yf(z))
\]
(5)
From equations (4) and (5), we get,
\[
yd(x)(f(z)) = yd(x)z
\]
(6)
We replace \( z \) by \([z, y]\) in equation (6), we get,
\[
yd(x)(f([z, y])) = 0
\]
(7)
By replacing \( y \) by \([f([z, y])][z, y]]d(x), r \in R \), we get,
\[
d(x)(f([z, y])) - [z, y]) = 0
\]
(8)
Since \( R \) is semiprime, we have,
\[
d(x)(f([z, y])) - [z, y]) = 0
\]
Since $d \neq 0$, we have,
$$\Rightarrow (f([z,y]) - [z,y]) = 0$$
$$\Rightarrow f([z,y]) = [z,y]$$
$$\Rightarrow [f(z), f(y)] = [z,y]$$
Hence $f$ is a strong commutivity preserving on $U$.

**Theorem 2:** Let $R$ be a semiprime ring and $f$ be a Generalized reverse derivation on a non-zero left ideal $U$ of $R$. If $f$ acts as a homomorphism on $U$, then $f$ is Commuting on $U$.

**Proof:** From the above Theorem1, $f$ is strong commutativity preserving. Then for all $x, y \in U$, we have,
$$\Rightarrow [x, y] = [f(x), f(y)]$$
$$\Rightarrow [x, y'][x] = [x', y]'(x)$$
$$\Rightarrow [x, y]f(x) - [x, y]x = 0$$
$$\Rightarrow [x, y]f(x) = 0$$
(7)

From $[x, y] = [f(x), f(y)]$ we can similarly show that $(f(x) - x)[x, y] = 0$, for all $x, y \in U$
We replace $y$ by $y'$, then we get,
$$\Rightarrow [x, y']f(x) = 0$$
This implies that $[x, y]U(f(x) - x) = 0$ and so, $[x, y]RU(f(x) - x) = 0$

Since $R$ is semiprime, it must contain a family $w = \{P_{x}/\alpha \in \Lambda\}$ of prime ideals such that $\cap P_{x} = 0$. If $P$ is a member of $w$ and $x \in U$, then from the last equation, we have, $U(f(x) - x) \subseteq P$ (or) $[x, y] \subseteq P$.
Suppose there exists $y \in U$ such that $[y, R] \not\subseteq P$. This implies that $U(f(y) - y) \subseteq P$.
Let $z$ be any element of $U$ such that $[y + z, R] \subseteq P$. This means that $[z, R] \not\subseteq P$ and hence $(z - z) \subseteq P$. On the other hand if $[y + z, R] \subseteq P$, then $U(f(y + z) - (y + z)) \subseteq P$. This implies that $U(f(z) - z) \subseteq P$.
Thus we conclude that $U(f(z) - z) \subseteq P$ for all $z \in U$ and hence $[U, U](f(z) - z) \subseteq P$ for all $z \in U$.
Since $P$ is arbitrary and $\cap P_{x} = 0$, we have, $[U, U](f(z) - z) = \{0\}$, for all $z \in U$.
Similarly, we can show that $(f(z) - z)[U, U] = \{0\}$.
This implies that $(f(z) - z) \in C_{U}[U, U]$, for all $z \in U$.
By Lemma 2 and by [7], we have $(f(x) - x) \in C_{U}(U)$, for all $z \in U$. Thus we have $(f(x) - x, x) = 0$, for all $x \in U$. This implies that $(f(x), x) = 0$, for all $x \in U$.
Hence $f$ is commuting on $U$.

**Theorem 3:** Let $R$ be a semiprime ring and $d: R \to R$ be a non-zero derivation. If $f$ is a Generalized reverse derivation on a left ideal $U$ of $R$, then $R$ is commutative.

**Proof:** Since $f$ is commuting on $U$, by the above Theorem 2, then we have,
$$[f(x), x] = 0$$
for all $x \in U$.
We replace $x$ by $x + y$, in the above equation, then we get,
$$\Rightarrow [f(x), y] + [f(y), x] = 0$$
(9)
Now by substituting $y = xy$, then we get,
$$\Rightarrow [y, x][y, x] = 0$$
(10)
We replace $y$ by $ry$, then we get,
$$\Rightarrow [r, x][y, x] = 0$$
$$\Rightarrow [r, x]Ud(x) = 0$$
for all $x \in U$ and $r \in R$.
Since $R$ is semiprime ring, it must contain a family $w = \{P_{x}/\alpha \in \Lambda\}$ of prime ideals such that $\cap P_{x} = 0$. 


If $P$ is a member of $w$ and $x \in U$, then from the last equation $[R,x] \subseteq P$ or $d(x) \subseteq P$. Since $d$ is non-zero on $R$, then by [8], $d$ is non-zero on $U$. Suppose $d(x) \not\subseteq P$, for some $x \in U$, then $[R,x] \subseteq P$. Suppose $z \in U$ is such that $z \not\in Z$, then $d(z) \subseteq P$ and $x + z \not\in Z$. This implies $d(x + z) \subseteq P$ and so $d(x) \subseteq P$, is a contradiction to our assumption that $d(x) \not\subseteq P$. So, this implies $z \in Z$, for all $z \in U$.

Thus $U$ is commutative and hence by [8], $R$ is commutative.

**Theorem 4:** Let $U$ be a left ideal of a semiprime ring $R$ such that $U \cap Z \neq 0$. Let $d$ be a non-zero derivation and $f$ be a Generalized reverse derivation on $R$ such that $f$ is Centralizing on $U$. Then $R$ is commutative.

**Proof:** We assume that $Z \neq 0$ because $f$ is commuting on $U$, then there is nothing to prove. Since $f$ is centralizing on $U$, we have $[f(x),x] \in Z$, for all $x \in U$. If we replace $x$ by $(x + y)$, then $[f(x + y),x + y] \in Z$, for all $x,y \in U$.

$$\Rightarrow [f(x),y] + [f(y),x] \in Z, \text{ for all } x,y \in U$$

(11)

We replace $x$ by $yz$ in equ.(11), then $[f(z),y] + z[d(y),y] + [f(y),y]z \in Z$.

Now, by Lemma 1, $f(z) \in Z$ and therefore, the above equation becomes,

$$z[d(y),y] + [f(y),y]z \in Z$$

But $f$ is centralizing on $U$, we have,

$$[f(y),y]z \in Z$$

and consequently $z[d(y),y] \in Z$.

Since $z$ is non-zero, it follows from the Remark 1 that $[d(y),y] \in Z$.

This implies that $d$ is centralizing on $U$ and hence by [3], we conclude that $R$ is commutative.

**References:**


