GENERALIZED REVERSE DERIVATIONS ON SEMIPRIME RINGS

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ABSTRACT: In this paper we extend our ideas from reverse derivation towards the Generalized reverse derivations on semiprime rings. In this Paper, we prove that if \( d \) is a non-zero reverse derivation of a semi prime ring \( R \) and \( f \) is a generalized reverse derivation, then \( f \) is a strong commutativity preserving. Using this, we prove that \( R \) is commutative.

INTRODUCTION: Bell and Martindale [3] studied centralizing mappings of semiprime rings and proved that if \( d \) is a non-zero derivation of a prime ring \( R \) such that \([d(x),x]=0\), for all \( x \) in a non-zero left ideal of \( R \), then \( R \) is commutative. Bell and Daif [2] investigated commutativity in prime and semiprime rings admitting a derivation or an endomorphism which is strong commutativity preserving on a non-zero right ideal. Ali and Shah [1] extended some results of Bell and Martindale [3] for generalized derivations. Bresar [6] studied centralizing mappings and derivations in prime rings and proved that if \( U \) be a non-zero left ideal of a prime ring \( R \) and \( d \) and \( g \) are derivations of \( R \) satisfying \( d(u)u - ug(u) \in Z \), for all \( u \in U \) and \( d \neq 0 \) then \( R \) is commutative. Vukman [11] studied some properties of generalised derivations of semiprime rings. Bresar and Vukman [5] have studied the notion of reverse derivation and some properties of reverse derivations. M.Samman and N.Alyamani [9] have studied some properties of reverse derivations on semiprime rings and proved that a mapping \( d \) on a semiprime ring \( R \) is a reverse derivation if and only if, it is a central derivation. Also proved that if a prime ring \( R \) admits a non-zero reverse derivation, then \( R \) is commutative.

K.Suvarna and D.S.Irfana [10] have studied some properties of prime and semiprime rings with generalized derivations on a non-zero left ideal of \( R \).

PRELIMINARIES: An additive map \( d \) from a ring \( R \) to \( R \) is called a derivation if \( d(xy) = d(x)y + xd(y) \) for all \( x, y \) in \( R \). An additive map \( d \) from a ring \( R \) to \( R \) is called a reverse derivation if \( d(xy) = d(y)x + yd(x) \) for all \( x, y \) in \( R \). An additive mapping \( f: R \rightarrow R \) is said to be a right generalized derivation if there exists a derivation \( d: R \rightarrow R \) such that \( f(xy) = f(x)y + xd(y) \) for all \( x, y \) in \( R \). An additive mapping \( f: R \rightarrow R \) is said to be a left generalized derivation if there exists a derivation \( d: R \rightarrow R \) such that \( f(xy) = d(x)y + xf(y) \), for all \( x, y \) in \( R \). An additive mapping \( f: R \rightarrow R \) is said to be a generalized derivation if it is both right and left generalized derivation exists. We know that an additive mapping \( f: R \rightarrow R \) is a right generalized reverse derivation if there exists a derivation \( d: R \rightarrow R \) such that \( f(xy) = f(x)y + xd(y) \), for all \( x, y \) in \( R \) and \( f \) is a left generalized reverse derivation if there exists a derivation \( d: R \rightarrow R \) such that \( f(xy) = d(y)x + yf(x) \), for all \( x, y \) in \( R \). Finally, \( f \) is a generalized reverse derivation of \( R \) associated with \( d \) if it is both right and left generalized reverse derivation of \( R \). A mapping \( f: R \rightarrow R \) is called centralizing reverse derivation if \([xf(x)] \in Z\), for all \( x \) in \( R \). If \([x, f(x)] = 0\), for all \( x \) in \( R \), then \( f \) is called commuting reverse derivation. A mapping \( f: R \rightarrow R \) is called strong commutativity preserving if \([f(x), f(y)] = [x, y]\) for all \( x, y \) in \( R \). Throughout this paper, \( R \) will denote a semiprime ring and \( Z \) its center.

Remark 1: For a non-zero elementa \( a \in Z \), if \( ab \in Z \), then \( b \in Z \).

To prove the main results we require the following lemmas:

Lemma 1: If \( f \) is an additive mapping from \( R \) to \( R \) such that \( f \) is centralizing on a left ideal \( U \) of \( R \), then \( f(x) \in Z \), for all \( x \in U \cap Z \).
Proof: Since $f$ is centralizing on $U$, we have $[f(x+y),x+y] \in Z$, for all $x, y \in U$. This implies that
\[ [f(x),y]+[f(y),x] \in Z \quad (1) \]
Now if $x \in Z$, then from equation (1), we have,
\[ [f(x),y] \in Z \]
We replace $y$ by $f(x)y$, then
\[ f(x)[f(x)y,y] \in Z \]
If $[f(x),y] = 0$, then $f(x) \in C_R(U)$, the centralizer of $U$ in $R$ and by [3] belongs to $Z$. But on the other hand, if $[f(x),y] \neq 0$, it again follows from the Remark 1 that $f(x) \in Z$. \hfill \blacksquare

**Lemma 2:** Let $R$ be a semiprime ring and $U$ a non-zero ideal of $R$. If $Z$ in $R$ centralizes the set $[U,U]$, then $Z$ centralizes $U$.

**Proof:** Let $Z$ centralizes $[U,U]$. Then for all $x,y$ in $U$, we have,
\[ z[x,y]=x[y,z], \text{ which can be rewritten as } z[x,y]=x[y,z]. \]
Hence $[z,x][y,z]=0$, for all $x, y$ in $U$ \quad (2)
If we replace $y$ by $yz$, then we get,
\[ [z,y]U=[z,x]=\{0\} \]
Since $U$ is an ideal, it follows that,
\[ [z,x]UR[z,x]=U[z,x]=\{0\} \]
Thus $[[x,y],x]=0$, for all $x$ in $U$ and by [4], $z$ centralizes $U$. \hfill \blacksquare

Now we prove the following results:

**Theorem 1:** Let $d: R \rightarrow R$ be a non-zero derivation of a semiprime ring $R$ and $f$ be a Generalized reverse derivation on a non-zero left ideal $U$ of $R$. If $f$ acts as a homomorphism on $U$, then $f$ is strong commutativity preserving on $U$.

**Proof:** We assume that $f$ acts as a homomorphism on $U$ and $f$ be a generalized reverse derivation on $U$. Then,
\[ f(xy) = f(x)f(y) = f(y)x + yd(x), \text{ for all } x, y \text{ in } U \quad (3) \]
We replace $y$ by $yz$, $z \in U$, the second equality of equation (3), we have,
\[ f(yz)=f(y)z+yd(x)=f(y)f(z)x+yd(x) \quad (4) \]
Since $f$ is a homomorphism, on the other hand, we have,
\[ f(x)f(yz) = f(x)f(y)f(z) = f(xy)f(z) = f((y)x+yd(x))f(z) = f(y)f((x+z)y)d(x)f(z) = f(y)(x+z)d(x)f(z) \quad (5) \]
From equations (4) and (5), we get,
\[ yd(x)f(z)=yd(x)z \quad (6) \]
We replace $z$ by $[x,y]$ in equation (6), we get,
\[ yd(x)f([x,y])-y([x,y])=0 \]
\[ =d(x)(f([x,y])-y([x,y]))=0 \]
By replacing $y$ by $(f([x,y])-y([x,y])r)Rd(x), r \in R$, we get,
\[ d(x)(f([x,y])-y([x,y]))Rd(x)(f([x,y])-y([x,y]))=0 \]
\[ =d(x)(f([x,y])-y([x,y]))Rd(x)(f([x,y])-y([x,y]))=0 \]
Since $R$ is semiprime, we have,
\[ d(x)(f([x,y])-y([x,y]))=0 \]
\[ =d(x)(f([x,y])-y([x,y]))=0 \]
Since $d \neq 0$, we have,
\[ \Rightarrow (f([z,y])-[z,y])=0 \]
\[ \Rightarrow f([z,y])=[z,y] \]
\[ \Rightarrow [f(z),f(y)]=[z,y] \]
Hence $f$ is a strong commutativity preserving on $U$.

**Theorem 2:** Let $R$ be a semiprime ring and $f$ be a Generalized reverse derivation on a non-zero left ideal $U$ of $R$. If $f$ acts as a homomorphism on $U$, then $f$ is Commuting on $U$.

**Proof:** From the above Theorem 1, $f$ is strong commutativity preserving. Then for all $x,y \in U$, we have,
\[ \Rightarrow [x,y]=f([x,y]) \]
\[ \Rightarrow [x,y]=f(x)f(y) \]
\[ \Rightarrow [x,y]-[x,y]=0 \]
\[ \Rightarrow [x,y]=0 \]

From $[x,y]=f([x,y])$ we can similarly show that $f([x]-x)[x,y]=0$, for all $x,y \in U$
We replace $y$ by $ry$, then we get,
\[ \Rightarrow [x,r]=f([x]-x)=0 \]
This implies that $[x,r]U(f(x)-x)\subseteq 0$ and so, $[x,r] RU(f(x)-x)= 0$

Since $R$ is semiprime, it must contain a family $w = \{P_{\alpha} / \alpha \in \Lambda \}$ of prime ideals such that $\cap P_{\alpha} = 0$. If $P$ is a member of $w$ and $x \in U$, then from the last equation, we have, $U(f(x)-x) \subseteq P$ (or) $[x,R] \subseteq P$.
Suppose there exists $y \in U$ such that $[y,R] \not\subseteq P$. This implies that $U(f(y)-y) \subseteq P$.
Let $z$ be any element of $U$ such that $[y+z,R] \subseteq P$. This means that $[z,R] \not\subseteq P$ and hence $(f(z)-z) \subseteq P$. On the other hand if $[y+z,R] \subseteq P$, then $U(f(y+z)-(y+z)) \subseteq P$. This implies that $U(f(z)-z) \subseteq P$.
Thus we conclude that $U(f(z)-z) \subseteq P$ for all $z \in U$ and hence $[U,U](f(z)-z) \subseteq P$ for all $z \in U$.
Since $P$ is arbitrary and $\cap P_{\alpha} = 0$, we have, $[U,U](f(z)-z) = \{0\}$, for all $z \in U$.
Similarly, we can show that $(f(z)-z)U \subseteq P$, for all $z \in U$.
This implies that $(f(z)-z) \in C_R[U,U]$, for all $z \in U$.
By Lemma 2 and by [7], we have $([x,x]-x) \in C_R(U)$, for all $x \in U$. Thus we have $[f(x)-x,x]=0$, for all $x \in U$.
This implies that $[f(x),x]=0$, for all $x \in U$.
Hence $f$ is commuting on $U$.

**Theorem 3:** Let $R$ be a semiprime ring and $d:R \rightarrow R$ be a non-zero derivation. If $f$ is a Generalized reverse derivation on a left ideal $U$ of $R$, then $R$ is commutative.

**Proof:** Since $f$ is commuting on $U$, by the above Theorem 2, then we have,
\[ [f(x),x]=0 \quad \text{for all } x \in U \]
We replace $x$ by $x+y$, in the above equation, then we get,
\[ \Rightarrow [f(x),y]+[f(y),x]=0 \]
Now by substituting $y=xy$, then we get,
\[ \Rightarrow [yv(x),x]=0 \]
We replace $y$ by $ry$, then we get,
\[ \Rightarrow [rx,v(x)]=0 \]
\[ \Rightarrow [r,x]Ud(x)=0 \quad \text{for all } x \in U \text{ and } r \in R. \]
Since $R$ is semiprime ring, it must contain a family $w = \{P_{\alpha} / \alpha \in \Lambda \}$ of prime ideals such that $\cap P_{\alpha} = 0$.
If $P$ is a member of $w$ and $x \in U$, then from the last equation $[R,x] \subseteq P$ or $d(x) \subseteq P$. Since $d$ is non-zero on $R$, then by [8], $d$ is non-zero on $U$. Suppose $d(x) \not\subseteq P$, for some $x \in U$, then $[R,x] \subseteq P$. Suppose $z \in U$ is such that $z \notin Z$, then $d(z) \subseteq P$ and $x + z \notin Z$. This implies $d(x + z) \subseteq P$ and so $d(x) \subseteq P$, is a contradiction to our assumption that $d(x) \not\subseteq P$. So, this implies $z \in Z$, for all $z \in U$.

Thus $U$ is commutative and hence by [8], $R$ is commutative.

**Theorem 4:** Let $U$ be a left ideal of a semiprime ring $R$ such that $U \cap Z \neq 0$. Let $d$ be a non-zero derivation and $f$ be a Generalized reverse derivation on $R$ such that $f$ is Centralizing on $U$. Then $R$ is commutative.

**Proof:** We assume that $Z \neq 0$ because $f$ is commuting on $U$, then there is nothing to prove.

Since $f$ is centralizing on $U$, we have $[f(x),x] \in Z$, for all $x \in U$. If we replace $x$ by $(x+y)$, then $[f(x+y),x+y] \in Z$, for all $x,y \in U$.

$$\Rightarrow [f(x),y] + [f(y),x] \in Z,$$  \hspace{1cm} (11)

We replace $x$ by $yz$ in equ.(11), then

$$\Rightarrow [f(z),y]y + z[d(y),y] + [f(y),y]z \in Z$$

Now, by Lemma 1, $f(z) \in Z$ and therefore, the above equation becomes,

$$\Rightarrow z[d(y),y] + [f(y),y]z \in Z$$

But $f$ is centralizing on $U$, we have,

$$\Rightarrow [f(y),y]z \in Z \text{ and consequently } z[d(y),y] \in Z.$$  

Since $z$ is non-zero, it follows from the Remark 1 that $[d(y),y] \in Z$.

This implies that $d$ is centralizing on $U$ and hence by [3], we conclude that $R$ is commutative.

**References:**


