

## GENERALIZED REVERSE DERIVATIONS ON SEMIPRIME RINGS

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**ABSTRACT:** In this paper we extend our ideas from reverse derivation towards the Generalized reverse derivations on semiprime rings. In this Paper, we prove that if  $d$  is a non-zero reverse derivation of a semi prime ring  $R$  and  $f$  is a generalized reverse derivation, then  $f$  is a strong commutativity preserving. Using this, we prove that  $R$  is commutative.

**INTRODUCTION:** Bell and Martindale [3] studied centralizing mappings of semiprime rings and proved that if  $d$  is a non-zero derivation of a prime ring  $R$  such that  $[d(x),x]=0$ , for all  $x$  in a non-zero left ideal of  $R$ , then  $R$  is commutative. Bell and Daif [2] investigated commutativity in prime and semiprime rings admitting a derivation or an endomorphism which is strong commutativity preserving on a non-zero right ideal. Ali and Shah [1] extended some results of Bell and Martindale [3] for generalized derivations. Bresar [6] studied centralizing mappings and derivations in prime rings and proved that if  $U$  be a non-zero left ideal of a prime ring  $R$  and  $d$  and  $g$  are derivations of  $R$  satisfying  $d(u)u - ug(u) \in Z$ , for all  $u \in U$  and  $d \neq 0$  then  $R$  is commutative. Vukman [11] studied some properties of generalized derivations of semiprime rings. Bresar and Vukman [5] have studied the notion of reverse derivation and some properties of reverse derivations. M.Samman and N.Alyamani [9] have studied some properties of reverse derivations on semiprime rings and proved that a mapping  $d$  on a semiprime ring  $R$  is a reverse derivation if and only if, it is a central derivation. Also proved that if a prime ring  $R$  admits a non-zero reverse derivation, then  $R$  is commutative. K.Suvarna and D.S.Irfana [10] have studied some properties of prime and semiprime rings with generalized derivations on a non-zero left ideal of  $R$ .

**PRELIMINARIES:** An additive map  $d$  from a ring  $R$  to  $R$  is called a derivation if  $d(xy) = d(x)y + xd(y)$  for all  $x, y$  in  $R$ . An additive map  $d$  from a ring  $R$  to  $R$  is called a reverse derivation if  $d(xy) = d(y)x + yd(x)$  for all  $x, y$  in  $R$ . An additive mapping  $f: R \rightarrow R$  is said to be a right generalized derivation if there exists a derivation  $d: R \rightarrow R$  such that  $f(xy) = f(x)y + xd(y)$  for all  $x, y \in R$ . An additive mapping  $f: R \rightarrow R$  is said to be a left generalized derivation if there exists a derivation  $d: R \rightarrow R$  such that  $f(xy) = d(x)y + xf(y)$ , for all  $x, y$  in  $R$ . An additive mapping  $f: R \rightarrow R$  is said to be a generalized derivation if it is both right and left generalized derivation exists. We know that an additive mapping  $f: R \rightarrow R$  is a right generalized reverse derivation if there exists a derivation  $d: R \rightarrow R$  such that  $f(xy) = f(y)x + yd(x)$ , for all  $x, y \in R$  and  $f$  is a left generalized reverse derivation if there exists a derivation  $d: R \rightarrow R$  such that  $f(xy) = d(y)x + yf(x)$ , for all  $x, y$  in  $R$ . Finally,  $f$  is a generalized reverse derivation of  $R$  associated with  $d$  if it is both right and left generalized reverse derivation of  $R$ . A mapping  $f: R \rightarrow R$  is called centralizing reverse derivation if  $[x, f(x)] \in Z$ , for all  $x \in R$ . If  $[x, f(x)] = 0$ , for all  $x$  in  $R$ , then  $f$  is called commuting reverse derivation. A mapping  $f: R \rightarrow R$  is called strong commutativity preserving if  $[f(x), f(y)] = [x, y]$  for all  $x, y \in R$ . Throughout this paper,  $R$  will denote a semi prime ring and  $Z$  its center.

**Remark 1:** For a non-zero element  $a \in Z$ , if  $ab \in Z$ , then  $b \in Z$ .

To prove the main results we require the following lemmas:

**Lemma 1:** If  $f$  is an additive mapping from  $R$  to  $R$  such that  $f$  is centralizing on a left ideal  $U$  of  $R$ , then  $f(x) \in Z$ , for all  $x \in U \cap Z$ .

**Proof:** Since  $f$  is centralizing on  $U$ , we have  $[f(x+y), x+y] \in Z$ , for all  $x, y \in U$ . This implies that

$$\Rightarrow [f(x), y] + [f(y), x] \in Z \quad (1)$$

Now if  $x \in Z$ , then from equation (1), we have,

$$\Rightarrow [f(x), y] \in Z$$

We replace  $y$  by  $f(x)y$ , then

$$\Rightarrow f(x)[f(x), y] \in Z$$

If  $[f(x), y] = 0$ , then  $f(x) \in C_R(U)$ , the centralizer of  $U$  in  $R$  and by [3] belongs to  $Z$ . But on the other hand, if  $[f(x), y] \neq 0$ , it again follows from the Remark 1 that  $f(x) \in Z$ . ■

**Lemma 2:** Let  $R$  be a semiprime ring and  $U$  a non-zero ideal of  $R$ . If  $Z$  in  $R$  centralizes the set  $[U, U]$ , then  $Z$  centralizes  $U$ .

**Proof:** Let  $Z$  centralizes  $[U, U]$ . Then for all  $x, y$  in  $U$ , we have,

$$\Rightarrow z[x, xy] = [x, xy]z, \text{ which can be rewritten as } zx[x, y] = x[x, y]z.$$

Hence  $[z, x][x, y] = 0$ , for all  $x, y$  in  $U$  (2)

If we replace  $y$  by  $yz$ , then we get,

$$\Rightarrow [z, x] U [z, x] = \{0\}$$

Since  $U$  is an ideal, it follows that,

$$\Rightarrow [z, x] UR [z, x] U = \{0\} = U[z, x]RU[z, x]$$

So that,  $[z, x]U = U[z, x] = \{0\}$ .

Thus  $[[z, x], x] = 0$ , for all  $x$  in  $U$  and by [4],  $z$  centralizes  $U$ . ■

Now we prove the following results:

**Theorem 1:** Let  $d: R \rightarrow R$  be a non-zero derivation of a semiprime ring  $R$  and  $f$  be a Generalized reverse derivation on a non-zero left ideal  $U$  of  $R$ . If  $f$  acts as a homomorphism on  $U$ , then  $f$  is strong commutativity preserving on  $U$ .

**Proof:** We assume that  $f$  acts as a homomorphism on  $U$  and  $f$  be a generalized reverse derivation on  $U$ . Then,

$$\Rightarrow f(xy) = f(x)f(y) = f(y)x + yd(x), \text{ for all } x, y \text{ in } U \quad (3)$$

We replace  $y$  by  $yz$ ,  $z \in U$ , the second equality of equation (3), we have,

$$\Rightarrow f(x)f(yz) = f(yz)x + yzd(x) = f(y)f(z)x + yzd(x) \quad (4)$$

Since  $f$  is a homomorphism, on the other hand, we have,

$$\Rightarrow f(x)f(yz) = f(x)f(y)f(z)$$

$$= f(xy)f(z)$$

$$= (f(y)x + yd(x))f(z)$$

$$= f(y)xf(z) + yd(x)f(z)$$

$$= f(y)f(z)x + yd(x)f(z) \quad (5)$$

From equations (4) and (5), we get,

$$\Rightarrow yd(x)f(z) = yd(x)z$$

$$\Rightarrow yd(x)(f(z) - z) = 0 \quad (6)$$

We replace  $z$  by  $[z, y]$  in equation (6), we get,

$$\Rightarrow yd(x)(f([z, y]) - [z, y]) = 0$$

$$\Rightarrow d(x)y(f([z, y]) - [z, y]) = 0$$

By replacing  $y$  by  $(f([z, y]) - [z, y])rd(x)$ ,  $r \in R$ , we get,

$$\Rightarrow d(x)(f([z, y]) - [z, y])rd(x)(f([z, y]) - [z, y]) = 0$$

$$\Rightarrow d(x)(f([z, y]) - [z, y])Rd(x)(f([z, y]) - [z, y]) = 0$$

Since  $R$  is semiprime, we have,

$$\Rightarrow d(x)(f([z, y]) - [z, y]) = 0$$

Since  $d \neq 0$ , we have,

$$\Rightarrow (f([z,y]) - [z,y]) = 0$$

$$\Rightarrow f([z,y]) = [z,y]$$

$$\Rightarrow [f(z), f(y)] = [z,y]$$

Hence  $f$  is a strong commutativity preserving on  $U$ . ■

**Theorem 2:** Let  $R$  be a semiprime ring and  $f$  be a Generalized reverse derivation on a non-zero left ideal  $U$  of  $R$ . If  $f$  acts as a homomorphism on  $U$ , then  $f$  is Commuting on  $U$ .

**Proof:** From the above Theorem 1,  $f$  is strong commutativity preserving. Then for all  $x, y \in U$ , we have,

$$\Rightarrow [x, yx] = [f(x), f(yx)]$$

$$\Rightarrow [x, y]x = [x, y]f(x)$$

$$\Rightarrow [x, y]f(x) - [x, y]x = 0$$

$$\Rightarrow [x, y](f(x) - x) = 0 \tag{7}$$

From  $[x, xy] = [f(x), f(xy)]$  we can similarly show that  $(f(x) - x)[x, y] = 0$ , for all  $x, y \in U$

We replace  $y$  by  $ry$ , then we get,

$$\Rightarrow [x, r]y(f(x) - x) = 0$$

This implies that  $[x, r]U(f(x) - x) = 0$  and so,  $[x, r]RU(f(x) - x) = 0$

Since  $R$  is semiprime, it must contain a family  $w = \{P_\alpha / \alpha \in \Lambda\}$  of prime ideals such that  $\bigcap P_\alpha = 0$ . If  $P$  is a member of  $w$  and  $x \in U$ , then from the last equation, we have,  $U(f(x) - x) \subseteq P$  (or)  $[x, R] \subseteq P$ . Suppose there exists  $y \in U$  such that  $[y, R] \not\subseteq P$ . This implies that  $U(f(y) - y) \subseteq P$ .

Let  $z$  be any element of  $U$  such that  $[y + z, R] \subseteq P$ . This means that  $[z, R] \not\subseteq P$  and hence  $(f(z) - z) \subseteq P$ . On the other hand if  $[y + z, R] \subseteq P$ , then  $U(f(y + z) - (y + z)) \subseteq P$ . This implies that  $U(f(z) - z) \subseteq P$ .

Thus we conclude that  $U(f(z) - z) \subseteq P$  for all  $z \in U$  and hence  $[U, U](f(z) - z) \subseteq P$  for all  $z \in U$ .

Since  $P$  is arbitrary and  $\bigcap P_\alpha = 0$ , we have,  $[U, U](f(z) - z) = \{0\}$ , for all  $z \in U$ .

Similarly, we can show that  $(f(z) - z)[U, U] = \{0\}$ .

This implies that  $(f(z) - z) \in C_R[U, U]$ , for all  $z \in U$ .

By Lemma 2 and by [7], we have  $(f(x) - x) \in C_R(U)$ , for all  $x \in U$ . Thus we have  $[f(x) - x, x] = 0$ , for all  $x \in U$ . This implies that  $[f(x), x] = 0$ , for all  $x \in U$ .

Hence  $f$  is commuting on  $U$ . ■

**Theorem 3:** Let  $R$  be a semiprime ring and  $d: R \rightarrow R$  be a non-zero derivation. If  $f$  is a Generalized reverse derivation on a left ideal  $U$  of  $R$ , then  $R$  is commutative.

**Proof:** Since  $f$  is commuting on  $U$ , by the above Theorem 2, then we have,

$$[f(x), x] = 0, \text{ for all } x \in U \tag{8}$$

We replace  $x$  by  $x + y$ , in the above equation, then we get,

$$\Rightarrow [f(x), y] + [f(y), x] = 0 \tag{9}$$

Now by substituting  $y = xy$ , then we get,

$$\Rightarrow [yd(x), x] = 0 \tag{10}$$

We replace  $y$  by  $ry$ , then we get,

$$\Rightarrow [r, x]yd(x) = 0$$

$$\Rightarrow [r, x]Ud(x) = 0, \text{ for all } x \in U \text{ and } r \in R.$$

Since  $R$  is semiprime ring, it must contain a family  $w = \{P_\alpha / \alpha \in \Lambda\}$  of prime ideals such that  $\bigcap P_\alpha = 0$ .

If  $P$  is a member of  $w$  and  $x \in U$ , then from the last equation  $[R, x] \subseteq P$  or  $d(x) \subseteq P$ . Since  $d$  is non-zero on  $R$ , then by [8],  $d$  is non-zero on  $U$ . Suppose  $d(x) \not\subseteq P$ , for some  $x \in U$ , then  $[R, x] \subseteq P$ . Suppose  $z \in U$  is such that  $z \notin Z$ , then  $d(z) \subseteq P$  and  $x + z \notin Z$ . This implies  $d(x + z) \subseteq P$  and so  $d(x) \subseteq P$ , is a contradiction to our assumption that  $d(x) \not\subseteq P$ . So, this implies  $z \in Z$ , for all  $z \in U$ .

Thus  $U$  is commutative and hence by [8],  $R$  is commutative. ■

**Theorem 4:** Let  $U$  be a left ideal of a semiprime ring  $R$  such that  $U \cap Z \neq 0$ . Let  $d$  be a non-zero derivation and  $f$  be a Generalized reverse derivation on  $R$  such that  $f$  is Centralizing on  $U$ . Then  $R$  is commutative.

**Proof:** We assume that  $Z \neq 0$  because  $f$  is commuting on  $U$ , then there is nothing to prove.

Since  $f$  is centralizing on  $U$ , we have  $[f(x), x] \in Z$ , for all  $x \in U$ . If we replace  $x$  by  $(x + y)$ , then  $[f(x + y), x + y] \in Z$ , for all  $x, y \in U$ .

$$\Rightarrow [f(x), y] + [f(y), x] \in Z, \text{ for all } x, y \in U \quad (11)$$

We replace  $x$  by  $yz$  in equ.(11), then

$$\Rightarrow [f(z), y]y + z[d(y), y] + [f(y), y]z \in Z$$

Now, by Lemma 1,  $f(z) \in Z$  and therefore, the above equation becomes,

$$\Rightarrow z[d(y), y] + [f(y), y]z \in Z$$

But  $f$  is centralizing on  $U$ , we have,

$$\Rightarrow [f(y), y]z \in Z \text{ and consequently } z[d(y), y] \in Z.$$

Since  $z$  is non-zero, it follows from the Remark 1 that  $[d(y), y] \in Z$ .

This implies that  $d$  is centralizing on  $U$  and hence by [3], we conclude that  $R$  is commutative. ■

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