

GENERALIZED REVERSE DERIVATIONS ON SEMIPRIME RINGS

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ABSTRACT: In this paper we extend our ideas from reverse derivation towards the Generalized reverse derivations on semiprime rings. In this Paper, we prove that if d is a non-zero reverse derivation of a semi prime ring R and f is a generalized reverse derivation, then f is a strong commutativity preserving. Using this, we prove that R is commutative.

INTRODUCTION: Bell and Martindale [3] studied centralizing mappings of semiprime rings and proved that if d is a non-zero derivation of a prime ring R such that $[d(x),x]=0$, for all x in a non-zero left ideal of R , then R is commutative. Bell and Daif [2] investigated commutativity in prime and semiprime rings admitting a derivation or an endomorphism which is strong commutativity preserving on a non-zero right ideal. Ali and Shah [1] extended some results of Bell and Martindale [3] for generalized derivations. Bresar [6] studied centralizing mappings and derivations in prime rings and proved that if U be a non-zero left ideal of a prime ring R and d and g are derivations of R satisfying $d(u)u - ug(u) \in Z$, for all $u \in U$ and $d \neq 0$ then R is commutative. Vukman [11] studied some properties of generalized derivations of semiprime rings. Bresar and Vukman [5] have studied the notion of reverse derivation and some properties of reverse derivations. M.Samman and N.Alyamani [9] have studied some properties of reverse derivations on semiprime rings and proved that a mapping d on a semiprime ring R is a reverse derivation if and only if, it is a central derivation. Also proved that if a prime ring R admits a non-zero reverse derivation, then R is commutative. K.Suvarna and D.S.Irfana [10] have studied some properties of prime and semiprime rings with generalized derivations on a non-zero left ideal of R .

PRELIMINARIES: An additive map d from a ring R to R is called a derivation if $d(xy) = d(x)y + xd(y)$ for all x, y in R . An additive map d from a ring R to R is called a reverse derivation if $d(xy) = d(y)x + yd(x)$ for all x, y in R . An additive mapping $f: R \rightarrow R$ is said to be a right generalized derivation if there exists a derivation $d: R \rightarrow R$ such that $f(xy) = f(x)y + xd(y)$ for all $x, y \in R$. An additive mapping $f: R \rightarrow R$ is said to be a left generalized derivation if there exists a derivation $d: R \rightarrow R$ such that $f(xy) = d(x)y + xf(y)$, for all x, y in R . An additive mapping $f: R \rightarrow R$ is said to be a generalized derivation if it is both right and left generalized derivation exists. We know that an additive mapping $f: R \rightarrow R$ is a right generalized reverse derivation if there exists a derivation $d: R \rightarrow R$ such that $f(xy) = f(y)x + yd(x)$, for all $x, y \in R$ and f is a left generalized reverse derivation if there exists a derivation $d: R \rightarrow R$ such that $f(xy) = d(y)x + yf(x)$, for all x, y in R . Finally, f is a generalized reverse derivation of R associated with d if it is both right and left generalized reverse derivation of R . A mapping $f: R \rightarrow R$ is called centralizing reverse derivation if $[x, f(x)] \in Z$, for all $x \in R$. If $[x, f(x)] = 0$, for all x in R , then f is called commuting reverse derivation. A mapping $f: R \rightarrow R$ is called strong commutativity preserving if $[f(x), f(y)] = [x, y]$ for all $x, y \in R$. Throughout this paper, R will denote a semi prime ring and Z its center.

Remark 1: For a non-zero element $a \in Z$, if $ab \in Z$, then $b \in Z$.

To prove the main results we require the following lemmas:

Lemma 1: If f is an additive mapping from R to R such that f is centralizing on a left ideal U of R , then $f(x) \in Z$, for all $x \in U \cap Z$.

Proof: Since f is centralizing on U , we have $[f(x+y), x+y] \in Z$, for all $x, y \in U$. This implies that

$$\Rightarrow [f(x), y] + [f(y), x] \in Z \quad (1)$$

Now if $x \in Z$, then from equation (1), we have,

$$\Rightarrow [f(x), y] \in Z$$

We replace y by $f(x)y$, then

$$\Rightarrow f(x)[f(x), y] \in Z$$

If $[f(x), y] = 0$, then $f(x) \in C_R(U)$, the centralizer of U in R and by [3] belongs to Z . But on the other hand, if $[f(x), y] \neq 0$, it again follows from the Remark 1 that $f(x) \in Z$. ■

Lemma 2: Let R be a semiprime ring and U a non-zero ideal of R . If Z in R centralizes the set $[U, U]$, then Z centralizes U .

Proof: Let Z centralizes $[U, U]$. Then for all x, y in U , we have,

$$\Rightarrow z[x, xy] = [x, xy]z, \text{ which can be rewritten as } zx[x, y] = x[x, y]z.$$

Hence $[z, x][x, y] = 0$, for all x, y in U (2)

If we replace y by yz , then we get,

$$\Rightarrow [z, x] U [z, x] = \{0\}$$

Since U is an ideal, it follows that,

$$\Rightarrow [z, x] UR [z, x] U = \{0\} = U [z, x] RU [z, x]$$

So that, $[z, x] U = U [z, x] = \{0\}$.

Thus $[[z, x], x] = 0$, for all x in U and by [4], z centralizes U . ■

Now we prove the following results:

Theorem 1: Let $d: R \rightarrow R$ be a non-zero derivation of a semiprime ring R and f be a Generalized reverse derivation on a non-zero left ideal U of R . If f acts as a homomorphism on U , then f is strong commutativity preserving on U .

Proof: We assume that f acts as a homomorphism on U and f be a generalized reverse derivation on U . Then,

$$\Rightarrow f(xy) = f(x)f(y) = f(y)x + yd(x), \text{ for all } x, y \text{ in } U \quad (3)$$

We replace y by yz , $z \in U$, the second equality of equation (3), we have,

$$\Rightarrow f(x)f(yz) = f(yz)x + yzd(x) = f(y)f(z)x + yzd(x) \quad (4)$$

Since f is a homomorphism, on the other hand, we have,

$$\Rightarrow f(x)f(yz) = f(x)f(y)f(z)$$

$$= f(xy)f(z)$$

$$= (f(y)x + yd(x))f(z)$$

$$= f(y)xf(z) + yd(x)f(z)$$

$$= f(y)f(z)x + yd(x)f(z) \quad (5)$$

From equations (4) and (5), we get,

$$\Rightarrow yd(x)f(z) = yd(x)z$$

$$\Rightarrow yd(x)(f(z) - z) = 0 \quad (6)$$

We replace z by $[z, y]$ in equation (6), we get,

$$\Rightarrow yd(x)(f([z, y]) - [z, y]) = 0$$

$$\Rightarrow d(x)y(f([z, y]) - [z, y]) = 0$$

By replacing y by $(f([z, y]) - [z, y])rd(x)$, $r \in R$, we get,

$$\Rightarrow d(x)(f([z, y]) - [z, y])rd(x)(f([z, y]) - [z, y]) = 0$$

$$\Rightarrow d(x)(f([z, y]) - [z, y])Rd(x)(f([z, y]) - [z, y]) = 0$$

Since R is semiprime, we have,

$$\Rightarrow d(x)(f([z, y]) - [z, y]) = 0$$

Since $d \neq 0$, we have,

$$\Rightarrow (f([z,y]) - [z,y]) = 0$$

$$\Rightarrow f([z,y]) = [z,y]$$

$$\Rightarrow [f(z), f(y)] = [z,y]$$

Hence f is a strong commutativity preserving on U . ■

Theorem 2: Let R be a semiprime ring and f be a Generalized reverse derivation on a non-zero left ideal U of R . If f acts as a homomorphism on U , then f is Commuting on U .

Proof: From the above Theorem 1, f is strong commutativity preserving. Then for all $x, y \in U$, we have,

$$\Rightarrow [x, yx] = [f(x), f(yx)]$$

$$\Rightarrow [x, y]x = [x, y]f(x)$$

$$\Rightarrow [x, y]f(x) - [x, y]x = 0$$

$$\Rightarrow [x, y](f(x) - x) = 0 \tag{7}$$

From $[x, xy] = [f(x), f(xy)]$ we can similarly show that $(f(x) - x)[x, y] = 0$, for all $x, y \in U$

We replace y by ry , then we get,

$$\Rightarrow [x, r]y(f(x) - x) = 0$$

This implies that $[x, r]U(f(x) - x) = 0$ and so, $[x, r]RU(f(x) - x) = 0$

Since R is semiprime, it must contain a family $w = \{P_\alpha / \alpha \in \Lambda\}$ of prime ideals such that $\bigcap P_\alpha = 0$. If P is a member of w and $x \in U$, then from the last equation, we have, $U(f(x) - x) \subseteq P$ (or) $[x, R] \subseteq P$. Suppose there exists $y \in U$ such that $[y, R] \not\subseteq P$. This implies that $U(f(y) - y) \subseteq P$.

Let z be any element of U such that $[y + z, R] \subseteq P$. This means that $[z, R] \not\subseteq P$ and hence $(f(z) - z) \subseteq P$. On the other hand if $[y + z, R] \subseteq P$, then $U(f(y + z) - (y + z)) \subseteq P$. This implies that $U(f(z) - z) \subseteq P$.

Thus we conclude that $U(f(z) - z) \subseteq P$ for all $z \in U$ and hence $[U, U](f(z) - z) \subseteq P$ for all $z \in U$.

Since P is arbitrary and $\bigcap P_\alpha = 0$, we have, $[U, U](f(z) - z) = \{0\}$, for all $z \in U$.

Similarly, we can show that $(f(z) - z)[U, U] = \{0\}$.

This implies that $(f(z) - z) \in C_R[U, U]$, for all $z \in U$.

By Lemma 2 and by [7], we have $(f(x) - x) \in C_R(U)$, for all $x \in U$. Thus we have $[f(x) - x, x] = 0$, for all $x \in U$. This implies that $[f(x), x] = 0$, for all $x \in U$.

Hence f is commuting on U . ■

Theorem 3: Let R be a semiprime ring and $d: R \rightarrow R$ be a non-zero derivation. If f is a Generalized reverse derivation on a left ideal U of R , then R is commutative.

Proof: Since f is commuting on U , by the above Theorem 2, then we have,

$$[f(x), x] = 0, \text{ for all } x \in U \tag{8}$$

We replace x by $x + y$, in the above equation, then we get,

$$\Rightarrow [f(x), y] + [f(y), x] = 0 \tag{9}$$

Now by substituting $y = xy$, then we get,

$$\Rightarrow [yd(x), x] = 0 \tag{10}$$

We replace y by ry , then we get,

$$\Rightarrow [r, x]yd(x) = 0$$

$$\Rightarrow [r, x]Ud(x) = 0, \text{ for all } x \in U \text{ and } r \in R.$$

Since R is semiprime ring, it must contain a family $w = \{P_\alpha / \alpha \in \Lambda\}$ of prime ideals such that $\bigcap P_\alpha = 0$.

If P is a member of w and $x \in U$, then from the last equation $[R, x] \subseteq P$ or $d(x) \subseteq P$. Since d is non-zero on R , then by [8], d is non-zero on U . Suppose $d(x) \not\subseteq P$, for some $x \in U$, then $[R, x] \subseteq P$. Suppose $z \in U$ is such that $z \notin Z$, then $d(z) \subseteq P$ and $x + z \notin Z$. This implies $d(x + z) \subseteq P$ and so $d(x) \subseteq P$, is a contradiction to our assumption that $d(x) \not\subseteq P$. So, this implies $z \in Z$, for all $z \in U$.

Thus U is commutative and hence by [8], R is commutative. ■

Theorem 4: Let U be a left ideal of a semiprime ring R such that $U \cap Z \neq 0$. Let d be a non-zero derivation and f be a Generalized reverse derivation on R such that f is Centralizing on U . Then R is commutative.

Proof: We assume that $Z \neq 0$ because f is commuting on U , then there is nothing to prove.

Since f is centralizing on U , we have $[f(x), x] \in Z$, for all $x \in U$. If we replace x by $(x + y)$, then $[f(x + y), x + y] \in Z$, for all $x, y \in U$.

$$\Rightarrow [f(x), y] + [f(y), x] \in Z, \text{ for all } x, y \in U \quad (11)$$

We replace x by yz in equ.(11), then

$$\Rightarrow [f(z), y]y + z[d(y), y] + [f(y), y]z \in Z$$

Now, by Lemma 1, $f(z) \in Z$ and therefore, the above equation becomes,

$$\Rightarrow z[d(y), y] + [f(y), y]z \in Z$$

But f is centralizing on U , we have,

$$\Rightarrow [f(y), y]z \in Z \text{ and consequently } z[d(y), y] \in Z.$$

Since z is non-zero, it follows from the Remark 1 that $[d(y), y] \in Z$.

This implies that d is centralizing on U and hence by [3], we conclude that R is commutative. ■

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