

# The simplest Meijer's G-function $G_{0,1}^{1,0}$ as the radial functions of the Hydrogen atom

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**Abstract:** The purpose of this paper is to show that how the sub-family of Meijer's G-functions,  $G_{0,1}^{1,0}$ , deduces all the radial states of the Hydrogen atom. By introducing two postulates and using operator method similar with the method used for the simple harmonic oscillator we obtain all the excited radial states of the Hydrogen atom from the ground state. It is just needed to use two properties of Meijer's G-functions.

## 1 INTRODUCTION

In the recent decades, Meijer's G-function (MGFs) has found various applications in different areas close to applied mathematics, such as mathematical physics, theoretical physics, mathematical statistics etc. Due to the important properties of the MGFs, it is possible to represent the solutions of many problems in terms of MGFs [1]. Recently Pishkoo and Darus obtain G-function solutions for Reaction-diffusion equation [2], Schrodinger equation [3], Diffusion equation, and Laplace's equation [4,5], respectively.

An evidence showing the importance of MGFs is given by the fact that the basic elementary functions and most of the special functions of mathematical physics, including the generalized hypergeometric functions, follow as its particular cases [1].

In this paper we show that how all the excited states of radial functions belong to the Hydrogen atom can be made by using the ground state, and two important properties of Meijer's G-functions.

## 2 MEIJER'S G-FUNCTIONS

We begin with the definition of Meijer's G-function as the following:

**Definition 2.1.** A definition of the Meijer's G-function is given by the following path integral in the complex plane, called Mellin-Barnes type integral [1,6-9]:

$$G_{p,q}^{m,n} \left( \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| z \right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} z^s ds. \quad (1)$$

Here, the integers  $m; n; p; q$  are called "orders" of the G-function, or the components of the order  $(m; n; p; q)$ ;  $a_j$  and  $b_j$  are called "parameters" and in general, they are complex numbers. The definition holds under the following assumptions:  $0 \leq m \leq q$  and  $0 \leq n \leq p$ , where  $m; n; p$ ; and  $q$  are integer numbers. Subtracting parameters  $a_j - b_k \neq 1, 2, 3, \dots$  for  $k = 1, \dots, n$  and  $j = 1, 2, \dots, m$  imply that no pole of any  $\Gamma(b_j - s), j = 1, \dots, m$  coincides with any pole of any  $\Gamma(1 - a_k + s), k = 1, \dots, n$ .

Choosing  $m = 1; n = 0; p = 0$  and  $q = 1$ , we have

$$G_{0,1}^{1,0}(\bar{b}_1 | z) = \frac{1}{2\pi i} \int_L \Gamma(b_1 - s) z^s ds. \quad (2)$$

Or in more general in terms of elementary function is (see [1]-Appendix)

$$G_{0,1}^{1,0}(\frac{-}{\alpha}|\eta z^\alpha) = \eta^{\frac{\beta}{\alpha}} z^\beta e^{-\eta z^\alpha} \tag{3}$$

Based on the definition, the following basic properties are easily derived:

$$z^\alpha G_{p,q}^{m,n}(\frac{a_p}{b_q}|z) = G_{p,q}^{m,n}(\frac{a_p+\alpha}{b_q+\alpha}|z), \tag{4}$$

where the multiplying term  $z^\alpha$  changes the parameters of the G-function. The derivatives of arbitrary order  $k$  can change the G-function's orders and parameters:

$$z^k \frac{d^k}{dz^k} G_{p,q}^{m,n}(\frac{a_p}{b_q}|z) = G_{p+1,q+1}^{m,n+1}(\frac{0,a_p}{b_q,k}|z). \tag{5}$$

### 3 THE HYDROGEN ATOM

The hydrogen atom consists of an electron and a proton interacting through the potential

$$V(r) = -\frac{e^2}{4\pi\epsilon_0 r} \tag{6}$$

Here  $r$  is the separation of the two particles. Since the potential (6) is central, the Schrodinger equation is solved by using  $\psi(r, \theta, \varphi) = R(r)Y_{lm}(\theta, \varphi)$ . Then the radial functions are obtained [10].

$$\left\{ -\frac{\hbar^2}{2m} \left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{l(l+1)}{r^2} \right] - \frac{e^2}{4\pi\epsilon_0 r} \right\} R(r) = ER(r) \tag{7}$$

$$R_{10} = \left(\frac{1}{r_B}\right)^{\frac{3}{2}} 2e^{-\frac{r}{r_B}} = c_1 e^{-\frac{r}{r_B}}, \tag{8}$$

With  $r_B = \frac{\hbar^2}{m} \frac{4\pi\epsilon_0}{e^2}$  where  $r_B$  is the Bohr radius.

$$R_{20} = \left(\frac{1}{2r_B}\right)^{\frac{3}{2}} \left(2 - \frac{r}{r_B}\right) e^{-\frac{r}{2r_B}} = (c_2 + c_3 r) e^{-\frac{r}{2r_B}}.$$

$$R_{21} = \left(\frac{1}{2r_B}\right)^{\frac{3}{2}} \frac{1}{\sqrt{3}} \frac{r}{r_B} e^{-\frac{r}{2r_B}} = c_4 r e^{-\frac{r}{2r_B}}.$$

$$R_{30} = \left(\frac{1}{3r_B}\right)^{\frac{3}{2}} 2 \left[ 1 - \frac{2}{3} \frac{r}{r_B} + \frac{2}{27} \left(\frac{r}{r_B}\right)^2 \right] e^{-\frac{r}{3r_B}} = (c_5 + c_6 r + c_7 r^2) e^{-\frac{r}{3r_B}}.$$

$$R_{31} = \left(\frac{1}{3r_B}\right)^{\frac{3}{2}} \frac{8}{9\sqrt{2}} \frac{r}{r_B} \left(1 - \frac{r}{6r_B}\right) e^{-\frac{r}{3r_B}} = (c_8 r + c_9 r^2) e^{-\frac{r}{3r_B}}.$$

$$R_{32} = \left(\frac{1}{3r_B}\right)^{\frac{3}{2}} \frac{4}{27\sqrt{10}} \left(\frac{r}{r_B}\right)^2 e^{-\frac{r}{3r_B}} = c_{10} r^2 e^{-\frac{r}{3r_B}}.$$

In this paper we show that radial functions of the Hydrogen atom are related together through properties (4) and (5) of G-function  $G_{0,1}^{1,0}$ . Starting with  $R_{10}$  in terms of  $G_{0,1}^{1,0}$ , by using properties (4), and (5) we may deduce  $R_{20}$  and  $R_{21}$  from  $R_{10}$ ; and then  $R_{30}$ ,  $R_{31}$ , and  $R_{32}$  from  $R_{20}$  and  $R_{21}$ .

#### 4 MAIN RESULTS

Let start with function (8). Choosing  $\alpha = 1$ ;  $\beta = 0$ ;  $\eta = \frac{1}{r_B}$  gives

$$G_{0,1}^{1,0}\left(\bar{\mid} \frac{r}{r_B}\right) \propto R_{10} \quad (9)$$

In order to obtain excited states of radial function from the ground state and in the language of Meijer's G-functions, we give the following interpretations in the format of two postulates. The postulates:

1- When the operator  $r$  is multiplied by  $\mathbf{G}_{0,1}^{1,0}$  it acts as "photon" that gives energy to electron to shift from the initial state (lower energy) to the final state (higher energy). So the Bohr radius  $nr_B$  changes to  $(n + 1)r_B$ .

( $n$  is positive integer)

2- When the operator  $r \frac{d}{dr}$  is multiplied by  $\mathbf{G}_{0,1}^{1,0}$  it acts as "photon" that gives energy to electron to shift from the initial state (lower energy) to the final state (higher energy). So the Bohr radius  $nr_B$  changes to  $(n + 1)r_B$ .

( $n$  is positive integer)

From (4) and (8) and for the second Bohr radius,  $r_B \rightarrow 2r_B$ , we have

$$rG_{0,1}^{1,0}\left(\bar{\mid} \frac{r}{r_B}\right) = G_{0,1}^{1,0}\left(\bar{\mid} \frac{r}{2r_B}\right) = \frac{1}{2r_B} r e^{-\frac{r}{2r_B}} \propto R_{21} \quad (10)$$

From (5) and (8) we have

$$r \frac{d}{dr} G_{0,1}^{1,0}\left(\bar{\mid} \frac{r}{r_B}\right) =_{r_B \rightarrow 2r_B} G_{1,2}^{1,1}\left(\bar{\mid} \frac{r}{2r_B}\right). \quad (11)$$

On the other hand we have

$$\begin{aligned} G_{1,2}^{1,1}\left(\bar{\mid} \frac{r}{2r_B}\right) &= \frac{1}{2\pi i} \int_L \frac{\Gamma(0-s)\Gamma(1-0+s)}{\Gamma(1-1+s)} \left(\frac{r}{2r_B}\right)^s ds \\ &= \frac{1}{2\pi i} \int_L \Gamma(-s)(1+s) \left(\frac{r}{2r_B}\right)^s ds \\ &= G_{0,1}^{1,0}\left(\bar{\mid} \frac{r}{2r_B}\right) + G_{0,1}^{1,0}\left(\bar{\mid} \frac{r}{2r_B}\right) \\ &= e^{-\frac{r}{2r_B}} + \left(\frac{1}{2r_B}\right) r e^{-\frac{r}{2r_B}}. \end{aligned}$$

$$r \frac{d}{dr} G_{0,1}^{1,0}\left(\bar{\mid} \frac{r}{r_B}\right) \propto R_{20} \quad (12)$$

Therefore, the operator " $r$ " (photon) shifts the electron from the radial state  $R_{10}$  to  $R_{21}$ , and the operator " $r \frac{d}{dr}$ " (photon) shifts the electron from the radial state  $R_{10}$  to  $R_{20}$ .

In the same way for the second time we repeat this operator method and it gives

$$r^2 G_{0,1}^{1,0}\left(\bar{\mid} \frac{r}{r_B}\right) = r G_{0,1}^{1,0}\left(\bar{\mid} \frac{r}{2r_B}\right) = G_{0,1}^{1,0}\left(\bar{\mid} \frac{r}{3r_B}\right)$$

$$r^2 G_{0,1}^{1,0}\left(\bar{\mid} \frac{r}{r_B}\right) \propto R_{32} \quad (13)$$

$$\begin{aligned} (r) \left(r \frac{d}{dr}\right) G_{0,1}^{1,0}\left(\bar{\mid} \frac{r}{r_B}\right) &= (r) \left[G_{0,1}^{1,0}\left(\bar{\mid} \frac{r}{2r_B}\right) + G_{0,1}^{1,0}\left(\bar{\mid} \frac{r}{2r_B}\right)\right] \\ &= G_{0,1}^{1,0}\left(\bar{\mid} \frac{r}{3r_B}\right) + G_{0,1}^{1,0}\left(\bar{\mid} \frac{r}{3r_B}\right) \\ &= \left(\frac{1}{3r_B}\right) r e^{-\frac{r}{3r_B}} + \left(\frac{1}{3r_B}\right)^2 r^2 e^{-\frac{r}{3r_B}} \end{aligned}$$

$$(r) \left(r \frac{d}{dr}\right) G_{0,1}^{1,0}\left(\bar{\mid} \frac{r}{r_B}\right) \propto R_{31} \quad (14)$$

$$\begin{aligned}
 \left(r \frac{d}{dr}\right)^2 G_{0,1}^{1,0}\left(\bar{\phantom{r}} \left| \frac{r}{r_B} \right.\right) &= \left(r \frac{d}{dr}\right) \left[ G_{0,1}^{1,0}\left(\bar{\phantom{r}} \left| \frac{r}{2r_B} \right.\right) + G_{0,1}^{1,0}\left(\bar{\phantom{r}} \left| \frac{r}{r_B} \right.\right) \right] \\
 &= G_{0,1}^{1,0}\left(\bar{\phantom{r}} \left| \frac{r}{3r_B} \right.\right) + G_{0,1}^{1,0}\left(\bar{\phantom{r}} \left| \frac{r}{2r_B} \right.\right) + G_{0,1}^{1,0}\left(\bar{\phantom{r}} \left| \frac{r}{r_B} \right.\right) \\
 &= e^{-\frac{r}{3r_B}} + \left(\frac{1}{3r_B}\right) r e^{-\frac{r}{3r_B}} + \left(\frac{1}{3r_B}\right)^2 r^2 e^{-\frac{r}{3r_B}} \\
 \left(r \frac{d}{dr}\right)^2 G_{0,1}^{1,0}\left(\bar{\phantom{r}} \left| \frac{r}{r_B} \right.\right) &\propto R_{30}
 \end{aligned}
 \tag{15}$$

The results are summarized in Table 1.

Table1: Derivation of excited radial states of the hydrogen atom from the ground state  $R_{10}$  in terms of Meijer's G-function  $G_{0,1}^{1,0}$

| Radial function | Meijer's G-function  | Operator                        |
|-----------------|--|---------------------------------|
| $R_{10}$        | $G_{0,1}^{1,0}\left(\bar{\phantom{r}} \left  \frac{r}{r_B} \right.\right)$   |                                 |
| $R_{21}$        | $G_{0,1}^{1,0}\left(\bar{\phantom{r}} \left  \frac{r}{2r_B} \right.\right)$  | $r$                             |
| $R_{20}$        | $G_{1,2}^{1,1}\left(0,1 \left  \frac{r}{2r_B} \right.\right) = G_{0,1}^{1,0}\left(\bar{\phantom{r}} \left  \frac{r}{2r_B} \right.\right) + G_{0,1}^{1,0}\left(\bar{\phantom{r}} \left  \frac{r}{r_B} \right.\right)$               | $r \frac{d}{dr}$                |
| $R_{32}$        | $G_{0,1}^{1,0}\left(\bar{\phantom{r}} \left  \frac{r}{3r_B} \right.\right)$  | $r^2$                           |
| $R_{31}$        | $G_{0,1}^{1,0}\left(\bar{\phantom{r}} \left  \frac{r}{3r_B} \right.\right) + G_{0,1}^{1,0}\left(\bar{\phantom{r}} \left  \frac{r}{2r_B} \right.\right)$  | $r\left(r \frac{d}{dr}\right)$  |
| $R_{30}$        | $G_{0,1}^{1,0}\left(\bar{\phantom{r}} \left  \frac{r}{3r_B} \right.\right) + G_{0,1}^{1,0}\left(\bar{\phantom{r}} \left  \frac{r}{2r_B} \right.\right) + G_{0,1}^{1,0}\left(\bar{\phantom{r}} \left  \frac{r}{r_B} \right.\right)$ | $\left(r \frac{d}{dr}\right)^2$ |

### 5 CONCLUSIONS

This paper applies the operator method to deduce excited radial states of the simplest atom from the simplest Meijer's G-function  $G_{0,1}^{1,0}\left(b_1 \left| \frac{r}{r_B} \right.\right)$  with  $b_1 = 0$ . There are two properties of Meijer's G-functions which include closed ness. All excited radial states of the Hydrogen atom are obtained by using these two properties. Now how these results can be generalized to other atoms or even nuclei?

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