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APPROXIMATION OF A FUNCTION BELONGING TO THE CLASS Lip (ψ (t), p) BY USING [s, α_n] MEANS

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Introduction

Meyer-Konig[6] introduces so called S_a method of summability which is one of the family of transformation including the Euler, Borel and Taylor (circle method) methods. Later Jakimovski[4] introduced $[F,d_n]$ transformation which methods the Euler method (E,q) Karmata method (K^{λ}) and Lotosky method as particular cases.
For the first time Meir and Sharma[5] introduced generalization of the S_a method and called it $[S, \alpha_n]$ method. They obtained sufficient condition for the regularity of this method. They also examined the behaviour of its Lebesgue constant.
Let $\{a\ j\}$ be a given sequence of real complex numbers. We shall say that $\{a\ j\}$f is the $[S,a_n]$,transformations of $\{S j\}$; i.e. the sequence of partial sums of the series $\sum a_n$ if

$$\{\sigma_n\} = \sum_{k=0}^{\infty} C_{nk} S_k ; (n = 0,1,2,3,\dots\dots\dots ..)$$

Converges, where (C_{nk}) is given by the identity .

$$\prod \frac{1-a}{1-a_j\theta} = \sum_{k=0}^{\infty} C_{nk}\theta^k$$

The sequence $\{S j\}$ is said to be $[S, \alpha_n]$ summable toσ if

$$\underset{n\to\infty}{Lim}\,\sigma = \sigma$$

Let f(x) ε L (0, 2π) and be periodic with period 2π outside this range. Let the Fourier series associated with the function be

$$\frac{a_0}{2} + \sum_{n=0}^{\infty}(a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} A_n(x)$$

and as usual we denote

$$\phi(t) = \phi_x(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}$$

Also

$$V_n = 1 + 2\sum_{j=0}^{n} \frac{a_j}{1-a_j}$$

$$T = 2\sum_{j=0}^{n} \frac{a_j}{(1-a_j)^2}$$

$m = \lfloor T_n \rfloor$, the integral part of T_n
and

$$a_n = \frac{2\pi}{m}$$

Meir and Sharma[5] while studying constant established that when V_n and T_n are bounded the [$S\ a_n$], method sums only convergent Fourier series and so here after we assume $T_n \to \infty$ and $V_n \to \infty$ with n.

A function f∈Lipα if

$$|f(x+t)-f(x)| = O\left(|t|^{\alpha}\right) \text{ for } 0<\alpha\le 1$$

and

f(x)∈Lip(α,p), for 0≤x≤2π, if

$$\left(\int_0^{2\pi}|f(x+t)-f(x)|^p\,dx\right)^{1/p} = O\left(|t|^{\alpha}\right) \text{ for } 0<\alpha\le 1,\ p\ge 1$$

A function f∈Lip (ϕ(t), p) class for p≥1 if

$$\left(\int_0^{2\pi}|f(x+t)-f(x)|^p\,dx\right)^{1/p} = O(\phi(t)),$$

Where ϕ(t) is positive increasing function and f∈Lip (ϕ(t), p) if

$$\left(\int_0^{2\pi}\left|[f(x+t)-f(x)]\sin^{\beta}x\right|^p\,dx\right)^{1/p} = O(\phi(t)),\ (\beta\ge 0)$$

We observe that

Lipα ⊆ Lip (α, p) ⊆ Lip (ϕ(t), p), for 0<α≤1, p≥1

To prove the theorem we need following auxiliary result:

Lemma 1: The following estimates hold:

If

$$K_n(t) = e^{it}\sum_{j=0}^{n}\frac{1-a_j}{1-a_j e^{it}}$$

Then

$$|K_n(t)| = O\left(\frac{1}{t\sqrt{T_n}}\right)$$

and

$$K_n(t) = \exp\left[V_n it - T_n t^2\right] + O\left(T_n t^3\right) \text{ for t to be very small.}$$

These are due to Meir and Sharma [5]

Lemma 2: If h(x, t) is a function of two variables defined for 0≤t≤2π, then

$$\left\|\int h(x,t)\,dt\right\|p \le \left\|\int h(x,t)\right\|p\,dt\ ;\ (p>1)$$

This is due to Hardy, Littlewood and Poly [3] .

THEOREM:

If f(x) is periodic with period 2π and belongs to the class Lip (ψ (t), p) for p>1, and if

$$\left\{\int_0^{1/\sqrt{n}}\left(\frac{\psi(t)}{t^{1/p}}\right)^p dt\right\}^{1/p} = \mathrm{O}\left(\psi\left(\frac{1}{\sqrt{n}}\right)\right)$$

and

$$\left\{\int_{1/\sqrt{n}}^{\pi}\left(\frac{\psi(t)}{t^{1/p+2}}\right)^{p}dt\right\}^{1/p}=\mathrm{O}\left(\psi\left(\frac{1}{\sqrt{n}}\right)n\right),$$

Then

$$\max_{0\le x\le 2\pi}\left|f(x)-\sigma_n(f,x)\right|=\mathrm{O}\left(\psi\left(\frac{1}{\sqrt{n}}\right)(n)^{1/2p}\right)$$

In the present chapter we have obtained the degree of approximation of a function belonging to the class Lip (ψ (t), p) by using [$S\,a_n$] means. Our theorem is as follows:-

THEOREM:

If f(x) is periodic with period 2π and belongs to class Lip (ψ (t), p) for p≥1, and β≥0 and $\frac{1}{3}<\eta<\frac{1}{2}$ and if

$$\left\{\int_0^{a_n}\left|\phi_x(t)\sin^{\beta}t\right|^{p}dt\right\}^{1/p}=O\left\{\phi\left(\frac{1}{m}\right)\right\}$$

$$\left\{\int_{a_n}^{(a_n)^{\beta}}\left|\frac{\phi_x(t)\sin^{\beta}t}{t^{2}}\right|^{p}dt\right\}^{1/p}=O\left\{m^{2}\phi\left(\frac{1}{m}\right)\right\}$$

$$\left\{\int_{(a_n)^{\beta}}^{\pi}\left|\frac{\phi_x(t)\sin^{\beta}t}{t^{2}}\right|^{p}dt\right\}^{1/p}=O\left\{m^{2\beta}\phi\left(\frac{1}{m^{\beta}}\right)\right\}$$

Then

$$\left|\sigma_n(f,x)-f(x)\right|=O\left\{m^{\beta+\frac{1}{p}}.\phi\left(\frac{1}{m^{\beta}}\right)\right\}$$

PROOF:

The [$S\,a_n$] transform of partial sums of Fourier series is given by

$$\sigma_n-f(x)=\frac{2}{\pi}\int_0^{\pi}\frac{\phi_x(t)}{t}\mathrm{Im}\left\{\sum_{k=0}^{n}C_{nk}\sin\left(k+\frac{1}{2}\right)t\right\}dt$$

$$=\frac{2}{\pi}\int_0^{\pi}\frac{\phi_x(t)}{t}\mathrm{Im}\left\{\exp\left(\frac{it}{2}\right)\sum_{k=0}^{n}C_{nk}\exp(ikt)\right\}dt$$

$$=\frac{2}{\pi}\left[\int_0^{a_n}+\int_{a_n}^{(a_n)^{\beta}}+\int_{(a_n)^{\beta}}^{\pi}\right]\frac{\phi_x(t)}{t}\mathrm{Im}\left\{\exp\left(\frac{it}{2}\right)\prod_{j=0}^{n}\frac{1-a_j}{1-a_j e^{it}}\right\}dt$$

$$=|I_1|+|I_2|+I_3\ \text{(say)}$$

where β is a number chosen such that $\frac{1+\eta}{3+\eta}\le\beta\le\frac{1}{2}$

By Minkowski inequality

$$\left|\sigma_n - f(x)\right| = \left|I_1\right| + \left|I_2\right| + \left|I_3\right|$$

By Lemma 1 and 2,

$$\left|I_1\right| = \mathrm{O}\ (1)\ \int_0^{a_n} \frac{\phi_x(t)}{t}\exp\left(\frac{-T_n t^2}{4}\right)\sin\frac{v_n t}{2}\,dt + \mathrm{O}\ (1)\ \int_0^{a_n}\frac{\phi_x(t)}{t}T_n t^3\,dt$$

$$= I_{1.1} + I_{1.2}$$

$$I_{1.1} = \mathrm{O}\ (1)\ \int_0^{a_n}\frac{\phi_x(t)}{t}\exp\left(\left(\frac{-T_n t^2}{4}\right)\sin\frac{V_n t}{2}\,dt\right)$$

Applying Hölder's inequality

$$\left|I_{1.1}\right| = \mathrm{O}\left[\left\{\int_0^{a_n}\left|\phi_x(t)\sin^{\beta}(t)\right|^p dt\right\}^{1/p}\right].O\left[\left\{\int_0^{a_n}\left|\frac{1}{\sin^{\beta} t}\right|^q dt\right\}^{1/q}\right]$$

$$= \mathrm{O}\left\{\phi\left(\frac{1}{m}\right)\right\}.O\left[\left\{\int_0^{a_n} t^{-\beta q}dt\right\}^{1/q}\right]$$

$$= \mathrm{O}\left[\phi\left(\frac{1}{m}\right)m^{\beta-\frac{1}{q}}\right]$$

$$= \mathrm{O}\left[m^{\beta+\frac{1}{P}}\phi\left(\frac{1}{m}\right)\right] \tag{1}$$

$$\left|I_{1.2}\right| = \mathrm{O}\left(T_n\right)\int_0^{a_n}\frac{\phi_n(t)}{t}t^3\,dt$$

$$= \mathrm{O}\left[\left\{\int_0^{a_n}\left|\phi_x(t)\sin^{\beta}(t)\right|^p dt\right\}^{1/p}\right].O\left[\left\{\int_0^{a_n}\left|\frac{t^2}{\sin^{\beta} t}\right|^q dt\right\}^{1/q}\right]$$

$$= \mathrm{O}\left[\phi\left(\frac{1}{m}\right)\right].O\left[\left\{\int_0^{a_n} t^{(2-\beta)q}\,dt\right\}^{1/q}\right]$$

$$= \mathrm{O}\left[\phi\left(\frac{1}{m}\right)\right].O\left[\left\{m^{(\beta-2)q-1}\right\}^{1/q}\right]$$

$$= \mathrm{O}\left(\phi\left(\frac{1}{m}\right)m^{\beta-3+1-\frac{1}{q}}\right)$$

$$= \mathrm{O}\left(m^{\beta+\frac{1}{p}}\phi\left(\frac{1}{m}\right)\right) \tag{2}$$

Further, by Lemma 1

$$I_2 = \frac{2}{\pi}\int_{\alpha_n}^{(a_n)^\beta} \frac{\phi_x(t)}{t}\exp\left(\frac{-T_n t^2}{4}\right)\sin\frac{v_n t}{2}\,dt + \frac{2}{\pi}\int_{\alpha_n}^{(a_n)^\beta}\frac{\phi_x(t)}{t}T_n t^3\,dt$$

$$= I_{2.1} + I_{2.2}$$

By the fact that |sin x| ≤1for all x, and applying Hölder's inequality

$$I_{2.1} = \mathrm{O}(1)\int_{a_n}^{(a_n)^\beta}\frac{\phi_x(t)}{t}dt$$

$$= \mathrm{O}\left[\left\{\int_{a_n}^{(a_n)^\beta}\left|\frac{\phi_x(t)\sin^\beta t}{t^2}\right|^p dt\right\}^{1/p}\right] O\left[\left\{\int_{a_n}^{(a_n)^\beta}\left|\frac{t}{\sin^\beta t}\right|^q dt\right\}^{1/q}\right]$$

$$= \mathrm{O}\left[m^2\phi\left(\frac{1}{m}\right)\right]\cdot\left[\left\{\int_{a_n}^{(a_n)^\beta} t^{(1-\beta)q}dt\right\}^{1/q}\right]$$

$$= \mathrm{O}\left[m^{\beta+\frac{1}{q}}\phi\left(\frac{1}{m}\right)\right] \tag{3}$$

$$I_{2.2} = \mathrm{O}(T_n)\int_{a_n}^{(a_n)^\beta}\phi_x(t)t^2\,dt$$

$$= \mathrm{O}\left[\left\{\int_{a_n}^{(a_n)^\beta}\left|\frac{\phi_x(t)\sin^\beta t}{t^2}\right|^p dt\right\}^{1/p}\right] O\left[\left\{\int_{a_n}^{(a_n)^\beta}\left|\frac{t^4}{\sin^\beta t}\right|^q dt\right\}^{1/q}\right]$$

$$= \mathrm{O}\left[m^2\phi\left(\frac{1}{m}\right)\right].O\left[\left\{\int_{a_n}^{(a_n)^\beta} t^{(4-\beta)q}dt\right\}^{1/q}\right]$$

$$= \mathrm{O}\left[m^{\beta+\frac{1}{q}}\phi\left(\frac{1}{m}\right)\right] \tag{4}$$

$$|I_3| = \mathrm{O}(1)\left\{\int_{(a_n)^\beta}^{\pi}\frac{|\phi_x(t)|^p}{t.t\sqrt{T_n}}dt\right\}^{1/p}$$

$$= \mathrm{O}(1)\left\{\int_{(a_n)^\beta}^{\pi}\left|\frac{\phi_x(t)}{t^2}\sin^\beta t\right|^p dt\right\}^{1/p} O\left[\left\{\int_{(a_n)^\beta}^{\pi}\left|\frac{t^2}{\sin^\beta t}\right|^q\frac{1}{t.t\sqrt{T_n}}dt\right\}\right]^{1/q}$$

Where $q = \dfrac{p}{p-1}$

$$= \mathrm{O}\left\{m^{2\beta}\phi\left(\frac{1}{m^\beta}\right)\right\}O\left[\left\{\int_{(a_n)^\beta}^{\pi}\frac{t^{(2-\beta)q}}{t^2\sqrt{T_n}}dt\right\}\right]^{1/q}$$

$$= \mathrm{O}\left\{m^{2\beta}\phi\left(\frac{1}{m^\beta}\right)\right\}O\left(\frac{1}{m^{1/2q}}\right)O\left[\left\{\int_{(a_n)^\beta}^{\pi}t^{(2-\beta)q-2}dt\right\}\right]^{1/q}$$

$$= O\left\{m^{2\beta}\phi\left(\frac{1}{m^{\beta}}\right)\right\}O\left\{m^{\beta^{2}-\beta-\frac{\beta}{p}}\right\}$$

$$= O\left\{m^{\beta^{2}+\beta-\frac{\beta}{p}}\phi\left(\frac{1}{m^{\beta}}\right)\right\}$$

$$= O\left\{m^{\beta+\frac{1}{P}}\phi\left(\frac{1}{m^{\beta}}\right)\right\} \tag{5}$$

Combining (1), (2), (3), (4) and (5), we get

$$\left|\sigma_{n}-f(x)\right| = O\left\{m^{\beta+\frac{1}{p}}\phi\left(\frac{1}{m^{\beta}}\right)\right\}$$

References:

[1]. Chui, C.K. and Holland, A.S.B., On the order of approximation by Euler and Taylor means, Journal of Approximation Theory, 39,24-38(1983).
[2]. Hardy, G.H.,Divergent Series, Oxford Uni. Press, (Clarendon) Oxford.
[3]. Hardy, G.H., Littleword, J.E. and Polya, G. Inequalities, Cambridge University Press, London/New York,(1967).
[4]. Jakimovski, A.,A Generalisation of Lototsky method of Summability, Michigan Math. J.6,277-290(1959).