APPROXIMATION OF A FUNCTION BELONGING TO THE CLASS Lip 
\((\psi(t), p)\) BY USING \([s, \alpha_n]\) MEANS 

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Introduction
Meyer-König\(^6\) introduces so called \(S_a\) method of summability which is one of the family of 
transformation including the Euler, Borel and Taylor (circle method) methods. Later Jakimovski\(^8\) 
introduced \([F,d_n]\) transformation which methods the Euler method (E,q) Karmata method (K\(^\lambda\)) and 
Lototsky method as particular cases. 
For the first time Meir and Sharma\(^5\) introduced generalization of the \(S_a\) method and called it \([S, \alpha_n]\) 
method. They obtained sufficient condition for the regularity of this method. They also examined 
the behaviour of its Lebesgue constant. 
Let \(\{a_j\}\) be a given sequence of real complex numbers. We shall say that \(\{a_j\}\) if is the 
\([S,\alpha_n]\) transformations of \(\{S_j\}\); i.e. the sequence of partial sums of the series \(\sum a_n\) if

\[\sigma_n = \sum_{i=0}^{\infty} C_n S_i (n = 0, 1, 2, \ldots, \ldots)\]

Converges, where \(\{C_n\}\) is given by the identity
\[\prod_{a_j \neq \theta} \frac{1 - a_j \theta}{1 - a_j} = \sum_{k=0}^{\infty} C_{nk} \theta^k\]

The sequence \(\{S_j\}\) is said to be \([S, \alpha_n]\) summable to \(\sigma\) if

\[\lim_{n \to \infty} \sigma_n = \sigma\]

Let \(f(x) \in L(0, 2\pi)\) and be periodic with period \(2\pi\) outside this range. Let the Fourier series 
associated with the function be
\[a_0 + \sum_{n=0}^{\infty} \left( a_n \cos nx + b_n \sin nx \right) = \sum_{n=0}^{\infty} A_n(x)\]

and as usual we denote
\[\phi(t) = \phi_x(t) = \frac{1}{2} [f(x + t) + f(x - t) - 2f(x)]\]

Also
\[V_n = 1 + 2 \sum_{j=0}^{n} \frac{a_j}{1 - a_j}\]
\[T = 2 \sum_{j=0}^{\infty} \frac{a_j}{(1 - a_j)^2}\]
\[m = \lfloor T \rfloor\] the integral part of \(T\) and
\[a_n = \frac{2\pi}{m}\]
Meir and Sharma while studying constant established that when $V_n$ and $T_n$ are bounded the $\{S a_n\}$, method sums only convergent Fourier series and so here after we assume $T_n \to \infty$ and $V_n \to \infty$ with $n$.

A function $f \in \text{Lip} \alpha$ if
\[
|f(x + t) - f(x)| = O(t^\alpha) \quad \text{for } 0 < \alpha \leq 1
\]
and
\[
f(x) \in \text{Lip}(\alpha, p), \quad \text{for } 0 \leq x \leq 2\pi, \text{ if }
\]
\[
\left( \int_0^{2\pi} |f(x + t) - f(x)|^p \, dx \right)^{1/p} = O(t^{\alpha}), \quad \text{for } 0 < \alpha \leq 1, \ p \geq 1
\]

A function $f \in \text{Lip}(\phi(t), p)$ class for $p \geq 1$ if
\[
\left( \int_0^{2\pi} |f(x + t) - f(x)|^p \, dx \right)^{1/p} = O(\phi(t))
\]

Where $\phi(t)$ is positive increasing function and $f \in \text{Lip}(\phi(t), p)$ if
\[
\left( \int_0^{2\pi} \left| f(x + t) - f(x) \right| \sin^\beta x \, dx \right)^{1/p} = O(\phi(t)), \ (\beta \geq 0)
\]

We observe that
\[
\text{Lip} \alpha \subseteq \text{Lip}(\alpha, p) \subseteq \text{Lip}(\phi(t), p), \text{ for } 0 < \alpha \leq 1, \ p \geq 1
\]

To prove the theorem we need following auxiliary result:

Lemma 1: The following estimates hold:

If
\[
K_\alpha(t) = e^{\sum_{j=1}^{n} \frac{1 - a_j}{a_j e^\beta}}
\]

Then
\[
\left| K_\alpha(t) \right| = O\left( \frac{1}{t^{\sqrt{T_n}}} \right)
\]

and
\[
K_\alpha(t) = \exp[\int \psi(t)^2 \, dt] + O(T_n, t^4) \quad \text{for } t \text{ to be very small}.
\]

These are due to Meir and Sharma.

Lemma 2: If $h(x, t)$ is a function of two variables defined for $0 \leq t \leq 2\pi$, then
\[
\left| \int_0^{2\pi} h(x, t) \, dt \right| = \int_0^{2\pi} |h(x, t)| \, dt : (p > 1)
\]

This is due to Hardy, Littlewood and Poly.

THEOREM:
If $f(x)$ is periodic with period $2\pi$ and belongs to the class $\text{Lip}(\psi(t), p)$ for $p > 1$, and if
\[
\left( \int_0^{1/\sqrt{n}} \left( \frac{\psi(t)}{t} \right)^p \, dt \right)^{1/p} = O\left( \frac{1}{\sqrt{n}} \right)
\]
and
\[ \left\{ \int_{1/\sqrt{n}}^{1} \left( \frac{\psi(t)}{t^{1/p+2}} \right)^p \, dt \right\}^{1/p} = O \left( \psi \left( \frac{1}{\sqrt{n}} \right)^{p-1} \right), \]

Then
\[ \max_{0 \leq x \leq 2\pi} |f(x) - \sigma_n(f,x)| = O \left( \psi \left( \frac{1}{\sqrt{n}} \right)^{2/p} \right) \]

In the present chapter we have obtained the degree of approximation of a function belonging to the class Lip (ψ(t), p) by using \( S a_n \) means. Our theorem is as follows:

**THEOREM:**

If \( f(x) \) is periodic with period \( 2\pi \) and belongs to class Lip (ψ(t), p) for \( p \geq 1 \), and \( \beta \geq 0 \) and if
\[ \left\{ \int_{a}^{\pi} \phi(x) \sin^p \frac{t}{t^{1/2}} \, dt \right\}^{1/p} = O \left\{ \frac{1}{m^{2(1-\beta)}} \right\}, \]

Then
\[ \left| \sigma_n(f,x) - f(x) \right| = O \left\{ \frac{1}{m^{2(1-\beta)}} \right\}. \]

**PROOF:**
The \( S a_n \) transform of partial sums of Fourier series is given by
\[ \sigma_n - f(x) = \sum_{k=0}^{\infty} C_{nk} \sin \left( k + \frac{1}{2} \right) \phi(kt) \right\} \, dt \]
\[ = 2 \int_{0}^{\pi} \phi(t) \left( \int_{0}^{\infty} \sum_{k=0}^{n} C_{nk} \sin \left( k + \frac{1}{2} \right) \, dt \right) \]
\[ = 2 \int_{0}^{\pi} \phi(t) \left( \int_{0}^{\infty} \sum_{k=0}^{n} C_{nk} \exp(ikt) \, dt \right) \]
\[ = 2 \int_{0}^{\pi} \phi(t) \left( \int_{0}^{\infty} \sum_{k=0}^{n} C_{nk} \exp(ikt) \, dt \right) \]
\[ = |I_1| + |I_2| + |I_3| \] (say)

where \( \beta \) is a number chosen such that \( \frac{1+\eta}{3+\eta} \leq \beta \leq \frac{1}{2} \)
By Minkowski inequality
\[ |\sigma_n - f(x)| = |I_1| + |I_2| + |I_3| \]

By Lemma 1 and 2,
\[ |I_1| = O(1) \int_0^{a_n} \frac{\phi_x(t)}{t} \exp \left( -\frac{T_n t^2}{4} \right) \sin \frac{V_n t}{2} dt + O(1) \int_0^{a_n} \frac{\phi_x(t)}{t} T_n t \sin \frac{V_n t}{2} dt \]
\[ = I_{1,1} + I_{1,2} \]
\[ I_{1,1} = O(1) \int_0^{a_n} \frac{\phi_x(t)}{t} \exp \left( -\frac{T_n t^2}{4} \right) \sin \frac{V_n t}{2} dt \]

Applying Hölder’s inequality
\[ |I_1| = O \left[ \left( \int_0^{a_n} \phi_x(t) \sin^p(t) \frac{d}{dt} \right)^{1/p} \right] \cdot O \left[ \left( \int_0^{a_n} \frac{1}{\sin^q(t)} \frac{d}{dt} \right)^{1/q} \right] \]
\[ = O \left( \frac{1}{m} \right) \cdot O \left( \int_0^{a_n} t^{\frac{1}{p}} \frac{d}{dt} \right)^{1/q} \]
\[ = O \left( \frac{1}{m} \right) m^{\frac{1}{p} + \frac{1}{q} - 1} \]
\[ = O \left( \frac{1}{m} \right) m^{\frac{1}{p} + \frac{1}{q} - 1} \] (1)

\[ |I_{1,2}| = O \left( T_n \right) \int_0^{a_n} \frac{\phi_x(t)}{t} T_n t \sin \frac{V_n t}{2} dt \]
\[ = O \left[ \left( \int_0^{a_n} \phi_x(t) \sin^p(t) \frac{d}{dt} \right)^{1/p} \right] \cdot O \left[ \left( \int_0^{a_n} \frac{1}{\sin^q(t)} \frac{d}{dt} \right)^{1/q} \right] \]
\[ = O \left( \frac{1}{m} \right) \cdot O \left( \int_0^{a_n} t^{2-p-q} \frac{d}{dt} \right)^{1/q} \]
\[ = O \left( \frac{1}{m} \right) \cdot O \left( \int_0^{a_n} \frac{1}{\sin^{2-p-q}} \frac{d}{dt} \right)^{1/q} \]
\[ = O \left( \frac{1}{m} \right) m^{\frac{1}{p} - \frac{1}{q}} \]
\[ = O \left( \frac{1}{m} \right) m^{\frac{1}{p} - \frac{1}{q}} \] (2)
Further, by Lemma 1

\[ I_3 = \frac{2}{\pi} \frac{\phi(t)}{t} \exp\left( -\frac{T_s}{4} \right) \int \sin \frac{t^2}{2} dt + \frac{2}{\pi} \frac{\phi(t)}{t} t^3 dt = I_{\text{first}} + I_{\text{second}} \]

By the fact that \( |\sin x| \leq 1 \) for all \( x \), and applying Hölder’s inequality

\[
I_{\text{first}} = O(1) \left[ \frac{a}{a_s} \int \frac{\phi(t)}{t} \right] dt
\]

\[
= O \left[ \left[ \frac{a_s}{a} \int \frac{\phi(t) \sin^\frac{t^2}{2}}{t^2} dt \right]^{1/p} \left[ \frac{a_s}{a} \int \frac{t^q}{\sin^q t} dt \right]^{1/q} \right]
\]

\[
= O \left[ m^2 \Phi \left( \frac{1}{m} \right) \right]
\]

\[
= O \left[ \left( \frac{a_s}{a} \int t^{4-s} dt \right)^{1/q} \right]
\]

\[
= O \left[ m^{\frac{2-s}{q}} \Phi \left( \frac{1}{m} \right) \right]
\]

\[
I_{\text{second}} = O(T_s) \left[ \frac{a_s}{a} \int \phi(t) t^2 dt \right]
\]

\[
= O \left[ \left[ \frac{a_s}{a} \int \frac{\phi(t) \sin^\frac{t^2}{2}}{t^2} dt \right]^{1/p} \left[ \frac{a_s}{a} \int \frac{t^q}{\sin^q t} dt \right]^{1/q} \right]
\]

\[
= O \left[ m^2 \Phi \left( \frac{1}{m} \right) \right]
\]

\[
= O \left[ \left( \frac{a_s}{a} \int t^{4-s} dt \right)^{1/q} \right]
\]

\[
= O \left[ m^{\frac{2-s}{q}} \Phi \left( \frac{1}{m} \right) \right]
\]

Where \( q = \frac{p}{p-1} \)

\[
= O \left[ m^{2-s} \Phi \left( \frac{1}{m} \right) \right]^{1/q} \left[ \left( \frac{1}{a_s} \int t^{2-s} \sqrt{T_s} dt \right)^{1/q} \right]
\]

\[
= O \left[ m^{2-s} \Phi \left( \frac{1}{m} \right) \right] O \left( \frac{1}{m^{1/2}} \right) \left[ \left( \frac{1}{a_s} \int t^{2-s} \sqrt{T_s} dt \right)^{1/q} \right]
\]
Combining (1), (2), (3), (4) and (5), we get

\[
\frac{1}{m} \left[ \phi \left( \frac{1}{m} \right) + m^{\beta-1+\frac{1}{p}} \phi \left( \frac{1}{m} \right) \right]
\]

\[
\frac{1}{m} \left[ \phi \left( \frac{1}{m} \right) + m^{\beta+\frac{1}{p}} \phi \left( \frac{1}{m} \right) \right]
\]

Combining (1), (2), (3), (4) and (5), we get

\[
\left| \sigma_n - f(x) \right| = \mathcal{O} \left( m^{\beta+\frac{1}{p}} \phi \left( \frac{1}{m} \right) \right)
\]

References: