APPROXIMATION OF A FUNCTION BELONGING TO THE CLASS Lip 
(ψ (t), p) BY USING [s, α_n] MEANS 

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Introduction

Meyer-König introduces so called S_0 method of summability which is one of the family of transformation including the Euler, Borel and Taylor (circle method) methods. Later Jakimovski introduced [F,d_n] transformation which methods the Euler method (E,q) Karmata method (K^λ) and Lotosky method as particular cases. For the first time Meir and Sharma introduced generalization of the S_n method and called it [S, α_n] method. They obtained sufficient condition for the regularity of this method. They also examined the behaviour of its Lebesgue constant.

Let \{a_j\} be a given sequence of real complex numbers. We shall say that \{a_j\}f is the \[S, α_n\] transformations of \{S_j\}; i.e. the sequence of partial sums of the series \[\sum a_n\] if

\[\sigma_n = \sum_{n=0}^{\infty} C_{n,k} \theta^k\]

where \(C_{n,k}\) is given by the identity

\[\prod_{j=1}^{n} \frac{1-a_j}{1-a_j \theta} = \sum_{k=0}^{\infty} C_{n,k} \theta^k\]

The sequence \{S_j\} is said to be \[S, α_n\] summable to \(σ\) if

\[\lim_{n \to \infty} S_j = σ\]

Let \(f(x) \in L (0, 2π)\) and be periodic with period 2π outside this range. Let the Fourier series associated with the function be

\[\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} A_n(x)\]

and as usual we denote

\[\phi (t) = \frac{1}{2}[f(x+t) + f(x-t) - 2f(x)]\]

Also

\[V_n = 1 + 2 \sum_{j=1}^{n} \frac{a_j}{1-a_j}\]

\[T = 2 \sum_{j=0}^{n} \frac{a_j}{(1-a_j)^2}\]

\[m = \lfloor T \rfloor\]

and

\[a_n = \frac{2π}{m}\]
Meir and Sharma while studying constant established that when $V_n$ and $T_n$ are bounded the $[S \ a_n]$, method sums only convergent Fourier series and so here after we assume $T_n \to \infty$ and $V_n \to \infty$ with $n$.

A function $f \in \text{Lip} \alpha$ if
\[
|f(x+t) - f(x)| = O(t^\alpha) \quad \text{for} \ 0<\alpha \leq 1
\]
and
\[
f(x) \in \text{Lip}(\alpha, p), \text{ for } 0 \leq x \leq 2\pi, \text{ if}
\]
\[
\left( \int_0^{2\pi} |f(x+t) - f(x)|^p \ dx \right)^{1/p} = O(t^\alpha), \text{ for } 0<\alpha \leq 1, \ p \geq 1
\]
A function $f \in \text{Lip} (\phi(t), p)$ class for $p \geq 1$ if
\[
\left( \int_0^{2\pi} |f(x+t) - f(x)|^p \ dx \right)^{1/p} = O(\phi(t)),
\]
Where $\phi(t)$ is positive increasing function and $f \in \text{Lip} (\phi(t), p)$ if
\[
\left( \int_0^{2\pi} \left| f(x+t) - f(x) \right|^p \ dx \right)^{1/p} = O(\phi(t)), \ (\beta \geq 0)
\]
We observe that
\[
\text{Lip} \alpha \subseteq \text{Lip} (\alpha, p) \subseteq \text{Lip} (\phi(t), p), \text{ for } 0<\alpha \leq 1, \ p \geq 1
\]
To prove the theorem we need following auxiliary result:

Lemma 1: The following estimates hold:

If
\[
K\alpha(t) = e^{\alpha t} \sum_{\nu=1}^{\infty} \frac{1 - a_{\nu}}{a_{\nu}}
\]
Then
\[
|K\alpha(t)| = O\left( \frac{1}{t^{\sqrt{T_n}}} \right)
\]
and
\[
K\alpha(t) = e^{\alpha t} \left[ V \alpha(t) + O(T, t^4) \right] \text{ for } t \text{ to be very small.}
\]
These are due to Meir and Sharma.

Lemma 2: If $h(x, t)$ is a function of two variables defined for $0 \leq t \leq 2\pi$, then
\[
\left\| \int h(x, t) dt \right\|_p \leq \left\| h(x, t) \right\|_p dt, \ (p > 1)
\]
This is due to Hardy, Littlewood and Poly.

THEOREM: If $f(x)$ is periodic with period $2\pi$ and belongs to the class $\text{Lip} (\psi(t), p)$ for $p > 1$, and if
\[
\left( \int_0^{1/\sqrt{n}} \left( \frac{1}{\psi(t)} \right)^{1/p} dt \right)^{1/p} = O\left( \psi\left( \frac{1}{\sqrt{n}} \right) \right)
\]
and
\[
\left\{ \int_{1/\sqrt{n}}^{1} \left( \frac{\psi(t)}{t} \right)^p dt \right\}^{1/p} = O\left( \frac{1}{\sqrt{n}} \right),
\]

Then
\[
\max_{0 \leq x \leq 2 \pi} |f(x) - \sigma_n(f,x)| = O\left( \frac{1}{\sqrt{n}} \right)\left( n^{1/2} \right).
\]

In the present chapter we have obtained the degree of approximation of a function belonging to the class Lip \((\psi(t), p)\) by using \([S a_n]\) means. Our theorem is as follows:

**THEOREM:**

If \(f(x)\) is periodic with period \(2\pi\) and belongs to class Lip \((\psi(t), p)\) for \(p \geq 1\), and \(\beta \geq 0\) and if

\[
\frac{1}{\phi_{\alpha}(t)} \int_{\alpha \phi_{\alpha}(t)}^\phi \phi_{\alpha}(t) \sin \frac{\theta}{t} \frac{dt}{t} = O\left\{ m^{2} \phi \left( \frac{1}{m} \right) \right\}
\]

Then
\[
|\sigma_n(f,x) - f(x)| = O\left\{ m^{-\beta} \phi \left( \frac{1}{m} \right) \right\}
\]

**PROOF:**

The \([S a_n]\) transform of partial sums of Fourier series is given by

\[
\sigma_n - f(x) = \frac{2}{\pi} \int_0^{\pi} \frac{\phi(t)}{t} \text{Im} \left\{ \sum_{k=0}^{\infty} C_{nk} \sin \left( k + \frac{1}{2} \right) \right\} dt
\]

\[
= \frac{2}{\pi} \int_0^{\pi} \frac{\phi(t)}{t} \text{Im} \left\{ \exp \left( \frac{it}{2} \right) \sum_{k=0}^{n} C_{nk} \exp(ikt) \right\} dt
\]

\[
= \frac{2}{\pi} \int_0^{\pi} \frac{\phi(t)}{t} \text{Im} \left\{ \exp \left( \frac{it}{2} \right) \prod_{j=0}^{n} \frac{1-a_j e^{i\theta}}{1-a_j e^{i\theta}} \right\} dt
\]

\[
= |I_1| + |I_2| + I_3 \quad \text{(say)}
\]

where \(\beta\) is a number chosen such that \(\frac{1+\eta}{3+\eta} < \beta \leq \frac{1}{2}\).
By Minkowski inequality
\[
\left| \sigma_n - f(x) \right| = |I_1| + |I_2| + |I_3|
\]

By Lemma 1 and 2,
\[
|I_1| = o(1) \int_0^{\frac{a}{t}} \phi_n(t) \exp \left( -\frac{T_n t^2}{4} \right) \frac{\sin V_n t}{2} dt + o(1) \int_0^{\frac{a}{t}} \phi_n(t) T_n t^3 dt
\]
\[
= I_{1.1} + I_{1.2}
\]
\[
I_{1.1} = o(1) \int_0^{\frac{a}{t}} \phi_n(t) \exp \left( -\frac{T_n t^2}{4} \right) \frac{\sin V_n t}{2} dt
\]

Applying Hölder’s inequality
\[
|I_1| = o\left[ \left( \int_0^{\frac{a}{t}} \phi_n(t) \sin^\beta(t) dt \right)^{\frac{1}{p}} \right] \cdot o\left[ \int_0^{\frac{a}{t}} \frac{1}{\sin^\beta t} dt \right]^{\frac{1}{q}}
\]
\[
= o\left( \frac{1}{m} \right)^{\frac{1}{q}} \cdot o\left( \int_0^{\frac{a}{t}} \frac{1}{\sin^\beta t} dt \right)^{\frac{1}{q}}
\]
\[
= o\left( \frac{1}{m} \right)^{\frac{1}{q}} m^{\frac{\beta - 1}{q}}
\]
\[
= o\left( \frac{1}{m} \right)^{\frac{1}{q}} m^{\frac{\beta - 1}{q}}
\]

\[
|I_{1.2}| = o\left( T_n \right) \left( \int_0^{\frac{a}{t}} \phi_n(t) T_n t^3 dt \right)
\]
\[
= o\left( \frac{1}{m} \right)^{\frac{1}{q}} \cdot o\left( \int_0^{\frac{a}{t}} \frac{t^2}{\sin^\beta t} dt \right)^{\frac{1}{q}}
\]
\[
= o\left( \frac{1}{m} \right)^{\frac{1}{q}} \cdot o\left( \int_0^{\frac{a}{t}} \frac{1}{\sin^\beta t} dt \right)^{\frac{1}{q}}
\]
\[
= o\left( \frac{1}{m} \right)^{\frac{1}{q}} m^{\frac{\beta - 1}{q}}
\]
\[
= o\left( \frac{1}{m} \right)^{\frac{1}{q}} m^{\frac{\beta - 1}{q}}
\]

(1)

(2)
Further, by Lemma 1

\[ I_2 = \frac{\frac{\partial \phi(t)}{\partial t} \exp \left( -\frac{T_1 t}{2} \right)}{t^2} \sin \frac{V_t}{2} dt + \frac{\frac{\partial \phi(t)}{\partial t} \sin \frac{V_t}{2}}{t^2} \frac{1}{T_1 t^3} dt \]

\[ = I_{13} - I_{22} \]

By the fact that \(|\sin x| \leq 1\) for all \(x\), and applying Hölder’s inequality

\[ I_{23} = O(1) \left[ \int_{a_s}^{b_s} \frac{\phi(t)}{t} dt \right] \]

\[ = O \left[ \frac{\phi(t)}{t} \int_{a_s}^{b_s} \left( \frac{1}{t^2} \right) dt \right] \]

\[ = O \left[ \frac{\phi(t)}{t} \int_{a_s}^{b_s} \sin \left( \frac{V_t}{2} \right) dt \right] \]

\[ = O \left[ \frac{\phi(t)}{t} \int_{a_s}^{b_s} \sin \left( \frac{V_t}{2} \right) dt \right] \]

(3)

\[ I_{22} = O \left( \int_{a_s}^{b_s} \phi(t) t^2 dt \right) \]

\[ = O \left[ \frac{\phi(t)}{t} \int_{a_s}^{b_s} \left( \frac{1}{t} \right) dt \right] \]

\[ = O \left[ \frac{\phi(t)}{t} \int_{a_s}^{b_s} \sin \left( \frac{V_t}{2} \right) dt \right] \]

\[ = O \left[ \frac{\phi(t)}{t} \int_{a_s}^{b_s} \sin \left( \frac{V_t}{2} \right) dt \right] \]

(4)

\[ \left| I_2 \right| = O(1) \left[ \int_{\frac{\phi(t)}{t}}^{\frac{\phi(t)}{t}} \left( \frac{1}{t} \right) dt \right] \]

\[ = O \left[ \frac{\phi(t)}{t} \int_{a_s}^{b_s} \sin \left( \frac{V_t}{2} \right) dt \right] \]

\[ = O \left[ \frac{\phi(t)}{t} \int_{a_s}^{b_s} \sin \left( \frac{V_t}{2} \right) dt \right] \]

\[ \text{Where } q = \frac{p}{p - 1} \]

\[ = O \left[ \frac{\phi(t)}{t} \int_{a_s}^{b_s} \sin \left( \frac{V_t}{2} \right) dt \right] \]

\[ = O \left[ \frac{\phi(t)}{t} \int_{a_s}^{b_s} \sin \left( \frac{V_t}{2} \right) dt \right] \]

\[ = O \left[ \frac{\phi(t)}{t} \int_{a_s}^{b_s} \sin \left( \frac{V_t}{2} \right) dt \right] \]
\[ = O \left( m^{2\beta} \phi \left( \frac{1}{m^{\beta}} \right) \right) \]
\[ = O \left( m^{\beta - \frac{\rho - \beta}{p}} \phi \left( \frac{1}{m^{\beta}} \right) \right) \]
\[ = O \left( m^{\beta + \frac{1}{p}} \phi \left( \frac{1}{m^{\beta}} \right) \right) \]

Combining (1), (2), (3), (4) and (5), we get

\[ |\sigma_n - f(x)| = O \left( m^{\beta + \frac{1}{p}} \phi \left( \frac{1}{m^{\beta}} \right) \right) \]

References: