

## APPROXIMATION OF A FUNCTION BELONGING TO THE CLASS $Lip(\psi(t), p)$ BY USING $[s, \alpha_n]$ MEANS

SANJAY MUKHERJEE

### Introduction

Meyer-Konig<sup>6</sup> introduces so called  $S_a$  method of summability which is one of the family of transformation including the Euler, Borel and Taylor (circle method) methods. Later Jakimovski<sup>4</sup> introduced  $[F, d_n]$  transformation which methods the Euler method (E,q) Karmata method ( $K^\lambda$ ) and Lotosky method as particular cases.

For the first time Meir and Sharma<sup>5</sup> introduced generalization of the  $S_a$  method and called it  $[S, \alpha_n]$  method. They obtained sufficient condition for the regularity of this method. They also examined the behaviour of its Lebesgue constant.

Let  $\{a_j\}$  be a given sequence of real complex numbers. We shall say that  $\{a_j\}$  is the  $[S, \alpha_n]$  transformations of  $\{S_j\}$ ; i.e. the sequence of partial sums of the series  $\sum a_n$  if

$$\{\sigma_n\} = \sum_{k=0}^{\infty} C_{nk} S_k; (n=0,1,2,3,\dots)$$

Converges, where  $(C_{nk})$  is given by the identity .

$$\prod_{j=0}^{\infty} \frac{1-a_j \theta}{1-a_j} = \sum_{k=0}^{\infty} C_{nk} \theta^k$$

The sequence  $\{S_j\}$  is said to be  $[S, \alpha_n]$  summable to  $\sigma$  if

$$\lim_{n \rightarrow \infty} \sigma_n = \sigma$$

Let  $f(x) \in L(0, 2\pi)$  and be periodic with period  $2\pi$  outside this range. Let the Fourier series associated with the function be

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} A_n(x)$$

and as usual we denote

$$\phi(t) = \phi_x(t) = \frac{1}{2} \{f(x+t) + f(x-t) - 2f(x)\}$$

Also

$$V_n = 1 + 2 \sum_{j=0}^n \frac{a_j}{1-a_j}$$

$$T = 2 \sum_{j=0}^n \left( \frac{a_j}{1-a_j} \right)^2$$

$m = [T_n]$ , the integral part of  $T_n$

and

$$a_n = \frac{2\pi}{m}$$

Meir and Sharma<sup>5</sup> while studying constant established that when  $V_n$  and  $T_n$  are bounded the  $[S a_n]$ , method sums only convergent Fourier series and so here after we assume  $T_n \rightarrow \infty$  and  $V_n \rightarrow \infty$  with  $n$ .

A function  $f \in \text{Lip} \alpha$  if

$$|f(x+t) - f(x)| = O(|t|^\alpha) \text{ for } 0 < \alpha \leq 1$$

and

$f(x) \in \text{Lip}(\alpha, p)$ , for  $0 \leq x \leq 2\pi$ , if

$$\left( \int_0^{2\pi} |f(x+t) - f(x)|^p dx \right)^{1/p} = O(|t|^\alpha), \text{ for } 0 < \alpha \leq 1, p \geq 1$$

A function  $f \in \text{Lip}(\phi(t), p)$  class for  $p \geq 1$  if

$$\left( \int_0^{2\pi} |f(x+t) - f(x)|^p dx \right)^{1/p} = O(\phi(t)),$$

Where  $\phi(t)$  is positive increasing function and  $f \in \text{Lip}(\phi(t), p)$  if

$$\left( \int_0^{2\pi} |f(x+t) - f(x)| \sin^\beta x dx \right)^{1/p} = O(\phi(t)), (\beta \geq 0)$$

We observe that

$\text{Lip} \alpha \subseteq \text{Lip}(\alpha, p) \subseteq \text{Lip}(\phi(t), p)$ , for  $0 < \alpha \leq 1, p \geq 1$

To prove the theorem we need following auxiliary result:

Lemma 1: The following estimates hold:

If

$$K_n(t) = e^{it} \sum_{j=0}^n \frac{1 - a_j}{1 - a_j e^{it}}$$

Then

$$|K_n(t)| = O\left(\frac{1}{t\sqrt{T_n}}\right)$$

and

$$K_n(t) = \exp[V_n it - T_n t^2] + O(T_n t^3) \text{ for } t \text{ to be very small.}$$

These are due to Meir and Sharma<sup>5</sup>

Lemma 2: If  $h(x, t)$  is a function of two variables defined for  $0 \leq t \leq 2\pi$ , then

$$\left\| \int h(x, t) dt \right\|_p \leq \left\| \int h(x, t) \right\|_p dt; (p > 1)$$

This is due to Hardy, Littlewood and Poly<sup>3</sup>.

**THEOREM:**

If  $f(x)$  is periodic with period  $2\pi$  and belongs to the class  $\text{Lip}(\psi(t), p)$  for  $p > 1$ , and if

$$\left\{ \int_0^{1/\sqrt{n}} \left( \frac{\psi(t)}{t^{1/p}} \right)^p dt \right\}^{1/p} = O\left( \psi\left(\frac{1}{\sqrt{n}}\right) \right)$$

and

$$\left\{ \int_{1/\sqrt{n}}^{\pi} \left( \frac{\psi(t)}{t^{1/p+2}} \right)^p dt \right\}^{1/p} = O \left( \psi \left( \frac{1}{\sqrt{n}} \right) n \right),$$

Then

$$\max_{0 \leq x \leq 2\pi} |f(x) - \sigma_n(f, x)| = O \left( \psi \left( \frac{1}{\sqrt{n}} \right) (n)^{1/2p} \right)$$

In the present chapter we have obtained the degree of approximation of a function belonging to the class Lip (ψ (t), p) by using [ S a n ] means. Our theorem is as follows:-

**THEOREM:**

If f(x) is periodic with period 2π and belongs to class Lip (ψ (t), p) for p≥1, and β≥0 and  $\frac{1}{3} < \eta < \frac{1}{2}$  and if

$$\left\{ \int_0^{a_n} |\phi_x(t) \sin^\beta t|^p dt \right\}^{1/p} = O \left\{ \phi \left( \frac{1}{m} \right) \right\}$$

$$\left\{ \int_{a_n}^{(a_n)^\beta} \left| \frac{\phi_x(t) \sin^\beta t}{t^2} \right|^p dt \right\}^{1/p} = O \left\{ m^2 \phi \left( \frac{1}{m} \right) \right\}$$

$$\left\{ \int_{(a_n)^\beta}^{\pi} \left| \frac{\phi_x(t) \sin^\beta t}{t^2} \right|^p dt \right\}^{1/p} = O \left\{ m^{2\beta} \phi \left( \frac{1}{m^\beta} \right) \right\}$$

Then

$$|\sigma_n(f, x) - f(x)| = O \left\{ m^{\beta + \frac{1}{p}} \phi \left( \frac{1}{m^\beta} \right) \right\}$$

**PROOF:**

The [ S a n ] transform of partial sums of Fourier series is given by

$$\begin{aligned} \sigma_n - f(x) &= \frac{2}{\pi} \int_0^\pi \frac{\phi_x(t)}{t} \operatorname{Im} \left\{ \sum_{k=0}^n C_{nk} \sin \left( k + \frac{1}{2} \right) t \right\} dt \\ &= \frac{2}{\pi} \int_0^\pi \frac{\phi_x(t)}{t} \operatorname{Im} \left\{ \exp \left( \frac{it}{2} \right) \sum_{k=0}^n C_{nk} \exp(ikt) \right\} dt \\ &= \frac{2}{\pi} \left[ \int_0^{a_n} + \int_{a_n}^{(a_n)^\beta} + \int_{(a_n)^\beta}^\pi \right] \frac{\phi_x(t)}{t} \operatorname{Im} \left\{ \exp \left( \frac{it}{2} \right) \prod_{j=0}^n \frac{1 - a_j}{1 - a_j e^{it}} \right\} dt \\ &= |I_1| + |I_2| + I_3 \quad (\text{say}) \end{aligned}$$

where β is a number chosen such that  $\frac{1+\eta}{3+\eta} \leq \beta \leq \frac{1}{2}$

By Minkowski inequality

$$|\sigma_n - f(x)| = |I_1| + |I_2| + |I_3|$$

By Lemma 1 and 2,

$$\begin{aligned} |I_1| &= O(1) \int_0^{a_n} \frac{\phi_x(t)}{t} \exp\left(\frac{-T_n t^2}{4}\right) \sin \frac{V_n t}{2} dt + O(1) \int_0^{a_n} \frac{\phi_x(t)}{t} T_n t^3 dt \\ &= I_{1.1} + I_{1.2} \\ I_{1.1} &= O(1) \int_0^{a_n} \frac{\phi_x(t)}{t} \exp\left(\left(\frac{-T_n t^2}{4}\right) \sin \frac{V_n t}{2} dt\right) \end{aligned}$$

Applying Hölder's inequality

$$\begin{aligned} |I_{1.1}| &= O\left[\left\{\int_0^{a_n} |\phi_x(t) \sin^\beta(t)|^p dt\right\}^{1/p}\right] \cdot O\left[\left\{\int_0^{a_n} \left|\frac{1}{\sin^\beta t}\right|^q dt\right\}^{1/q}\right] \\ &= O\left[\phi\left(\frac{1}{m}\right)\right] \cdot O\left[\left\{\int_0^{a_n} t^{-\beta q} dt\right\}^{1/q}\right] \\ &= O\left[\phi\left(\frac{1}{m}\right) m^{\beta - \frac{1}{q}}\right] \\ &= O\left[m^{\beta + \frac{1}{p}} \phi\left(\frac{1}{m}\right)\right] \end{aligned} \tag{1}$$

$$\begin{aligned} |I_{1.2}| &= O(T_n) \int_0^{a_n} \frac{\phi_x(t)}{t} t^3 dt \\ &= O\left[\left\{\int_0^{a_n} |\phi_x(t) \sin^\beta(t)|^p dt\right\}^{1/p}\right] \cdot O\left[\left\{\int_0^{a_n} \left|\frac{t^2}{\sin^\beta t}\right|^q dt\right\}^{1/q}\right] \\ &= O\left[\phi\left(\frac{1}{m}\right)\right] \cdot O\left[\left\{\int_0^{a_n} t^{(2-\beta)q} dt\right\}^{1/q}\right] \\ &= O\left[\phi\left(\frac{1}{m}\right)\right] \cdot O\left[\left\{m^{(\beta-2)q-1}\right\}^{1/q}\right] \\ &= O\left[\phi\left(\frac{1}{m}\right) m^{\beta-3+1-\frac{1}{q}}\right] \\ &= O\left[m^{\beta + \frac{1}{p}} \phi\left(\frac{1}{m}\right)\right] \end{aligned} \tag{2}$$

Further, by Lemma 1

$$I_2 = \frac{2}{\pi} \int_{\alpha_n}^{(\alpha_n)^\beta} \frac{\phi_x(t)}{t} \exp\left(\frac{-T_n t^2}{4}\right) \sin \frac{\nu_n t}{2} dt + \frac{2}{\pi} \int_{\alpha_n}^{(\alpha_n)^\beta} \frac{\phi_x(t)}{t} T_n t^3 dt$$

$$= I_{2.1} + I_{2.2}$$

By the fact that  $|\sin x| \leq 1$  for all  $x$ , and applying Hölder's inequality

$$I_{2.1} = O(1) \int_{\alpha_n}^{(\alpha_n)^\beta} \frac{\phi_x(t)}{t} dt$$

$$= O \left[ \left\{ \int_{\alpha_n}^{(\alpha_n)^\beta} \left| \frac{\phi_x(t) \sin^\beta t}{t^2} \right|^p dt \right\}^{1/p} \right] \left[ \left\{ \int_{\alpha_n}^{(\alpha_n)^\beta} \left| \frac{t}{\sin^\beta t} \right|^q dt \right\}^{1/q} \right]$$

$$= O \left[ m^2 \phi\left(\frac{1}{m}\right) \right] \left[ \left\{ \int_{\alpha_n}^{(\alpha_n)^\beta} t^{(1-\beta)q} dt \right\}^{1/q} \right]$$

$$= O \left[ m^{\beta + \frac{1}{q}} \phi\left(\frac{1}{m}\right) \right]$$
(3)

$$I_{2.2} = O(T_n) \int_{\alpha_n}^{(\alpha_n)^\beta} \phi_x(t) t^2 dt$$

$$= O \left[ \left\{ \int_{\alpha_n}^{(\alpha_n)^\beta} \left| \frac{\phi_x(t) \sin^\beta t}{t^2} \right|^p dt \right\}^{1/p} \right] \left[ \left\{ \int_{\alpha_n}^{(\alpha_n)^\beta} \left| \frac{t^4}{\sin^\beta t} \right|^q dt \right\}^{1/q} \right]$$

$$= O \left[ m^2 \phi\left(\frac{1}{m}\right) \right] \left[ \left\{ \int_{\alpha_n}^{(\alpha_n)^\beta} t^{(4-\beta)q} dt \right\}^{1/q} \right]$$

$$= O \left[ m^{\beta + \frac{1}{q}} \phi\left(\frac{1}{m}\right) \right]$$
(4)

$$|I_3| = O(1) \left\{ \int_{\alpha_n}^{\pi} \frac{|\phi_x(t)|^p}{t \cdot t \sqrt{T_n}} dt \right\}^{1/p}$$

$$= O(1) \left\{ \int_{\alpha_n}^{\pi} \left| \frac{\phi_x(t)}{t^2} \sin^\beta t \right|^p dt \right\}^{1/p} \left[ \left\{ \int_{\alpha_n}^{\pi} \left| \frac{t^2}{\sin^\beta t} \frac{1}{t \cdot t \sqrt{T_n}} \right|^q dt \right\}^{-1/q} \right]$$

Where  $q = \frac{p}{p-1}$

$$= O \left\{ m^{2\beta} \phi\left(\frac{1}{m^\beta}\right) \right\} \left[ \left\{ \int_{\alpha_n}^{\pi} \frac{t^{(2-\beta)q}}{t^2 \sqrt{T_n}} dt \right\}^{-1/q} \right]$$

$$= O \left\{ m^{2\beta} \phi\left(\frac{1}{m^\beta}\right) \right\} \left( \frac{1}{m^{1/2q}} \right) \left[ \left\{ \int_{\alpha_n}^{\pi} t^{(2-\beta)q-2} dt \right\}^{-1/q} \right]$$

$$\begin{aligned}
&= O \left\{ m^{2\beta} \phi \left( \frac{1}{m^\beta} \right) \right\} O \left\{ m^{\beta^2 - \beta - \frac{\beta}{p}} \right\} \\
&= O \left\{ m^{\beta^2 + \beta - \frac{\beta}{p}} \phi \left( \frac{1}{m^\beta} \right) \right\} \\
&= O \left\{ m^{\beta + \frac{1}{p}} \phi \left( \frac{1}{m^\beta} \right) \right\}
\end{aligned} \tag{5}$$

Combining (1), (2), (3), (4) and (5), we get

$$|\sigma_n - f(x)| = O \left\{ m^{\beta + \frac{1}{p}} \phi \left( \frac{1}{m^\beta} \right) \right\}$$

### References:

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