Blow-up phenomena and global existence to a weakly dissipative shallow water equation

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Abstract. We first establish the local well-posedness for a weakly dissipative shallow water equation which includes both the weakly dissipative Camassa-Holm equation and the weakly dissipative Degasperis-Procesi equation as its special cases. Then two blow-up results are derived for certain initial profiles. Finally, We study the long time behavior of the solutions.

1 Introduction

Degasperis and Procesi [1] studied the following family of third-order dispersive nonlinear equations

$$u_t + c_0 u_x + \gamma u_{xxx} - \alpha^2 u_{xxt} = \left(c_1 u^2 + c_2 u_x^2 + c_3 u u_{xx} \right)_x ,$$
(1.1)

where $\alpha, c_0, c_1, c_2, c_3$ and γ are real constants. They found that there are only three equations that satisfy the asymptotic integrability condition, namely, the Korteweg-de Vries(KdV) equation, the Camassa-Holm(CH) equation and the Degasperis-Procesi(DP) equation.

For $\alpha = c_0 = c_2 = c_3 = 0$, Eq. (1.1) becomes the well-known KdV equation

$$u_t - 6uu_x + u_{xxx} = 0 ,$$
(1.2)

which describes the unidirectional propagation of waves at the free surface of shallow

water under the influence of gravity [2,3]. The Cauchy problem of Eq. (1.2) has been studied widely. Bourgain [4] proved that the Cauchy problem associated with the equation is globally well-posed in H^s, H^s $(s \geq 0)$. Kenig [3] and Tao [5] showed that it is globally well-posed for $u \in L^2(S)$ The local well-posedness of Eq. (1.2) was pushed down to $H^s, s \geq -\frac{1}{2}$ by Kenig et al. [6]. Whitham [7] found that the equation does not accommodate wave breaking.

For $c_0 = 0$, $c_1 = -\frac{3c_3}{2\alpha^2}$ and $c_2 = \frac{c_3}{2}$, Eq. (1.1) becomes the CH equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx},$$
(1.3)

which is a model describing the unidirectional propagation of shallow water waves over a flat bottom [8]. The CH equation has a bi-Hamiltonian structure [9] and is completely integrable [10,11]. It admits, in addition to smooth waves, a multitude of traveling wave solutions with singularities:peakons, cuspons, stumpons and composite waves [8,12]. Its solitary waves are stable solitons[13,14], retaining their shape and form after interactions[15].

The Cauchy problem of Eq. (1.3) has been studied extensively. Constantin [16] and Rodriguez-Blanco [17]studied the locally well-posed for initial data $u_0 \in H^s(R)$ with $s > \frac{3}{2}$. More interestingly, it has strong solutions that are global in time [18,19] as well as solutions that blow up in finite time [18,20,21]. On the other hand, Bressan [22] and Xin [23] showed that the Eq. (1.3) has global weak solutions with initial data $u_0 \in H^1$

For $c_1 = -\frac{3c_3}{\alpha^2}$ and $c_2 = c_3$, Eq. (1.1) becomes the DP equation

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx},$$

(1.4)

which can be used as a model for nonlinear shallow water dynamics and its asymptotic accuracy is the same as Eq. (1.3) [24]. Degasperis et al. [24] also showed that Eq. (1.4) has a bi-Hamiltonian structure with an infinite sequence of conserved quantities and admits exact peakon solutions which are analogous to Eq. (1.3) peakons [8,13,14]. Dullin etal. [25] showed that Eq. (1.4) can be obtained from the shallow water elevation equation by an appropriate Kodama transformation. The numerical stability of solitons and peakons, the multi-soliton solutions and their peakon limits, together with an inverse scattering method to compute N-peakon solutions to DP equation have been investigated respectively in [26]-[28]. After Eq. (1.4) appeared, it has attracted many researchers to discover its dynamics (see[29]-[36]). Yin [29,30] proved the local well-posedness of Eq. (1.4) with initial data $u_0 \in H^s(R)\left(s > \frac{3}{2}\right)$ on the line and on the circle, and derived the precise blow-up scenario and a blow-up result. The global existence of strong solutions and global weak solutions to Eq. (1.4) were studied in [31,32]. Similar to Eq. (1.3) [18]-[23], Eq. (1.4) has not only global strong solutions [31,33], but also blow-up solutions [33]-[35]. Apart from these, Coclite and Karlsen [36] proved that it has global entropy weak solutions in $L^1(R) \cap BV(R)$ and $L^2(R) \cap L^4(R)$

For $\gamma = 0$ and $c_1 = c_2 = c_3 \neq 0$ under suitable mathematical transforms and several restrictions on its coefficients[37], Eq. (1.1) becomes

$$u_t - u_{xxt} + (a+b)uu_x = au_x u_{xx} + buu_{xxx},$$

(1.5)

where a > 0andb > 0are arbitrary positive constants. Obviously, Eq. (1.5) is a generalization of both the CH and DP equations. We can see that in models (1.3) and(1.4) , the coefficient of x uu is equal to the coefficient of $u_x u_{xx}$ plus the coefficient of uu_{xxx}, that is

      3=2+1, 4=3+1.

Recently, Lai and Wu [38,39] obtained the existence of the strong solution and the global existence of its weak solutions.

In general, the energy dissipation mechanisms are difficult to avoid in a real world, many authors modified those models with dissipation. Ott and Sudan [40] investigated the KdV equation with the presence of dissipation and their effect on solution of the KdV equation. The long time behavior of solutions to the weakly dissipative KdV equation was studied by Ghidaglia [41]. Recently, Wu and Yin investigated the weakly dissipative CH equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} - \lambda(u - u_{xx}),$$

(1.6)

on the line [42] and on the circle [43]. They also studied the weakly dissipative DP equation

$$u_t - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx} - \lambda(u - u_{xx}),$$

(1.7)

on the line in [44,45] and on the circle [46], where $\lambda > 0$ is a constant. In [47], Yan, Li and Zhang considered the global existence and blow-up for the weakly dissipative Novikov equation

$$u_t - u_{xxt} + 4u^2 u_x = 3uu_x u_{xx} + u^2 u_{xxx} - \lambda(u - u_{xx}), \quad \lambda > 0.$$
(1.8)

Motivated by all the above works, in the paper, we'll consider the Cauchy problem of the following weakly dissipative shallow water equation

$$\begin{cases} u_t - u_{xxt} + (a+b)uu_x + \lambda(u - u_{xx}) = au_x u_{xx} + buu_{xxx} \\ u(0,x) = u_0(x) \end{cases}$$
(1.9)

Where $a > 0, b > 0$ and $\lambda > 0$ are arbitrary constants.

Although Eq. (1.9) is very similar to Eq. (1.5) in the form and the short time behaviors, there are some essential differences between them. The global solutions of Eq. (1.9) decay to zero as time goes to infinity provided the potential $m_0 = (1 - \partial_x^2)u_0$ is of one sign. This long time behavior is an important feature that Eq. (1.5) does not possess. On the other hand, the weakly dissipative term breaks the conservation law of Eq. (1.5) :

$$\int_R u(t,x)\,dx = \int_R u(0,x)\,dx = \text{cons}\tan t,$$
(2.0)

which play an important role in the study of Eq. (1.5) .

2 Local well-posedness and blow-up scenario

In this section, we prove the local well-posedness and the precise blow-up scenario of Eq. (1.9) .
We now apply Kato's theory to establish the local well-posedness. For convenience, we reformulate problem (1.8) as follows:

$$\begin{cases} m_t + au_x m + bum_x + \lambda m = 0 \\ m(0,x) = u_0(x) - u_{0,xx}(x) \end{cases},$$
(2.1)

Where $m = u - u_{xx}$, $a > 0$, $b > 0$ and $\lambda > 0$ are arbitrary constants.

Note that if $p(x) := \frac{1}{2}e^{|x|}, x \in R$, then $(1 - \partial_x^2)^{-1} f = p * f$ for all $f \in L^2(R)$, and $p * m = u$.
Here we denote by $*$ the convolution. Using this identity, Eq. (2.1) takes the form of quasi-linear evolution equation of hyperbolic type:

$$\begin{cases} u_t + buu_x = -\partial_x p * \left(\frac{a}{2}u^2 + \frac{3b-a}{2}u_x^2 \right) - \lambda u \\ u(0,x) = u_0(x) \end{cases},$$
(2.2)

or in the equivalent form:

$$\begin{cases} u_t + buu_x = -\partial_x \left(1 - \partial_x^2\right)^{-1}\left(\frac{a}{2}u^2 + \frac{3b-a}{2}u_x^2 \right) - \lambda u \\ u(0,x) = u_0(x) \end{cases},$$
(2.3)

The local well-posedness of the Cauchy problem of Eq. (2.2) with initial data $u_0 \in H^s(R), s > \dfrac{3}{2}$ can be obtained by applying the Kato's theorem [48]. More precisely, we have the following local well-posedness result.

Theorem 2.1. Let $u_0 \in H^s(R), s > \dfrac{3}{2}$, there exist a maximum $T > 0$, and a unique solution u to Eq. (2.2) (or Eq. (1.9)) which depends continuously on initial data u_0 such that $u(t,x) \in C([0,T); H^s(R)) \cap C^1([0,T); H^{s-1}(R))$. Moreover, T is independent of s in the sense that if $u_0 \in H^{s'}(R)$ for some $s' \neq s, s' > \dfrac{3}{2}$, then, $u(t,x) \in C([0,T); H^{s'}(R)) \cap C^1([0,T); H^{s'-1}(R))$ with the same T. In particular, if $u_0 \in H^\infty(R)$, then $u \in C([0,T); H^\infty(R))$.

Proof. Set $H^{s-1}(R), Y = H^s(R), \Lambda = (1-\partial_x^2)^{\frac{1}{2}}, Q = \Lambda$,
$f = -\partial_x p * \left(\dfrac{a}{2} u^2 + \dfrac{3b-a}{2} u_x^2 \right) - \lambda u$ and $A(u) = bu\partial_x$ with constant $b > 0$. By applying the Kato's theorem [48] we can finish the proof of this theorem. One can see the similar proof in [17, 38] for details.

Now, we present the precise blow-up scenario for sufficiently regular solutions to Eq. (1.9) .

Theorem 2.2. Let $u_0 \in H^s(R), s > \dfrac{3}{2}$ be given and assume that T is the maximal existence time of the solution $u(\cdot, u_0)$ to Eq. (1.9) . Then the corresponding solution blows up in finite time if and only if

$$\liminf_{t \to T} \left\{ \inf_{x \in R} \left[u_x(t,x) \right] \right\} = -\infty$$

Proof. By Theorem 2.1 and a simple density argument, we only need to show that the above theorem holds for s = 3. Multiplying Eq.(1.9) by $m = u - u_{xx}$ and integrating by parts, we have

$$\frac{1}{2} \frac{d}{dt} \int_R m^2 dx = -a \int_R u_x m^2 dx - b \int_R u m_x m \, dx - \lambda \int_R m^2 dx$$

$$(2.4)$$

$$= \left(\frac{b}{2} - a \right) \int_R u_x m^2 dx - \lambda \int_R m^2 dx \, .$$

Now, we differentiate Eq. (2.1) with respect to the spatial variable x , then multiply theresultant equation by m $_x$. Applying m = u- u $_{xx}$ and integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\int_R m_x^2 dx = -\int_R \left(au_x m + bum_x + \lambda m \right)_x m_x dx$$

$$= -\int_R \left(au_{xx}m + (a+b)u_x m_x + bum_{xx} + \lambda m_x \right) m_x dx$$

$$= -\int_R a(u-m)mm_x dx - (a+b)\int_R u_x m_x^2 dx$$

$$\quad - b\int_R um_{xx}m_x dx - \lambda \int_R m_x^2 dx$$

$$= \frac{a}{2}\int_R u_x m^2 dx - \left(a+\frac{b}{2} \right)\int_R u_x m_x^2 dx - \lambda \int_R m_x^2 dx \; .$$

(2.5)

Here, we used $\int_R m^2 m_x dx = 0$ and $-\int_R um_{xx}m_x dx = \frac{1}{2}\int_R u_x m_x^2 dx$.

Combining (2.4) with (2.5), we have

$$\frac{1}{2}\frac{d}{dt}\int_R \left(m^2 + m_x^2 \right) dx = -\frac{a-b}{2}\int_R u_x m^2 dx - \frac{2a+b}{2}\int_R u_x m_x^2 dx$$

$$(2.6) \quad -\lambda \int_R \left(m^2 + m_x^2 \right) dx$$

If u_x is bounded from below on $[0,T) \times R$, i.e., there exists a constant $M > 0$ such that

$$-u_x(t,x) \le M \text{ on } [0,T) \times R \; .$$

(2.7)

It then follows from (2.6) that

$$\frac{d}{dt}\int_R \left(m^2 + m_x^2 \right) dx \le 2\left(|2a+b|M - \lambda \right)\int_R \left(m^2 + m_x^2 \right) dx = -a\int_R -a -b\int_R um_x \; .$$

(2.8)

By means of Gronwall's inequality, we have

$$\|m\|_{H^1}^2 \le e^{2(|2a+b|M-\lambda)t} \|m_0\|_{H^1}^2 \; .$$

(2.9)

The above inequality, Sobolev's imbedding theorem and Theorem 2.2 ensure that the solution does not blow up in finite time.

On the other hand, by Theorem 2.2 and Sobolev's imbedding theorem, we see that if

$$\liminf_{t \to T} \left\{ \inf_{x \in R} \left[u_x(t,x) \right] \right\} = -\infty$$

then the solution will blow up in finite time. This completes the proof of the theorem.

3 Blow-up

In this section, we will establish two blow-up results for strong solutions to Eq. . (1.9) , from which we can see the effect of weak dissipation to the blow-up phenomena of the solution

Consider now the following initial value equation

$$\begin{cases} q_t = bu(t,q) \\ q(0,x) = x \end{cases}, \quad t \in [0,T)$$

(3.1)

Lemma 3.1. [38] Let $u_0 \in H^s(R)$, $s > 3$, and let T > 0 be the maximal existence time of thesolution to Eq. (1.9) . Then Eq. (3.1) has a unique solution $q \in C^1([0,T) \times R, R)$ Moreover, the map q(t,·) is an increasing diffeomorphism of R with

$$q_x(t,x) = \exp\left(\int_0^t bu_x \left(s, q(s,x) \right) ds \right) > 0 \; ,$$

(3.2)

for $(t,x) \in [0,T) \times R$.

Lemma 3.2 . Let $u_0 \in H^s(R)$ with $s>3$, and let $T>0$ 0be the maximal existence time of the solution u to Eq. (1.9) Then we have

$$m(t,q(t,x))q_x^2(t,x)=m_0(x)e^{-\int_0^t[(a-2b)u_x+\lambda]d\tau} ,$$

(3.3)

where $(t,x)\in[0,T)\times R$.

Proof . Differentiating the left-hand side of Eq. (3.3) with respect to time variable t , in view of the relations (2.1) and (3.1) , we obtain

$$\frac{d}{dt}\left[m(t,q(t,x))q_x^2(t,x)\right]$$
$$= m_t q_x^2 + 2mq_x q_{xt} + m_t q_t q_x^2$$
$$=(m_t + 2bmu_x + bm_x u)q_x^2$$
$$=(m_t + au_x m + bum_x + \lambda m)q_x^2 - [(a-2b)u_x + \lambda]mq_x^2$$
$$-[(a-2b)u_x + \lambda]mq_x^2 .$$

Using $q_x(0,x)=1$ and solving the above equation, we have

$$m(t,q(t,x))q_x^2(t,x)=m_0(x)e^{-\int_0^t[(a-2b)u_x+\lambda]d\tau} .$$

This completes the proof of the lemma.

Theorem 3.3. Let $a>b>0$, $3b-a\geq0$ and $u_0 \in H^s(R)$ with $s>\dfrac{3}{2}$. Assume that $u_0(x)\backslash$ is odd and $u_0'(x)<-\dfrac{\lambda}{a-b}$ then the solution to Eq. (1.8) blows up in finite time.

Proof . Let $T>0$ be the existence time of the solution $u(t.)$ of Eq.(1.9) with the initial data $u_0(x)\in H^3(R)$. Differentiating Eq. (2.2) with respect to x , we have

$$u_{xt} = -bu_x^2 - buu_{xx} + \frac{a}{2}u^2 + \frac{3b-a}{2}u_x^2$$
$$-p*\left(\frac{a}{2}u^2 + \frac{3b-a}{2}u_x^2\right) - \lambda u_x .$$

Here, we used $\partial_x^2 p*f = p*f - f$.

Note that Eq.(2.2) is invariant under the symmetry transformation $u(x,t)\to(-u,-x)$. Thus, we deduce that if $u_0(x)$ is odd, then $u(x,t)$ is odd for any $t\in[0,T)$. By continuity with respect to x of u and u_{xx}, we have

$$u(t,0)=u_{xx}(t,0)=0 , \qquad \forall t\in[0,T) .$$

(3.5)

Hence, in view of (3.4) , we have

$$u_{xt}(t,0) = -bu_x^2(t,0) + \frac{3b-a}{2}u_x^2(t,0)$$

$$-p*\left(\frac{a}{2}u^2 + \frac{3b-a}{2}u_x^2\right)(t,0) - \lambda u_x(t,0) ,$$

(3.6)

Moreover, we have

$$u_{xt}(t,0) \le -\frac{a-b}{2}u_x^2(t,0) - \lambda u_x(t,0) .$$

(3.7)

Here, we use the relation $p*\left(\frac{a}{2}u^2 + \frac{3b-a}{2}u_x^2\right)(t,0) \ge 0$.

Let $m(t) = u_x(t,0)$ for $t \in [0,T)$, it follows from (3.7) that

$$\frac{dm(t)}{dt} \le -\frac{a-b}{2}m^2(t) - \lambda m(t) .$$

(3.8)

If $m(0) < -\frac{2\lambda}{a-b}$, then $m(t) < 0$ for all $t \in [0,T)$. From the above inequality we obtain

$$\frac{d}{dt}\left(\frac{1}{m(t)}\right) - \frac{\lambda}{m(t)} \ge \frac{a-b}{2} .$$

(3.9)

Solving (3.9), we have

$$\left(\frac{1}{m(0)} + \frac{a-b}{2\lambda}\right)e^{\lambda t} - \frac{a-b}{2\lambda} \le \frac{1}{m(t)} < 0 .$$

(3.10)

Moreover, we can conclude that $T < \infty$. In fact, if $T = \infty$, i.e., the solution exists globally in time. From the assumption of the theorem, we have $\frac{1}{m(0)} + \frac{a-b}{2\lambda} > 0$, then $\left(\frac{1}{m(0)} + \frac{a-b}{2\lambda}\right)e^{\lambda t} \to \infty$ as $t \to \infty$, which contradicts (3.10). This completes the proof of the theorem .

Theorem 3.4 . Let $a > b > 0, 3b - a \ge 0$,and $u_0 \in H^s(R)$ with $s > \frac{3}{2}$. Assume that $u_0(x)$ is odd and $u_0'(x) < -\frac{\lambda}{b}$ such that

$$\begin{cases} m_0(x) = u_0(x) - u_{0,xx}(x) \ge 0, for , x \le 0 \\ m_0(x) = u_0(x) - u_{0,xx}(x) \le 0, for , x \ge 0 \end{cases}$$

then the solution to Eq. (1.9) blows up in finite time.

Proof. Similar to Theorem 4.2, we know u(t,x) and u_{xx}(t,x) are odd, and the following equality still holds true

$$u_{xt}(t,0) = -bu_x^2(t,0) + \frac{3b-a}{2}u_x^2(t,0)$$

$$-p*\left(\frac{a}{2}u^2 + \frac{3b-a}{2}u_x^2\right)(t,0) - \lambda u_x(t,0) \tag{3.11}$$

$$= -bu_x^2(t,0) + \frac{3b-a}{2}u_x^2(t,0) - \frac{a-3b}{2}p*(u^2-u_x^2)(t,0)$$

$$-\frac{3b}{2}p*(u^2)(t,0) - \lambda u_x(t,0)$$

Note that $p*(u^2-u_x^2)(t,0) \geq -u_x^2(t,0)$ (see [49]) and $p*(u^2)(t,0) \geq 0$.

Then, we have

$$u_{xt}(t,0) \leq -bu_x^2(t,0) - \lambda u_x(t,0) . \tag{3.12}$$

Letting $m(t) = u_x(t,0)$ for $t \in [0,T)$, we have

$$\frac{dm(t)}{dt} \leq -bm^2(t) - \lambda m(t) . \tag{3.13}$$

if $m(0) < -\frac{\lambda}{b}$,then $m(t) < 0$ for all $t \in [0,T)$. From the above inequality we obtain

$$\frac{d}{dt}\left(\frac{1}{m(t)}\right) - \frac{\lambda}{m(t)} \geq b . \tag{3.14}$$

Solving (3.14) , we have

$$\left(\frac{1}{m(0)} + \frac{b}{\lambda}\right)e^{\lambda t} - \frac{b}{\lambda} \leq \frac{1}{m(t)} < 0 . \tag{3.15}$$

Moreover, we can conclude that $T < \infty$. Since if $T = \infty$, i.e., the solution exists globally in time. From the assumption of the theorem, we have $\frac{1}{m(0)} + \frac{b}{\lambda} > 0$, which implies $\left(\frac{1}{m(0)} + \frac{b}{\lambda}\right)e^{\lambda t} \to \infty$ as $T < \infty$. This completes the proof of the theorem. .

4 Global solution

In this section, we prove a global existence theorem for strong solutions to Eq. (1.9)

Lemma 4.1. Let $u_0 \in H^s(R)$ with $s > 3$, and let $T > 0$ be the maximal existence time of the solution u to Eq. (1.9) . Then we have

$$\|u(t,x)\|_{H^1}^2 = e^{-2\lambda t}\|u_0(x)\|_{H^1}^2 , \qquad \forall t \in [0,T) . \tag{4.1}$$

Proof . The proof is similar to that of lemma 2.5 in [42], we omit the details.

Theorem 4.2. Let $u_0 \in H^s(R)$ with $s > \frac{3}{2}$ is such that the associated potential $m(0) = (1-\partial_x^2)u_0$ satisfies $m(0) \leq 0$ on $(-\infty, x_0]$ and $m \geq 0$ on $[x_0,\infty)$ for some point $x_0 \in R$. Then the corresponding solution $u(x,t)$ to Eq. (1.8) exists globally in time. Moreover, the global solution decays to 0 in the H^s-norm as time goes to infinity.

Proof. Let T be the maximal existence time of the solution $u(\cdot,u_0)$, which is guaranteed by Theorem 2.1. Applying a simple density argument, we only need to show that the above theorem holds for some $s > \frac{3}{2}$. Here we assume $s = 3$ to prove the above theorem .

Note that $u(t,x) = p*m$ with $p = \frac{1}{2}e^{-|x|}$,$x \in R$,then

$$u(t,x) = \frac{1}{2} e^{-x} \int_{-\infty}^{x} e^{\xi} m(t,\xi) d\xi$$

$$+ \frac{1}{2} e^{x} \int_{x}^{\infty} e^{-\xi} m(t,\xi) d\xi \ , \ (t,x) \in [0,T) \times R \ .$$

(4.2)

Following the same argument in the proof of (3.6) in[42], we also have $u_x(t,x) \geq -\|u(t,\cdot)\|_{L^\infty}$ (4.3)

From the Agmon's inequality, we obtain

$$\|u\|_{L^\infty} \leq \|u\|_{L^2}^{\frac{1}{2}} \|\partial_x u\|_{L^2}^{\frac{1}{2}} \leq \frac{1}{\sqrt{2}} \sqrt{\|u\|_{L^2}^2 + \|u_x\|_{L^2}^2}$$

$$\leq \frac{1}{\sqrt{2}} \|u\|_{H^1} \ .$$

(4.4)

Thus, in view of (4.3) - (4.4) and Lemma 4.1, we get

$$u_x(t,x) \geq -\frac{e^{-2\lambda t}}{\sqrt{2}} \|u_0\|_{H^1} \geq -\frac{1}{\sqrt{2}} \|u_0\|_{H^1} \ .$$

(4.5)

It follows from(4.5) and Theorem 2.2 that $T = \infty$, i.e., the solution $u(t, x)$ exists globally in time.

From (4.5), we obtain that $-u_x(t,x) \leq \dfrac{\lambda}{2|2a+b|}$ for sufficiently large t. It then follows from (2.6) that

$$\frac{d}{dt} \int_{R} (m^2 + m_x^2) dx \leq -\lambda \int_{R} (m^2 + m_x^2) dx \ .$$

(4.6)

By means of Gronwall's inequality, we have

$$\|u\|_{H^3}^2 \leq \|m\|_{H^1}^2 \leq e^{-\lambda t} \|m_0\|_{H^1}^2 \ .$$

(4.7)

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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