The Bulletin of Society for Mathematical Services and Standards *Online: 2014-09-01*
ISSN: 2277-8020, Vol. 11, pp 31-39
doi:10.18052/www.scipress.com/BSMaSS.11.31
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ON SOLUTION OF TWO-POINT FUZZY BOUNDARY VALUE PROBLEMS

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Keywords: Second-order fuzzy differential equations, fuzzy boundary value problems, Hukuhara differentiability.

Abstract: In this paper, we investigate the existence and uniqueness of solutions of two-point fuzzy boundary value problems for second-order fuzzy differential equations. Some sufficient conditions are presented that guarantee the existence and uniqueness of solutions under the approach of Hukuhara differentiability.

1. Introduction

Fuzzy set theory is a powerful tool for modeling uncertainly and for processing vague or subjective information in mathematical models [3]. Particulary, the study of fuzzy differential equation forms a suitable setting for mathematical modeling of real world problems in which uncertainties or vagueness pervade. In fact, fuzzy differential equations is a very important topic from theoretical point of view [5], [6], [7], [9] and also it has applications, for example, in population models [4], civil engineering [8], medicine [1], [2].

In this paper, an investigation is made on the solution of two-point fuzzy boundary value problems by using the Hukuhara differentiability. To put it precisely, the two-point fuzzy boundary value problem is given as the form

$$Ly := P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x), \quad x \in (a,b) \tag{1.1}$$

$$B_1(y) := Ay(a) + By'(a) = [k_1]^\alpha \tag{1.2}$$

$$B_2(y) := Cy(b) + Dy'(b) = [k_2]^\alpha \tag{1.3}$$

where $[k_1]^\alpha = \left[\underline{k_1}_\alpha, \overline{k_1}_\alpha\right], [k_2]^\alpha = \left[\underline{k_2}_\alpha, \overline{k_2}_\alpha\right]$ are symmetric triangular fuzzy numbers, $A^2 + B^2 \neq 0$, $C^2 + D^2 \neq 0$, $P_0(x)$, $P_1(x)$, $P_2(x)$, $F(x)$ are continuous functions and $P_0(x)$ has no zeros on (a,b).

2. Preliminaries

In this section, we give some definitions and introduce the necessary notation which will be used throughout the paper.

Definition 2.1. A fuzzy number is a function $u : \mathbb{R} \to [0,1]$ satisfying the following properties:

u is normal, convex fuzzy set, upper semi-continuous on $\mathbb{R}$ and $cl\{x \in \mathbb{R} \mid u(x) > 0\}$ is compact, where cl denotes the closure of a subset.

Let $\mathbb{R}_F$ denote the space of fuzzy numbers.

Definition 2.2. Let $u \in \mathbb{R}_F$. The α-level set of u, denoted $[u]^\alpha, 0 < \alpha \le 1$, is $[u]^\alpha = \{x \in \mathbb{R} \mid u(x) \ge \alpha\}$. If α=0, the support of u is defined $[u]^0 = \mathrm{cl}\{x \in \mathbb{R} \mid u(x) > 0\}$. The notation $[u]^\alpha = [\underline{u}_\alpha, \overline{u}_\alpha]$, denotes explicitly the α-level set of u. We refer to $\underline{u}$ and $\overline{u}$ as the lower and upper branches of u, respectively.

The following remark shows when $[\underline{u}_\alpha, \overline{u}_\alpha]$ is a valid α-level set.

Remark 2.3. The sufficient and necessary conditions for $[\underline{u}_\alpha, \overline{u}_\alpha]$ to define the parametric form of a fuzzy number as follows:

(i) $\underline{u}_\alpha$ is bounded monotonic increasing (nondecreasing) left-continuous function on $(0,1]$ and right-continuous for $\alpha = 0$,

(ii) $\overline{u}_\alpha$ is bounded monotonic decreasing (nonincreasing) left-continuous function on $(0,1]$ and right-continuous for $\alpha = 0$,

(iii) $\underline{u}_\alpha \le \overline{u}_\alpha$, $0 \le \alpha \le 1$.

Definition 2.4. For $u, v \in \mathbb{R}_F$ and $\lambda \in \mathbb{R}$, the sum $u+v$ and the product λu are defined by $[u+v]^\alpha = [u]^\alpha + [v]^\alpha$, $[\lambda u]^\alpha = \lambda [u]^\alpha$, $\forall \alpha \in [0,1]$, where $[u]^\alpha + [v]^\alpha$ means the usual addition of two intervals (subsets) of $\mathbb{R}$ and $\lambda [u]^\alpha$ means the usual product between a scalar and a subset of $\mathbb{R}$.

The metric structure is given by the Hausdorff distance

$$D: \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_+ \cup \{0\},$$

by

$$D(u,v) = \sup_{\alpha \in [0,1]} \max\left\{ |\underline{u}_\alpha - \underline{v}_\alpha|, |\overline{u}_\alpha - \overline{v}_\alpha| \right\}.$$

Definition 2.5. Let $u, v \in \mathbb{R}_F$. If there exist $w \in \mathbb{R}_F$ such that $u = v + w$, then w is called the Hukuhara difference of u and v and it is denoted by $u \ominus v$.

Definition 2.6. Let I=(a,b) and $F: I \to \mathbb{R}_F$ be a fuzzy function. We say F is Hukuhara differentiable at $t_0 \in I$, if there exists an element $F'(t_0) \in \mathbb{R}_F$ such that, for all h>0 sufficiently near to 0, we have $F(t_0 + h) \ominus F(t_0)$, $F(t_0) \ominus F(t_0 - h)$ and the limits

$$\lim_{h \to 0^+} \frac{F(t_0 + h) \ominus F(t_0)}{h} = \lim_{h \to 0^+} \frac{F(t_0) \ominus F(t_0 - h)}{h} = F'(t_0).$$

Here the limits are taken in the metric space $(\mathbb{R}_F, D)$.

Theorem 2.7. Let $f: I \to \mathbb{R}_F$ be a function and denote $[f(t)]^\alpha = [\underline{f}_\alpha(t), \overline{f}_\alpha(t)]$, for each $\alpha \in [0,1]$. If f is Hukuhara differentiable, then $\underline{f}_\alpha(t)$ and $\overline{f}_\alpha(t)$ are differentiable functions and $[f'(t)]^\alpha = [\underline{f}'_\alpha(t), \overline{f}'_\alpha(t)]$.

3. Fuzzy Boundary Value Problem

In this section we concern with the existence and uniqueness of solution for the problem (1.1)-(1.3).

Theorem 3.1. Let $[\phi(x)]^\alpha = [\underline{\phi}_\alpha(x), \overline{\phi}_\alpha(x)], [\psi(x)]^\alpha = [\underline{\psi}_\alpha(x), \overline{\psi}_\alpha(x)]$. Let Φ and Ψ be solutions of $Ly = 0$ such that either $B_1(\phi) = B_1(\psi) = 0$ or $B_2(\phi) = B_2(\psi) = 0$, then $\{\underline{\phi}_\alpha, \underline{\psi}_\alpha\}$ and $\{\overline{\phi}_\alpha, \overline{\psi}_\alpha\}$ are linearly dependent.

Proof. Recall that $B_1(y) = Ay(a) + By'(a)$ and $A^2 + B^2 \neq 0$. Therefore, if $B_1(\phi) = B_1(\psi) = 0$ then A and B are nontrivial solutions of the systems

$$\begin{aligned} A\underline{\phi}_\alpha(a) + B\underline{\phi}'_\alpha(a) &= 0 \\ A\underline{\psi}_\alpha(a) + B\underline{\psi}'_\alpha(a) &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} A\overline{\phi}_\alpha(a) + B\overline{\phi}'_\alpha(a) &= 0 \\ A\overline{\psi}_\alpha(a) + B\overline{\psi}'_\alpha(a) &= 0. \end{aligned}$$

This implies that

$$\underline{\phi}_\alpha(a)\underline{\psi}'_\alpha(a) - \underline{\phi}'_\alpha(a)\underline{\psi}_\alpha(a) = 0 \quad \text{and} \quad \overline{\phi}_\alpha(a)\overline{\psi}'_\alpha(a) - \overline{\phi}'_\alpha(a)\overline{\psi}_\alpha(a) = 0.$$

So $\{\underline{\phi}_\alpha, \underline{\psi}_\alpha\}$ and $\{\overline{\phi}_\alpha, \overline{\psi}_\alpha\}$ are linearly dependent.

Corollary 3.2. If $\{\underline{\phi}_\alpha, \underline{\psi}_\alpha\}$ or $\{\overline{\phi}_\alpha, \overline{\psi}_\alpha\}$ is linearly independent, then

$B_1^2(\phi) + B_1^2(\psi) \neq 0$ and $B_2^2(\phi) + B_2^2(\psi) \neq 0$.

Theorem 3.3. The following statements are equivalent; that is, they are all true or all false.

(a) There is a fundamental set $\{z_1, z_2\}$ of solutions of $Ly = 0$ such that

$$B_1(z_1)B_2(z_2) - B_1(z_2)B_2(z_1) \neq 0. \qquad (3.1)$$

(b) If $\{y_1, y_2\}$ is a fundamental set of solutions of $Ly = 0$ then

$$B_1(y_1)B_2(y_2) - B_1(y_2)B_2(y_1) \neq 0. \qquad (3.2)$$

(c) For each continuous function F and pair of symmetric triangular fuzzy numbers $([k_1]^\alpha, [k_2]^\alpha)$, the fuzzy boundary value problem (1.1)-(1.3) has a unique solution except the α-level set.

(d) The homogeneous boundary value problem

$$Ly = 0,\ B_1(y) = 0,\ B_2(y) = 0 \qquad (3.3)$$

has only the trivial solution y=0.

(e) There is a fundamental set $\{z_1, z_2\}$ of solutions of $Ly = 0$ such that $B_1(z_1) = 0$ and $B_2(z_2) = 0$.

Proof. We show that

$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (a)$.

(a)⇒(b): Since $\{z_1, z_2\}$ is a fundamental set of solutions of $Ly = 0$, there are constants a_1, a_2, b_1, b_2 such that

$$\begin{aligned} y_1 &= a_1 z_1 + a_2 z_2, \\ y_2 &= b_1 z_1 + b_2 z_2. \end{aligned} \qquad (3.4)$$

Moreover,

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \neq 0 . \qquad (3.5)$$

Because if this determinant were zero, its rows would be linearly dependent and therefore $\{y_1, y_2\}$ would be linearly dependent, contrary to our assumption that $\{y_1, y_2\}$ is a fundamental set of solutions of $Ly = 0$. From (3.4),

$$\begin{bmatrix} B_1(y_1) & B_2(y_1) \\ B_1(y_2) & B_2(y_2) \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} B_1(z_1) & B_2(z_1) \\ B_1(z_2) & B_2(z_2) \end{bmatrix}.$$

Since the determinant of a product of matrices is equal to the product of the determinants of the matrices, (3.1) and (3.5) imply (3.2).

(b)⇒(c): Since $\{y_1, y_2\}$ is a fundamental set of solutions of $Ly = 0$, the general solution of Ly=F is

$$[y]^\alpha = \left[\underline{y}_{\alpha_p} + c_1(\alpha)y_1 + c_2(\alpha)y_2, \overline{y}_{\alpha_p} + c_3(\alpha)y_1 + c_4(\alpha)y_2\right],$$

where $\left[y_p\right]^\alpha = \left[\underline{y}_{\alpha_p}, \overline{y}_{\alpha_p}\right]$ is a particular solution of $Ly = F$. To satisfy the boundary conditions, we must choose

$$\underline{k_1}_\alpha = B_1\left(\underline{y}_{\alpha_p}\right) + c_1(\alpha)B_1(y_1) + c_2(\alpha)B_1(y_2),$$

$$\underline{k_2}_\alpha = B_2\left(\underline{y}_{\alpha_p}\right) + c_1(\alpha)B_2(y_1) + c_2(\alpha)B_2(y_2)$$

and

$$\overline{k}_{1\alpha} = B_1\left(\overline{y}_{\alpha_p}\right) + c_3(\alpha)B_1(y_1) + c_4(\alpha)B_1(y_2),$$

$$\overline{k}_{2\alpha} = B_2\left(\overline{y}_{\alpha_p}\right) + c_3(\alpha)B_2(y_1) + c_4(\alpha)B_2(y_2),$$

which are equivalent to

$$c_1(\alpha)B_1(y_1) + c_2(\alpha)B_1(y_2) = \underline{k_1}_\alpha - B_1\left(\underline{y}_{\alpha_p}\right),$$

$$c_1(\alpha)B_2(y_1) + c_2(\alpha)B_2(y_2) = \underline{k_2}_\alpha - B_2\left(\underline{y}_{\alpha_p}\right)$$

and

$$c_3(\alpha)B_1(y_1) + c_4(\alpha)B_1(y_2) = \overline{k}_{1\alpha} - B_1\left(\overline{y}_{\alpha_p}\right),$$

$$c_3(\alpha)B_2(y_1) + c_4(\alpha)B_2(y_2) = \overline{k}_{2\alpha} - B_2\left(\overline{y}_{\alpha_p}\right).$$

(c)⇒(d): Obviously, y=0 is a solution of (3.3). From (c) with F=0 and $[k_1]^\alpha = [k_2]^\alpha = 0$, it is the only solution.

(d)⇒(e): Let $\{y_1, y_2\}$ be a fundamental system for $Ly = 0$ and let

$$z_1 = B_1(y_2)y_1 - B_1(y_1)y_2 \quad \text{and} \quad z_2 = B_2(y_2)y_1 - B_2(y_1)y_2.$$

Then $B_1(z_1)=0$ and $B_2(z_2)$. To see that z_1 and z_2 are linearly independent, note that

$$a_1z_1+a_2z_2=a_1\left[B_1(y_2)y_1-B_1(y_1)y_2\right]+a_2\left[B_2(y_2)y_1-B_2(y_1)y_2\right]$$

$$=\left[B_1(y_2)a_1+B_2(y_2)a_2\right]y_1-\left[B_1(y_1)a_1+B_2(y_1)a_2\right]y_2.$$

Therefore, since y_1 and y_2 are linearly independent, $a_1z_1+a_2z_2=0$ if and only if

$$\begin{bmatrix} B_1(y_1) & B_2(y_1) \\ B_1(y_2) & B_2(y_2) \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

If this system has a nontrivial solution then so does the system

$$\begin{bmatrix} B_1(y_1) & B_1(y_2) \\ B_2(y_1) & B_2(y_2) \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This implies that $y=c_1z_1+c_2z_2$ is a nontrivial solution of (3.3), which contradicts (d).

(e)⇒(a): Theorem 3.1 implies that if $B_1(z_1)=0$ and $B_2(z_2)=0$ then $B_1(z_2)\neq 0$ and $B_2(z_1)\neq 0$. This implies (3.1), which completes the proof.

Theorem 3.4. Suppose the homogeneous boundary value problem (3.3) has only the trivial solution. Let y_1 and y_2 be a fundamental set of solutions of $Ly=0$ such that $B_1(y_1)=0$ and $B_2(y_2)=0$, and let $W=y_1y_2^{'}-y_1^{'}y_2$. Then the unique solution except the α-level set of

$$Ly=F,\ B_1(y)=[0]^\alpha,\ B_2(y)=[0]^\alpha \qquad (3.6)$$

is

$$[y(x)]^\alpha=\left[\underline{y}_\alpha(x),\overline{y}_\alpha(x)\right], \qquad (3.7)$$

where

$$\underline{y}_\alpha(x)=y_1(x)\left(\int_x^b\frac{F(t)y_2(t)}{P_0(t)W(t)}dt+\frac{(-1+\alpha)}{B_2(y_1)}\right)+y_2(x)\left(\int_a^x\frac{F(t)y_1(t)}{P_0(t)W(t)}dt+\frac{(-1+\alpha)}{B_1(y_2)}\right), \qquad (3.8)$$

$$\overline{y}_\alpha(x)=y_1(x)\left(\int_x^b\frac{F(t)y_2(t)}{P_0(t)W(t)}dt+\frac{(1-\alpha)}{B_2(y_1)}\right)+y_2(x)\left(\int_a^x\frac{F(t)y_1(t)}{P_0(t)W(t)}dt+\frac{(1-\alpha)}{B_1(y_2)}\right) \qquad (3.9)$$

and $[0]^\alpha=[-1+\alpha,1-\alpha]$.

Proof. Suppose the solution of Ly=F is

$$[y]^\alpha=\left[u_1(x,\alpha)y_1+u_2(x,\alpha)y_2,u_3(x,\alpha)y_1+u_4(x,\alpha)y_2\right].$$

Using the method of variation of parameters

$$u_1'(x,\alpha)y_1 + u_2'(x,\alpha)y_2 = 0 \qquad u_3'(x,\alpha)y_1 + u_4'(x,\alpha)y_2 = 0$$

$$u_1'(x,\alpha)y_1' + u_2'(x,\alpha)y_2' = \frac{F}{P_0} \quad \text{and} \quad u_3'(x,\alpha)y_1' + u_4'(x,\alpha)y_2' = \frac{F}{P_0}$$

are obtained. Solving for $u_1'(x,\alpha), u_2'(x,\alpha)$, $u_3'(x,\alpha)$ and $u_4'(x,\alpha)$ yields

$$u_1'(x,\alpha) = -\frac{Fy_2}{P_0W},\ u_2'(x,\alpha) = \frac{Fy_1}{P_0W},\ u_3'(x,\alpha) = -\frac{Fy_2}{P_0W},\ u_4'(x,\alpha) = \frac{Fy_1}{P_0W}.$$

Integrating these yields

$$u_1(x,\alpha) = \int_x^b \frac{F(t)y_2(t)}{P_0(t)W(t)}dt + u_1(b,\alpha), \quad u_2(x,\alpha) = \int_a^x \frac{F(t)y_1(t)}{P_0(t)W(t)}dt + u_2(a,\alpha),$$

$$u_3(x,\alpha) = \int_x^b \frac{F(t)y_2(t)}{P_0(t)W(t)}dt + u_3(b,\alpha), \quad u_4(x,\alpha) = \int_a^x \frac{F(t)y_1(t)}{P_0(t)W(t)}dt + u_4(a,\alpha).$$

Using $B_1(y) = Ay(a) + By'(a) = [-1+\alpha, 1-\alpha]$ and $B_2(y) = Cy(b) + Dy'(b) = [-1+\alpha, 1-\alpha]$, we solved $u_1(b,\alpha)$, $u_2(a,\alpha)$, $u_3(b,\alpha)$, $u_4(a,\alpha)$ as

$$u_1(b,\alpha) = \frac{(-1+\alpha)}{B_2(y_1)},\ u_2(a,\alpha) = \frac{(-1+\alpha)}{B_1(y_2)},\ u_3(b,\alpha) = \frac{(1-\alpha)}{B_2(y_1)}, u_4(a,\alpha) = \frac{(1-\alpha)}{B_1(y_2)}.$$

This completes the proof.

Theorem 3.5. Suppose the homogeneous boundary value problem (3.3) has a nontrivial solution y_1 and let y_2 be any solution of $Ly = 0$ that isn't a constant multiple of y_1. Let $W = y_1y_2' - y_1'y_2$. If $\alpha = 1$, then the fuzzy boundary value problem (3.6) has infinitely many solutions, all of the form $y = y_p + c_1y_1$, where

$$y_p = y_1(x)\int_x^b \frac{F(t)y_2(t)}{P_0(t)W(t)}dt + y_2(x)\int_a^x \frac{F(t)y_1(t)}{P_0(t)W(t)}dt$$

and c_1 is a arbitrary constant. Also $[0]^\alpha = [-1+\alpha, 1-\alpha]$. If $\alpha \neq 1$, then (3.6) has no solution.

Proof. From the proof of Theorem 3.4, $[y_p(x)]^\alpha = \left[\underline{y}_{\alpha_p}(x), \overline{y}_{\alpha_p}(x)\right]$ is a particular solution of $Ly = F$, where

$$\underline{y}_{\alpha_p}(x) = y_1(x)\left(\int_x^b \frac{F(t)y_2(t)}{P_0(t)W(t)}dt + \frac{(-1+\alpha)}{B_2(y_1)}\right) + y_2(x)\left(\int_a^x \frac{F(t)y_1(t)}{P_0(t)W(t)}dt + \frac{(-1+\alpha)}{B_1(y_2)}\right),$$

$$\overline{y}_{\alpha_p}(x) = y_1(x)\left(\int_x^b \frac{F(t)y_2(t)}{P_0(t)W(t)}dt + \frac{(1-\alpha)}{B_2(y_1)}\right) + y_2(x)\left(\int_a^x \frac{F(t)y_1(t)}{P_0(t)W(t)}dt + \frac{(1-\alpha)}{B_1(y_2)}\right).$$

Therefore the general solution of (3.6) is of the form

$$[y]^{\alpha}=\left[\underline{y}_{\alpha_p}+c_1(\alpha)y_1+c_2(\alpha)y_2,\overline{y}_{\alpha_p}+c_3(\alpha)y_1+c_4(\alpha)y_2\right].$$

Then from $B_1(y)=Ay(a)+By'(a)=[-1+\alpha,1-\alpha]$, we obtained $c_2(\alpha)=c_4(\alpha)=0$. From $B_2(y)=Cy(b)+Dy'(b)=[-1+\alpha,1-\alpha]$, we obtained $\alpha=1$. Therefore $y=y_p+c_1y_1$. This completes the proof.

Example 3.6. Consider the fuzzy boundary value problem

$$x^2y''-2xy'+2y=2x^3,\ y(1)=[3+\alpha,5-\alpha],\ y'(2)=[2+\alpha,4-\alpha]. \qquad (3.10)$$

Here $B_1(y)=y(1)$ and $B_2(y)=y'(2)$. Let $\{y_1,y_2\}=\{x^2,x\}$, which is a fundamental set of solutions of homogeneous differential equation in (3.10). Then

$$B_1(y_1)B_2(y_2)-B_1(y_2)B_2(y_1)=-3.$$

So Theorem 3.3 implies that (3.10) has a unique solution except the α-level set. The solution of differential equation in (3.10) is

$$[y(x)]^{\alpha}=\left[x^3+c_1(\alpha)x^2+c_2(\alpha)x,x^3+c_3(\alpha)x^2+c_4(\alpha)x\right].$$

Using boundary conditions, coefficients $c_1(\alpha)$, $c_2(\alpha)$, $c_3(\alpha)$, $c_4(\alpha)$ are obtained as

$$c_1(\alpha)=c_3(\alpha)=-4,\ c_2(\alpha)=6+\alpha,\ c_4(\alpha)=8-\alpha.$$

Example 3.7. Consider the fuzzy boundary value problem

$$y''+y=1,\ y(0)=[-1+\alpha,1-\alpha],\ y\left(\frac{\pi}{2}\right)=[-1+\alpha,1-\alpha]. \qquad (3.11)$$

Here $B_1(y)=y(0)$ and $B_2(y)=y\left(\frac{\pi}{2}\right)$. Let $\{y_1,y_2\}=\{\sin x,\cos x\}$, which is a fundamental set of solutions of homogeneous differential equation in (3.11). Then

$$B_1(y_1)B_2(y_2)-B_1(y_2)B_2(y_1)=-1.$$

So Theorem 3.3 implies that (3.11) has a unique solution except the α-level set. Also, the Wronskian of $\{y_1,y_2\}$ is

$$W(x)=\sin x(-\sin x)-\cos x\cos x=-1.$$

Then (3.7), (3.8) and (3.9) yield the solution

$$[y(x)]^{\alpha}=\left[(-2+\alpha)(\cos x+\sin x)+1,(-\alpha)(\cos x+\sin x)+1\right].$$

Example 3.8. If we take

$$y(0)=[-1+\alpha,1-\alpha],\ y(\pi)=[-1+\alpha,1-\alpha] \qquad (3.12)$$

as the boundary conditions for the fuzzy boundary value problem in (3.11), we have

$$B_1(y_1)B_2(y_2)-B_1(y_2)B_2(y_1)=0.$$

The solution of differential equation in (3.11) is

$$[y(x)]^{\alpha}=\left[\underline{y}_{\alpha}(x),\overline{y}_{\alpha}(x)\right],$$

Where

$\underline{y}_\alpha(x)=c_1(\alpha)\cos x+c_2(\alpha)\sin x+1$, $\overline{y}_\alpha(x)=c_3(\alpha)\cos x+c_4(\alpha)\sin x+1$.

From boundary conditions, we have

$\underline{y}_\alpha(0)=-1+\alpha \Rightarrow c_1(\alpha)=-2+\alpha$, $\overline{y}_\alpha(0)=1-\alpha \Rightarrow c_3(\alpha)=-\alpha$,

$\underline{y}_\alpha(\pi)=-1+\alpha \Rightarrow c_1(\alpha)=2-\alpha$, $\overline{y}_\alpha(\pi)=1-\alpha \Rightarrow c_3(\alpha)=\alpha$.

Therefore, differential equatian in (3.11) with boundary conditions (3.12) has no solution.

Example 3.9. Consider the fuzzy boundary value problem

$$y''+y=\sin 2x,\ y(0)=[-1+\alpha,1-\alpha],\ y(\pi)=[-1+\alpha,1-\alpha]. \qquad (3.13)$$

The solution of differential equation in (3.13) is

$$[y(x)]^\alpha=\left[c_1(\alpha)\cos x+c_2(\alpha)\sin x-\frac{\sin 2x}{3},c_3(\alpha)\cos x+c_4(\alpha)\sin x-\frac{\sin 2x}{3}\right].$$

From boundary conditions, we get

$\underline{y}_\alpha(0)=-1+\alpha \Rightarrow c_1(\alpha)=-1+\alpha$, $\overline{y}_\alpha(0)=1-\alpha \Rightarrow c_3(\alpha)=1-\alpha$,

$\underline{y}_\alpha(\pi)=-1+\alpha \Rightarrow c_1(\alpha)=1-\alpha$, $\overline{y}_\alpha(\pi)=1-\alpha \Rightarrow c_3(\alpha)=-1+\alpha$.

Therefore, if $\alpha=1$, (3.13) has infinitely many solutions, differing by constant multiples of sinx. If $\alpha\neq 1$, (3.13) has no solution.

Example 3.10. Consider the fuzzy boundary value problem

$$y''+y=F(x),\ y(0)+y'(0)=[-1+\alpha,1-\alpha],\ y(\pi)+y'(\pi)=[-1+\alpha,1-\alpha]. \qquad (3.14)$$

Here $B_1(y)=y(0)+y'(0)$ and $B_2(y)=y(\pi)+y'(\pi)$. Let $\{z_1,z_2\}=\{\cos x,\sin x\}$, which is a fundamental set of solutions of homogeneous differential equation in (3.14). Then

$B_1(z_1)B_2(z_2)-B_1(z_2)B_2(z_1)=2$.

So Theorem 3.3 implies that (3.14) has a unique solution except the α-level set. Let

$y_1=B_1(z_2)z_1-B_1(z_1)z_2=\cos x-\sin x$ and $y_2=B_2(z_2)z_1-B_2(z_1)z_2=\cos x+\sin x$.

Then $B_1(y_1)=0$, $B_2(y_2)=0$ and the Wronskian of $\{y_1,y_2\}$ is

$$W(x)=(\cos x-\sin x)(-\sin x+\cos x)-(\cos x+\sin x)(-\sin x-\cos x)=2.$$

So (3.7), (3.8) and (3.9) yield the solution

$$[y(x)]^\alpha=[\underline{y}_\alpha(x),\overline{y}_\alpha(x)],$$

Where

$$\underline{y}_\alpha(x)=\frac{\cos x-\sin x}{2}\left(\int_x^\pi F(t)(\cos t+\sin t)dt+(1-\alpha)\right)$$
$$+\frac{\cos x+\sin x}{2}\left(\int_0^x F(t)(\cos t-\sin t)dt+(\alpha-1)\right),$$

$$\bar{y}_{\alpha}(x)=\frac{\cos x-\sin x}{2}\left(\int_{x}^{\frac{\pi}{4}}F(t)(\cos t+\sin t)dt+(\alpha-1)\right)$$
$$+\frac{\cos x+\sin x}{2}\left(\int_{0}^{x}F(t)(\cos t-\sin t)dt+(1-\alpha)\right).$$

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