

ON SOLUTION OF TWO-POINT FUZZY BOUNDARY VALUE PROBLEMS

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Abstract: In this paper, we investigate the existence and uniqueness of solutions of two-point fuzzy boundary value problems for second-order fuzzy differential equations. Some sufficient conditions are presented that guarantee the existence and uniqueness of solutions under the approach of Hukuhara differentiability.

1. Introduction

Fuzzy set theory is a powerful tool for modeling uncertainly and for processing vague or subjective information in mathematical models [3]. Particularly, the study of fuzzy differential equation forms a suitable setting for mathematical modeling of real world problems in which uncertainties or vagueness pervade. In fact, fuzzy differential equations is a very important topic from theoretical point of view [5], [6], [7], [9] and also it has applications, for example, in population models [4], civil engineering [8], medicine [1], [2].

In this paper, an investigation is made on the solution of two-point fuzzy boundary value problems by using the Hukuhara differentiability. To put it precisely, the two-point fuzzy boundary value problem is given as the form

$$Ly := P_0(x)y' + P_1(x)y' + P_2(x)y = F(x), \quad x \in (a, b) \quad (1.1)$$

$$B_1(y) := Ay(a) + By'(a) = [k_1]^\alpha \quad (1.2)$$

$$B_2(y) := Cy(b) + Dy'(b) = [k_2]^\alpha \quad (1.3)$$

where $[k_1]^\alpha = [\underline{k}_{1\alpha}, \overline{k}_{1\alpha}]$, $[k_2]^\alpha = [\underline{k}_{2\alpha}, \overline{k}_{2\alpha}]$ are symmetric triangular fuzzy numbers, $A^2 + B^2 \neq 0$,

$C^2 + D^2 \neq 0$, $P_0(x)$, $P_1(x)$, $P_2(x)$, $F(x)$ are continuous functions and $P_0(x)$ has no zeros on

(a, b) .

2. Preliminaries

In this section, we give some definitions and introduce the necessary notation which will be used throughout the paper.

Definition 2.1. A fuzzy number is a function $u: \mathbb{R} \rightarrow [0, 1]$ satisfying the following properties:

u is normal, convex fuzzy set, upper semi-continuous on \mathbb{R} and $\text{cl}\{x \in \mathbb{R} \mid u(x) > 0\}$ is compact,

where cl denotes the closure of a subset.

Let \mathbb{F} denote the space of fuzzy numbers.

Definition 2.2. Let $u \in \square_F$. The α -level set of u , denoted $[u]^\alpha$, $0 < \alpha \leq 1$, is $[u]^\alpha = \{x \in \square \mid u(x) \geq \alpha\}$. If $\alpha = 0$, the support of u is defined $[u]^0 = \text{cl}\{x \in \square \mid u(x) > 0\}$. The notation $[u]^\alpha = [\underline{u}_\alpha, \bar{u}_\alpha]$, denotes explicitly the α -level set of u . We refer to \underline{u} and \bar{u} as the lower and upper branches of u , respectively.

The following remark shows when $[\underline{u}_\alpha, \bar{u}_\alpha]$ is a valid α -level set.

Remark 2.3. The sufficient and necessary conditions for $[\underline{u}_\alpha, \bar{u}_\alpha]$ to define the parametric form of a fuzzy number as follows:

(i) \underline{u}_α is bounded monotonic increasing (nondecreasing) left-continuous function on $(0, 1]$ and right-continuous for $\alpha = 0$,

(ii) \bar{u}_α is bounded monotonic decreasing (nonincreasing) left-continuous function on $(0, 1]$ and right-continuous for $\alpha = 0$,

(iii) $\underline{u}_\alpha \leq \bar{u}_\alpha$, $0 \leq \alpha \leq 1$.

Definition 2.4. For $u, v \in \square_F$ and $\lambda \in \square$, the sum $u+v$ and the product λu are defined by

$[u+v]^\alpha = [u]^\alpha + [v]^\alpha$, $[\lambda u]^\alpha = \lambda [u]^\alpha$, $\forall \alpha \in [0, 1]$, where $[u]^\alpha + [v]^\alpha$ means the usual addition of two

intervals (subsets) of \square and $\lambda [u]^\alpha$ means the usual product between a scalar and a subset of \square . The metric structure is given by the Hausdorff distance

$$D: \square_F \times \square_F \rightarrow \square_+ \cup \{0\},$$

by

$$D(u, v) = \sup_{\alpha \in (0, 1]} \max \left\{ |\underline{u}_\alpha - \underline{v}_\alpha|, |\bar{u}_\alpha - \bar{v}_\alpha| \right\}.$$

Definition 2.5. Let $u, v \in \square_F$. If there exist $w \in \square_F$ such that $u = v + w$, then w is called the Hukuhara difference of u and v and it is denoted by $u \dot{-} v$.

Definition 2.6. Let $I = (a, b)$ and $F: I \rightarrow \square_F$ be a fuzzy function. We say F is Hukuhara differentiable at $t_0 \in I$, if there exists an element $F'(t_0) \in \square_F$ such that, for all $h > 0$ sufficiently near to 0, we have

$F(t_0 + h) \dot{-} F(t_0)$, $F(t_0) \dot{-} F(t_0 - h)$ and the limits

$$\lim_{h \rightarrow 0^+} \frac{F(t_0 + h) \dot{-} F(t_0)}{h} = \lim_{h \rightarrow 0^+} \frac{F(t_0) \dot{-} F(t_0 - h)}{h} = F'(t_0).$$

Here the limits are taken in the metric space (\square_F, D) .

Theorem 2.7. Let $f: I \rightarrow \square_F$ be a function and denote $[f(t)]^\alpha = [\underline{f}_\alpha(t), \bar{f}_\alpha(t)]$, for each $\alpha \in [0, 1]$. If f is Hukuhara differentiable, then $\underline{f}_\alpha(t)$ and $\bar{f}_\alpha(t)$ are differentiable functions and $[f'(t)]^\alpha = [\underline{f}'_\alpha(t), \bar{f}'_\alpha(t)]$.

3. Fuzzy Boundary Value Problem

In this section we concern with the existence and uniqueness of solution for the problem (1.1)-(1.3).

Theorem 3.1. Let $[\phi(x)]^\alpha = [\underline{\phi}_\alpha(x), \bar{\phi}_\alpha(x)]$, $[\psi(x)]^\alpha = [\underline{\psi}_\alpha(x), \bar{\psi}_\alpha(x)]$. Let Φ and Ψ be solutions of $Ly = 0$ such that either $B_1(\phi) = B_1(\psi) = 0$ or $B_2(\phi) = B_2(\psi) = 0$, then $\{\underline{\phi}_\alpha, \underline{\psi}_\alpha\}$ and $\{\bar{\phi}_\alpha, \bar{\psi}_\alpha\}$ are linearly dependent.

Proof. Recall that $B_1(y) = Ay(a) + By'(a)$ and $A^2 + B^2 \neq 0$. Therefore, if $B_1(\phi) = B_1(\psi) = 0$ then A and B are nontrivial solutions of the systems

$$\begin{aligned} A\underline{\phi}_\alpha(a) + B\underline{\phi}'_\alpha(a) = 0 & \quad \text{and} \quad A\bar{\phi}_\alpha(a) + B\bar{\phi}'_\alpha(a) = 0 \\ A\underline{\psi}_\alpha(a) + B\underline{\psi}'_\alpha(a) = 0 & \quad \quad \quad A\bar{\psi}_\alpha(a) + B\bar{\psi}'_\alpha(a) = 0. \end{aligned}$$

This implies that

$$\underline{\phi}_\alpha(a)\underline{\psi}'_\alpha(a) - \underline{\phi}'_\alpha(a)\underline{\psi}_\alpha(a) = 0 \quad \text{and} \quad \bar{\phi}_\alpha(a)\bar{\psi}'_\alpha(a) - \bar{\phi}'_\alpha(a)\bar{\psi}_\alpha(a) = 0.$$

So $\{\underline{\phi}_\alpha, \underline{\psi}_\alpha\}$ and $\{\bar{\phi}_\alpha, \bar{\psi}_\alpha\}$ are linearly dependent.

Corollary 3.2. If $\{\underline{\phi}_\alpha, \underline{\psi}_\alpha\}$ or $\{\bar{\phi}_\alpha, \bar{\psi}_\alpha\}$ is linearly independent, then

$$B_1^2(\phi) + B_1^2(\psi) \neq 0 \quad \text{and} \quad B_2^2(\phi) + B_2^2(\psi) \neq 0.$$

Theorem 3.3. The following statements are equivalent; that is, they are all true or all false.

(a) There is a fundamental set $\{z_1, z_2\}$ of solutions of $Ly = 0$ such that

$$B_1(z_1)B_2(z_2) - B_1(z_2)B_2(z_1) \neq 0. \tag{3.1}$$

(b) If $\{y_1, y_2\}$ is a fundamental set of solutions of $Ly = 0$ then

$$B_1(y_1)B_2(y_2) - B_1(y_2)B_2(y_1) \neq 0. \tag{3.2}$$

(c) For each continuous function F and pair of symmetric triangular fuzzy numbers $([k_1]^\alpha, [k_2]^\alpha)$,

the fuzzy boundary value problem (1.1)-(1.3) has a unique solution except the α -level set.

(d) The homogeneous boundary value problem

$$Ly = 0, \quad B_1(y) = 0, \quad B_2(y) = 0 \tag{3.3}$$

has only the trivial solution $y=0$.

(e) There is a fundamental set $\{z_1, z_2\}$ of solutions of $Ly = 0$ such that $B_1(z_1) = 0$ and $B_2(z_2) = 0$.

Proof. We show that

$$(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d) \Rightarrow (e) \Rightarrow (a).$$

(a)⇒(b): Since $\{z_1, z_2\}$ is a fundamental set of solutions of $Ly = 0$, there are constants a_1, a_2, b_1, b_2 such that

$$\begin{aligned} y_1 &= a_1z_1 + a_2z_2, \\ y_2 &= b_1z_1 + b_2z_2. \end{aligned} \tag{3.4}$$

Moreover,

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \neq 0. \quad (3.5)$$

Because if this determinant were zero, its rows would be linearly dependent and therefore $\{y_1, y_2\}$ would be linearly dependent, contrary to our assumption that $\{y_1, y_2\}$ is a fundamental set of solutions of $Ly = 0$. From (3.4),

$$\begin{bmatrix} B_1(y_1) & B_2(y_1) \\ B_1(y_2) & B_2(y_2) \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} B_1(z_1) & B_2(z_1) \\ B_1(z_2) & B_2(z_2) \end{bmatrix}.$$

Since the determinant of a product of matrices is equal to the product of the determinants of the matrices, (3.1) and (3.5) imply (3.2).

(b)⇒(c): Since $\{y_1, y_2\}$ is a fundamental set of solutions of $Ly = 0$, the general solution of $Ly = F$ is

$$[y]^\alpha = \left[\underline{y}_{\alpha_p} + c_1(\alpha)y_1 + c_2(\alpha)y_2, \bar{y}_{\alpha_p} + c_3(\alpha)y_1 + c_4(\alpha)y_2 \right],$$

where $[\underline{y}_p]^\alpha = [\underline{y}_{\alpha_p}, \bar{y}_{\alpha_p}]$ is a particular solution of $Ly = F$. To satisfy the boundary conditions, we must choose

$$\underline{k}_{1\alpha} = B_1(\underline{y}_{\alpha_p}) + c_1(\alpha)B_1(y_1) + c_2(\alpha)B_1(y_2),$$

$$\underline{k}_{2\alpha} = B_2(\underline{y}_{\alpha_p}) + c_1(\alpha)B_2(y_1) + c_2(\alpha)B_2(y_2)$$

and

$$\bar{k}_{1\alpha} = B_1(\bar{y}_{\alpha_p}) + c_3(\alpha)B_1(y_1) + c_4(\alpha)B_1(y_2),$$

$$\bar{k}_{2\alpha} = B_2(\bar{y}_{\alpha_p}) + c_3(\alpha)B_2(y_1) + c_4(\alpha)B_2(y_2),$$

which are equivalent to

$$c_1(\alpha)B_1(y_1) + c_2(\alpha)B_1(y_2) = \underline{k}_{1\alpha} - B_1(\underline{y}_{\alpha_p}),$$

$$c_1(\alpha)B_2(y_1) + c_2(\alpha)B_2(y_2) = \underline{k}_{2\alpha} - B_2(\underline{y}_{\alpha_p})$$

and

$$c_3(\alpha)B_1(y_1) + c_4(\alpha)B_1(y_2) = \bar{k}_{1\alpha} - B_1(\bar{y}_{\alpha_p}),$$

$$c_3(\alpha)B_2(y_1) + c_4(\alpha)B_2(y_2) = \bar{k}_{2\alpha} - B_2(\bar{y}_{\alpha_p}).$$

(c)⇒(d): Obviously, $y=0$ is a solution of (3.3). From (c) with $F=0$ and $[k_1]^\alpha = [k_2]^\alpha = 0$, it is the only solution.

(d)⇒(e): Let $\{y_1, y_2\}$ be a fundamental system for $Ly = 0$ and let

$$z_1 = B_1(y_2)y_1 - B_1(y_1)y_2 \quad \text{and} \quad z_2 = B_2(y_2)y_1 - B_2(y_1)y_2.$$

Then $B_1(z_1) = 0$ and $B_2(z_2)$. To see that z_1 and z_2 are linearly independent, note that

$$\begin{aligned} a_1 z_1 + a_2 z_2 &= a_1 [B_1(y_2)y_1 - B_1(y_1)y_2] + a_2 [B_2(y_2)y_1 - B_2(y_1)y_2] \\ &= [B_1(y_2)a_1 + B_2(y_2)a_2]y_1 - [B_1(y_1)a_1 + B_2(y_1)a_2]y_2. \end{aligned}$$

Therefore, since y_1 and y_2 are linearly independent, $a_1 z_1 + a_2 z_2 = 0$ if and only if

$$\begin{bmatrix} B_1(y_1) & B_2(y_1) \\ B_1(y_2) & B_2(y_2) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

If this system has a nontrivial solution then so does the system

$$\begin{bmatrix} B_1(y_1) & B_1(y_2) \\ B_2(y_1) & B_2(y_2) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

This implies that $y = c_1 z_1 + c_2 z_2$ is a nontrivial solution of (3.3), which contradicts (d).

(e)⇒(a): Theorem 3.1 implies that if $B_1(z_1) = 0$ and $B_2(z_2) = 0$ then $B_1(z_2) \neq 0$ and $B_2(z_1) \neq 0$.

This implies (3.1), which completes the proof.

Theorem 3.4. Suppose the homogeneous boundary value problem (3.3) has only the trivial solution.

Let y_1 and y_2 be a fundamental set of solutions of $Ly = 0$ such that $B_1(y_1) = 0$ and $B_2(y_2) = 0$, and

let $W = y_1 y_2' - y_1' y_2$. Then the unique solution except the α -level set of

$$Ly = F, \quad B_1(y) = [0]^\alpha, \quad B_2(y) = [0]^\alpha \quad (3.6)$$

is

$$[y(x)]^\alpha = [\underline{y}_\alpha(x), \bar{y}_\alpha(x)], \quad (3.7)$$

where

$$\underline{y}_\alpha(x) = y_1(x) \left(\int_x^b \frac{F(t)y_2(t)}{P_0(t)W(t)} dt + \frac{(-1+\alpha)}{B_2(y_1)} \right) + y_2(x) \left(\int_a^x \frac{F(t)y_1(t)}{P_0(t)W(t)} dt + \frac{(-1+\alpha)}{B_1(y_2)} \right), \quad (3.8)$$

$$\bar{y}_\alpha(x) = y_1(x) \left(\int_x^b \frac{F(t)y_2(t)}{P_0(t)W(t)} dt + \frac{(1-\alpha)}{B_2(y_1)} \right) + y_2(x) \left(\int_a^x \frac{F(t)y_1(t)}{P_0(t)W(t)} dt + \frac{(1-\alpha)}{B_1(y_2)} \right) \quad (3.9)$$

and $[0]^\alpha = [-1+\alpha, 1-\alpha]$.

Proof. Suppose the solution of $Ly=F$ is

$$[y]^\alpha = [u_1(x, \alpha)y_1 + u_2(x, \alpha)y_2, u_3(x, \alpha)y_1 + u_4(x, \alpha)y_2].$$

Using the method of variation of parameters

$$\begin{aligned} u_1'(x, \alpha)y_1 + u_2'(x, \alpha)y_2 &= 0 & u_3'(x, \alpha)y_1 + u_4'(x, \alpha)y_2 &= 0 \\ u_1'(x, \alpha)y_1' + u_2'(x, \alpha)y_2' &= \frac{F}{P_0} & \text{and} & & u_3'(x, \alpha)y_1' + u_4'(x, \alpha)y_2' &= \frac{F}{P_0} \end{aligned}$$

are obtained. Solving for $u_1'(x, \alpha)$, $u_2'(x, \alpha)$, $u_3'(x, \alpha)$ and $u_4'(x, \alpha)$ yields

$$u_1'(x, \alpha) = -\frac{Fy_2}{P_0W}, \quad u_2'(x, \alpha) = \frac{Fy_1}{P_0W}, \quad u_3'(x, \alpha) = -\frac{Fy_2}{P_0W}, \quad u_4'(x, \alpha) = \frac{Fy_1}{P_0W}.$$

Integrating these yields

$$u_1(x, \alpha) = \int_x^b \frac{F(t)y_2(t)}{P_0(t)W(t)} dt + u_1(b, \alpha), \quad u_2(x, \alpha) = \int_a^x \frac{F(t)y_1(t)}{P_0(t)W(t)} dt + u_2(a, \alpha),$$

$$u_3(x, \alpha) = \int_x^b \frac{F(t)y_2(t)}{P_0(t)W(t)} dt + u_3(b, \alpha), \quad u_4(x, \alpha) = \int_a^x \frac{F(t)y_1(t)}{P_0(t)W(t)} dt + u_4(a, \alpha).$$

Using $B_1(y) = Ay(a) + By'(a) = [-1 + \alpha, 1 - \alpha]$ and $B_2(y) = Cy(b) + Dy'(b) = [-1 + \alpha, 1 - \alpha]$, we solved $u_1(b, \alpha)$, $u_2(a, \alpha)$, $u_3(b, \alpha)$, $u_4(a, \alpha)$ as

$$u_1(b, \alpha) = \frac{(-1 + \alpha)}{B_2(y_1)}, \quad u_2(a, \alpha) = \frac{(-1 + \alpha)}{B_1(y_2)}, \quad u_3(b, \alpha) = \frac{(1 - \alpha)}{B_2(y_1)}, \quad u_4(a, \alpha) = \frac{(1 - \alpha)}{B_1(y_2)}.$$

This completes the proof.

Theorem 3.5. Suppose the homogeneous boundary value problem (3.3) has a nontrivial solution y_1 and let y_2 be any solution of $Ly = 0$ that isn't a constant multiple of y_1 . Let $W = y_1y_2' - y_1'y_2$. If $\alpha = 1$, then the fuzzy boundary value problem (3.6) has infinitely many solutions, all of the form $y = y_p + c_1y_1$, where

$$y_p = y_1(x) \int_x^b \frac{F(t)y_2(t)}{P_0(t)W(t)} dt + y_2(x) \int_a^x \frac{F(t)y_1(t)}{P_0(t)W(t)} dt$$

and c_1 is a arbitrary constant. Also $[0]^\alpha = [-1 + \alpha, 1 - \alpha]$. If $\alpha \neq 1$, then (3.6) has no solution.

Proof. From the proof of Theorem 3.4, $[y_p(x)]^\alpha = [\underline{y}_{\alpha_p}(x), \bar{y}_{\alpha_p}(x)]$ is a particular solution of

$Ly = F$, where

$$\underline{y}_{\alpha_p}(x) = y_1(x) \left(\int_x^b \frac{F(t)y_2(t)}{P_0(t)W(t)} dt + \frac{(-1 + \alpha)}{B_2(y_1)} \right) + y_2(x) \left(\int_a^x \frac{F(t)y_1(t)}{P_0(t)W(t)} dt + \frac{(-1 + \alpha)}{B_1(y_2)} \right),$$

$$\bar{y}_{\alpha_p}(x) = y_1(x) \left(\int_x^b \frac{F(t)y_2(t)}{P_0(t)W(t)} dt + \frac{(1 - \alpha)}{B_2(y_1)} \right) + y_2(x) \left(\int_a^x \frac{F(t)y_1(t)}{P_0(t)W(t)} dt + \frac{(1 - \alpha)}{B_1(y_2)} \right).$$

Therefore the general solution of (3.6) is of the form

$$[y]^\alpha = \left[\underline{y}_{\alpha_p} + c_1(\alpha)y_1 + c_2(\alpha)y_2, \bar{y}_{\alpha_p} + c_3(\alpha)y_1 + c_4(\alpha)y_2 \right].$$

Then from $B_1(y) = Ay(a) + By'(a) = [-1 + \alpha, 1 - \alpha]$, we obtained $c_2(\alpha) = c_4(\alpha) = 0$. From $B_2(y) = Cy(b) + Dy'(b) = [-1 + \alpha, 1 - \alpha]$, we obtained $\alpha = 1$. Therefore $y = y_p + c_1y_1$. This completes the proof.

Example 3.6. Consider the fuzzy boundary value problem

$$x^2y' - 2xy' + 2y = 2x^3, \quad y(1) = [3 + \alpha, 5 - \alpha], \quad y'(2) = [2 + \alpha, 4 - \alpha]. \quad (3.10)$$

Here $B_1(y) = y(1)$ and $B_2(y) = y'(2)$. Let $\{y_1, y_2\} = \{x^2, x\}$, which is a fundamental set of solutions of homogeneous differential equation in (3.10). Then

$$B_1(y_1)B_2(y_2) - B_1(y_2)B_2(y_1) = -3.$$

So Theorem 3.3 implies that (3.10) has a unique solution except the α -level set. The solution of differential equation in (3.10) is

$$[y(x)]^\alpha = \left[x^3 + c_1(\alpha)x^2 + c_2(\alpha)x, x^3 + c_3(\alpha)x^2 + c_4(\alpha)x \right].$$

Using boundary conditions, coefficients $c_1(\alpha), c_2(\alpha), c_3(\alpha), c_4(\alpha)$ are obtained as

$$c_1(\alpha) = c_3(\alpha) = -4, \quad c_2(\alpha) = 6 + \alpha, \quad c_4(\alpha) = 8 - \alpha.$$

Example 3.7. Consider the fuzzy boundary value problem

$$y' + y = 1, \quad y(0) = [-1 + \alpha, 1 - \alpha], \quad y\left(\frac{\pi}{2}\right) = [-1 + \alpha, 1 - \alpha]. \quad (3.11)$$

Here $B_1(y) = y(0)$ and $B_2(y) = y\left(\frac{\pi}{2}\right)$. Let $\{y_1, y_2\} = \{\sin x, \cos x\}$, which is a fundamental set of solutions of homogeneous differential equation in (3.11). Then

$$B_1(y_1)B_2(y_2) - B_1(y_2)B_2(y_1) = -1.$$

So Theorem 3.3 implies that (3.11) has a unique solution except the α -level set. Also, the Wronskian of $\{y_1, y_2\}$ is

$$W(x) = \sin x(-\sin x) - \cos x \cos x = -1.$$

Then (3.7), (3.8) and (3.9) yield the solution

$$[y(x)]^\alpha = \left[(-2 + \alpha)(\cos x + \sin x) + 1, (-\alpha)(\cos x + \sin x) + 1 \right].$$

Example 3.8. If we take

$$y(0) = [-1 + \alpha, 1 - \alpha], \quad y(\pi) = [-1 + \alpha, 1 - \alpha] \quad (3.12)$$

as the boundary conditions for the fuzzy boundary value problem in (3.11), we have

$$B_1(y_1)B_2(y_2) - B_1(y_2)B_2(y_1) = 0.$$

The solution of differential equation in (3.11) is

$$[y(x)]^\alpha = \left[\underline{y}_\alpha(x), \bar{y}_\alpha(x) \right],$$

Where

$$\underline{y}_\alpha(x) = c_1(\alpha)\cos x + c_2(\alpha)\sin x + 1, \quad \bar{y}_\alpha(x) = c_3(\alpha)\cos x + c_4(\alpha)\sin x + 1.$$

From boundary conditions, we have

$$\underline{y}_\alpha(0) = -1 + \alpha \Rightarrow c_1(\alpha) = -2 + \alpha, \quad \bar{y}_\alpha(0) = 1 - \alpha \Rightarrow c_3(\alpha) = -\alpha,$$

$$\underline{y}_\alpha(\pi) = -1 + \alpha \Rightarrow c_1(\alpha) = 2 - \alpha, \quad \bar{y}_\alpha(\pi) = 1 - \alpha \Rightarrow c_3(\alpha) = \alpha.$$

Therefore, differential equation in (3.11) with boundary conditions (3.12) has no solution.

Example 3.9. Consider the fuzzy boundary value problem

$$y' + y = \sin 2x, \quad y(0) = [-1 + \alpha, 1 - \alpha], \quad y(\pi) = [-1 + \alpha, 1 - \alpha]. \quad (3.13)$$

The solution of differential equation in (3.13) is

$$[y(x)]^\alpha = \left[c_1(\alpha)\cos x + c_2(\alpha)\sin x - \frac{\sin 2x}{3}, c_3(\alpha)\cos x + c_4(\alpha)\sin x - \frac{\sin 2x}{3} \right].$$

From boundary conditions, we get

$$\underline{y}_\alpha(0) = -1 + \alpha \Rightarrow c_1(\alpha) = -1 + \alpha, \quad \bar{y}_\alpha(0) = 1 - \alpha \Rightarrow c_3(\alpha) = 1 - \alpha,$$

$$\underline{y}_\alpha(\pi) = -1 + \alpha \Rightarrow c_1(\alpha) = 1 - \alpha, \quad \bar{y}_\alpha(\pi) = 1 - \alpha \Rightarrow c_3(\alpha) = -1 + \alpha.$$

Therefore, if $\alpha = 1$, (3.13) has infinitely many solutions, differing by constant multiples of $\sin x$. If $\alpha \neq 1$, (3.13) has no solution.

Example 3.10. Consider the fuzzy boundary value problem

$$y' + y = F(x), \quad y(0) + y'(0) = [-1 + \alpha, 1 - \alpha], \quad y(\pi) + y'(\pi) = [-1 + \alpha, 1 - \alpha]. \quad (3.14)$$

Here $B_1(y) = y(0) + y'(0)$ and $B_2(y) = y(\pi) + y'(\pi)$. Let $\{z_1, z_2\} = \{\cos x, \sin x\}$, which is a fundamental set of solutions of homogeneous differential equation in (3.14). Then

$$B_1(z_1)B_2(z_2) - B_1(z_2)B_2(z_1) = 2.$$

So Theorem 3.3 implies that (3.14) has a unique solution except the α -level set. Let

$$y_1 = B_1(z_2)z_1 - B_1(z_1)z_2 = \cos x - \sin x \quad \text{and} \quad y_2 = B_2(z_2)z_1 - B_2(z_1)z_2 = \cos x + \sin x.$$

Then $B_1(y_1) = 0$, $B_2(y_2) = 0$ and the Wronskian of $\{y_1, y_2\}$ is

$$W(x) = (\cos x - \sin x)(-\sin x + \cos x) - (\cos x + \sin x)(-\sin x - \cos x) = 2.$$

So (3.7), (3.8) and (3.9) yield the solution

$$[y(x)]^\alpha = [\underline{y}_\alpha(x), \bar{y}_\alpha(x)],$$

Where

$$\begin{aligned} \underline{y}_\alpha(x) = & \frac{\cos x - \sin x}{2} \left(\int_x^\pi F(t)(\cos t + \sin t) dt + (1 - \alpha) \right) \\ & + \frac{\cos x + \sin x}{2} \left(\int_0^x F(t)(\cos t - \sin t) dt + (\alpha - 1) \right), \end{aligned}$$

$$\bar{y}_\alpha(x) = \frac{\cos x - \sin x}{2} \left(\int_x^{\pi} F(t)(\cos t + \sin t) dt + (\alpha - 1) \right) \\ + \frac{\cos x + \sin x}{2} \left(\int_0^x F(t)(\cos t - \sin t) dt + (1 - \alpha) \right).$$

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