ON SOLUTION OF TWO-POINT FUZZY BOUNDARY VALUE PROBLEMS

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Abstract: In this paper, we investigate the existence and uniqueness of solutions of two-point fuzzy boundary value problems for second-order fuzzy differential equations. Some sufficient conditions are presented that guarantee the existence and uniqueness of solutions under the approach of Hukuhara differentiability.

1. Introduction

Fuzzy set theory is a powerful tool for modeling uncertainly and for processing vague or subjective information in mathematical models [3]. Particularly, the study of fuzzy differential equation forms a suitable setting for mathematical modeling of real world problems in which uncertainties or vagueness pervade. In fact, fuzzy differential equations is a very important topic from theoretical point of view [5], [6], [7], [9] and also it has applications, for example, in population models [4], civil engineering [8], medicine [1], [2].

In this paper, an investigation is made on the solution of two-point fuzzy boundary value problems by using the Hukuhara differentiability. To put it precisely, the two-point fuzzy boundary value problem is given as the form

\[ L_y := P_0(x)y' + P_1(x)y' + P_2(x)y = F(x), \quad x \in (a, b) \]  \hspace{1cm} (1.1)

\[ B_1(y) := Ay(a) + By'(a) = [k_1]^a \]  \hspace{1cm} (1.2)

\[ B_2(y) := Cy(b) + Dy'(b) = [k_2]^b \]  \hspace{1cm} (1.3)

where \([k_1]^a = [k_{1a}, k_{1a}^a], [k_2]^b = [k_{2a}, k_{2a}^b]\) are symmetric triangular fuzzy numbers, \(A^2 + B^2 \neq 0, C^2 + D^2 \neq 0\), \(P_0(x), P_1(x), P_2(x), F(x)\) are continuous functions and \(P_0(x)\) has no zeros on \((a, b)\).

2. Preliminaries

In this section, we give some definitions and introduce the necessary notation which will be used throughout the paper.

Definition 2.1. A fuzzy number is a function \(u: \mathfrak{F} \rightarrow [0, 1]\) satisfying the following properties:

- \(u\) is normal, convex fuzzy set, upper semi-continuous on \(\mathfrak{F}\) and \(\text{cl}\{x \in \mathfrak{F} | u(x) > 0\}\) is compact,

where \(\text{cl}\) denotes the closure of a subset.

Let \(\mathfrak{F}\), denote the space of fuzzy numbers.
Definition 2.2. Let \( u \in \mathbb{F} \). The \( \alpha \)-level set of \( u \), denoted \([u]^\alpha, 0 < \alpha \leq 1\), is \([u]^\alpha = \{x \in \mathbb{F} | u(x) \geq \alpha\}\). If \( \alpha = 0 \), the support of \( u \) is defined \([u]^0 = \text{cl}\{x \in \mathbb{F} | u(x) > 0\}\). The notation \([u]^\alpha = [u_\alpha, u_\alpha]\), denotes explicitly the \( \alpha \)-level set of \( u \). We refer to \( u \) and \( \overline{u} \) as the lower and upper branches of \( u \), respectively.

The following remark shows when \([u_\alpha, u_\alpha]\) is a valid \( \alpha \)-level set.

Remark 2.3. The sufficient and necessary conditions for \([u_\alpha, u_\alpha]\) to define the parametric form of a fuzzy number as follows:

(i) \( u_\alpha \) is bounded monotonic increasing (nondecreasing) left-continuous function on \((0,1]\) and right-continuous for \( \alpha = 0 \),

(ii) \( \overline{u}_\alpha \) is bounded monotonic decreasing (nonincreasing) left-continuous function on \((0,1]\) and right-continuous for \( \alpha = 0 \),

(iii) \( u_\alpha \leq \overline{u}_\alpha \), \( 0 \leq \alpha \leq 1 \).

Definition 2.4. For \( u, v \in \mathbb{F} \) and \( \lambda \in \mathbb{F} \), the sum \( u + v \) and the product \( \lambda u \) are defined by \([u + v]^\alpha = [u]^\alpha + [v]^\alpha\), \( \lambda [u]^\alpha = \lambda [u]^\alpha\), \( \forall \alpha \in [0,1]\), where \([u]^\alpha + [v]^\alpha\) means the usual addition of two intervals (subsets) of \( \mathbb{F} \) and \( \lambda [u]^\alpha \) means the usual product between a scalar and a subset of \( \mathbb{F} \). The metric structure is given by the Hausdorff distance

\[
D: \mathbb{F} \times \mathbb{F} \rightarrow [0, \infty) \cup \{0\},
\]

by

\[
D(u, v) = \sup_{\alpha \in [0,1]} \max \left\{ |u_\alpha - v_\alpha|, \left| \overline{u}_\alpha - \overline{v}_\alpha \right| \right\}.
\]

Definition 2.5. Let \( u, v \in \mathbb{F} \). If there exist \( w \in \mathbb{F} \) such that \( u = v + w \), then \( w \) is called the Hukuhara difference of \( u \) and \( v \) and it is denoted by \( u \Delta v \).

Definition 2.6. Let \( l = (a, b) \) and \( F: l \rightarrow \mathbb{F} \) be a fuzzy function. We say \( F \) is Hukuhara differentiable at \( t_0 \in l \), if there exists an element \( F(t_0) \in \mathbb{F} \) such that, for all \( h > 0 \) sufficiently near to \( 0 \), we have \( F(t_0 + h) \Delta F(t_0) \), \( F(t_0) \Delta F(t_0 - h) \) and the limits

\[
\lim_{h \to 0^+} \frac{F(t_0 + h) \Delta F(t_0)}{h} = \lim_{h \to 0^-} \frac{F(t_0) \Delta F(t_0 - h)}{h} = F'(t_0).
\]

Here the limits are taken in the metric space \((\mathbb{F}, D)\).

Theorem 2.7. Let \( f: l \rightarrow \mathbb{F} \) be a function and denote \([f(t)]^\alpha = [f_\alpha(t), \overline{f}_\alpha(t)]\), for each \( \alpha \in [0,1] \). If \( f \) is Hukuhara differentiable, then \( f_\alpha(t) \) and \( \overline{f}_\alpha(t) \) are differentiable functions and

\[
[f(t)]^\alpha = [f_\alpha(t), \overline{f}_\alpha(t)].
\]
3. Fuzzy Boundary Value Problem

In this section we concern with the existence and uniqueness of solution for the problem (1.1)-(1.3).

**Theorem 3.1.** Let \([\phi(x)]^a = \[\phi_a(x), \phi_a(x)\], [\psi(x)]^a = [\psi_a(x), \psi_a(x)]\). Let \(\Phi\) and \(\Psi\) be solutions of \(L_y = 0\) such that either \(B_1(\phi) = B_1(\psi) = 0\) or \(B_2(\phi) = B_2(\psi) = 0\), then \(\{\psi_a, \psi_a\}\) and \(\{\psi_a, \psi_a\}\) are linearly dependent.

**Proof.** Recall that \(B_i(y) = Ay(a) + By'(a)\) and \(A^2 + B^2 \neq 0\). Therefore, if \(B_i(\phi) = B_i(\psi) = 0\) then \(A\) and \(B\) are nontrivial solutions of the systems

\[
A\phi_a(a) + B\phi_a'(a) = 0 \quad \text{and} \quad A\phi_a(a) + B\phi_a'(a) = 0
\]

\[
A\psi_a(a) + B\psi_a'(a) = 0 \quad \text{and} \quad A\psi_a(a) + B\psi_a'(a) = 0.
\]

This implies that

\[
\phi_a(a)\psi_a'(a) - \phi_a(a)\psi_a'(a) = 0 \quad \text{and} \quad \phi_a(a)\psi_a'(a) - \phi_a(a)\psi_a'(a) = 0.
\]

So \(\{\phi_a, \psi_a\}\) and \(\{\phi_a, \psi_a\}\) are linearly dependent.

**Corollary 3.2.** If \(\{\phi_a, \psi_a\}\) or \(\{\phi_a, \psi_a\}\) is linearly independent, then

\[B_1^2(\phi) + B_2^2(\psi) \neq 0 \quad \text{and} \quad B_1^2(\phi) + B_2^2(\psi) \neq 0.\]

**Theorem 3.3.** The following statements are equivalent; that is, they are all true or all false.

(a) There is a fundamental set \(\{z_1, z_2\}\) of solutions of \(L_y = 0\) such that

\[B_1(z_1)B_2(z_2) - B_1(z_2)B_2(z_1) \neq 0. \quad (3.1)\]

(b) If \(\{y_1, y_2\}\) is a fundamental set of solutions of \(L_y = 0\) then

\[B_1(y_1)B_2(y_2) - B_1(y_2)B_2(y_1) \neq 0. \quad (3.2)\]

(c) For each continuous function \(F\) and pair of symmetric triangular fuzzy numbers \([[k_1]], [[k_2]]\)), the fuzzy boundary value problem (1.1)-(1.3) has a unique solution except the \(\alpha\)-level set.

(d) The homogeneous boundary value problem

\[L_y = 0, \quad B_1(y) = 0, \quad B_2(y) = 0 \quad (3.3)\]

has only the trivial solution \(y = 0\).

(e) There is a fundamental set \(\{z_1, z_2\}\) of solutions of \(L_y = 0\) such that \(B_1(z_1) = 0\) and \(B_2(z_2) = 0\).

**Proof.** We show that

(a) \(\Rightarrow\) (b) \(\Rightarrow\) (c) \(\Rightarrow\) (d) \(\Rightarrow\) (e) \(\Rightarrow\) (a).

(a)\(\Rightarrow\)(b): Since \(\{z_1, z_2\}\) is a fundamental set of solutions of \(L_y = 0\), there are constants \(a_i, a_z, b_i, b_z\) such that

\[
y_1 = a_1z_1 + a_2z_2,
\]

\[
y_2 = b_1z_1 + b_2z_2. \quad (3.4)
\]
Moreover,
\[ \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \neq 0. \quad (3.5) \]

Because if this determinant were zero, its rows would be linearly dependent and therefore \( \{y_1, y_2\} \) would be linearly dependent, contrary to our assumption that \( \{y_1, y_2\} \) is a fundamental set of solutions of \( Ly = 0 \). From (3.4),
\[
\begin{bmatrix} B_1(y_1) & B_2(y_1) \\ B_1(y_2) & B_2(y_2) \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} B_1(z_1) & B_2(z_1) \\ B_1(z_2) & B_2(z_2) \end{bmatrix}.
\]

Since the determinant of a product of matrices is equal to the product of the determinants of the matrices, (3.1) and (3.5) imply (3.2).

(b)\(\Rightarrow\)(c): Since \( \{y_1, y_2\} \) is a fundamental set of solutions of \( Ly = 0 \), the general solution of \( Ly = F \) is
\[
\begin{bmatrix} y \end{bmatrix}^a = \begin{bmatrix} y_{a_1}^\alpha + c_1(\alpha)y_1 + c_2(\alpha)y_2, y_{a_2}^\alpha + c_3(\alpha)y_1 + c_4(\alpha)y_2 \end{bmatrix},
\]
where \( \begin{bmatrix} y_p \end{bmatrix}^a = \begin{bmatrix} y_{a_p}^\alpha, y_{a_p}^\alpha \end{bmatrix} \) is a particular solution of \( Ly = F \). To satisfy the boundary conditions, we must choose
\[
k_{1\alpha} = B_1(y_{a_1}^\alpha) + c_1(\alpha)B_1(y_1) + c_2(\alpha)B_1(y_2),
\]
\[
k_{2\alpha} = B_2(y_{a_2}^\alpha) + c_1(\alpha)B_2(y_1) + c_2(\alpha)B_2(y_2)
\]
and
\[
k_{1\alpha} = B_1(y_{a_1}^\alpha) + c_3(\alpha)B_1(y_1) + c_4(\alpha)B_1(y_2),
\]
\[
k_{2\alpha} = B_2(y_{a_2}^\alpha) + c_3(\alpha)B_2(y_1) + c_4(\alpha)B_2(y_2),
\]
which are equivalent to
\[
c_1(\alpha)B_1(y_1) + c_2(\alpha)B_1(y_2) = k_{1\alpha} - B_1(y_{a_1}^\alpha),
\]
\[
c_1(\alpha)B_2(y_1) + c_2(\alpha)B_2(y_2) = k_{2\alpha} - B_2(y_{a_2}^\alpha)
\]
and
\[
c_1(\alpha)B_1(y_1) + c_4(\alpha)B_1(y_2) = k_{1\alpha} - B_1(y_{a_1}^\alpha),
\]
\[
c_2(\alpha)B_2(y_1) + c_4(\alpha)B_2(y_2) = k_{2\alpha} - B_2(y_{a_2}^\alpha).
\]

(c)\(\Rightarrow\)(d): Obviously, \( y = 0 \) is a solution of (3.3). From (c) with \( F = 0 \) and \( \begin{bmatrix} k_1 \end{bmatrix}^a = \begin{bmatrix} k_2 \end{bmatrix}^a = 0 \), it is the only solution.

d)\(\Rightarrow\)(e): Let \( \{y_1, y_2\} \) be a fundamental system for \( Ly = 0 \) and let
\[ z_1 = B_1(y_2)y_1 - B_1(y_1)y_2 \quad \text{and} \quad z_2 = B_2(y_2)y_1 - B_2(y_1)y_2. \]
Then $B_1(z_1) = 0$ and $B_2(z_2)$. To see that $z_1$ and $z_2$ are linearly independent, note that

$$a_1 z_1 + a_2 z_2 = a_1 \left[ B_1 (y_1) v_1 - B_2 (y_1) w_1 \right] + a_2 \left[ B_1 (y_2) v_2 - B_2 (y_2) w_2 \right]$$

$$= \left[ B_1 (y_2) a_1 + B_2 (y_2) a_2 \right] v_2 - \left[ B_1 (y_1) a_1 + B_2 (y_1) a_2 \right] w_2.$$ 

Therefore, since $y_1$ and $y_2$ are linearly independent, $a_1 z_1 + a_2 z_2 = 0$ if and only if

$$\begin{bmatrix} B_1 (y_1) & B_2 (y_1) \\ B_1 (y_2) & B_2 (y_2) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$ 

If this system has a nontrivial solution then so does the system

$$\begin{bmatrix} B_1 (y_1) & B_1 (y_2) \\ B_2 (y_1) & B_2 (y_2) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$ 

This implies that $y = c_1 z_1 + c_2 z_2$ is a nontrivial solution of (3.3), which contradicts (d).

\((e) \Rightarrow (a):\) Theorem 3.1 implies that if $B_1(z_1) = 0$ and $B_2(z_2) = 0$ then $B_1(z_2) \neq 0$ and $B_2(z_2) \neq 0$.

This implies (3.1), which completes the proof.

**Theorem 3.4.** Suppose the homogeneous boundary value problem (3.3) has only the trivial solution. Let $y_1$ and $y_2$ be a fundamental set of solutions of $L y = 0$ such that $B_1 (y_1) = 0$ and $B_2 (y_2) = 0$, and let $W = y_1 y_2 - y_1' y_2$. Then the unique solution except the $\alpha$-level set of

$$L y = F, \quad B_1(y) = [0]^a, \quad B_2(y) = [0]^a$$

is

$$\begin{bmatrix} y(x) \end{bmatrix} = \begin{bmatrix} y_\alpha(x), y_\alpha(x) \end{bmatrix}, \quad (3.6)$$

where

$$y_\alpha(x) = y_1(x) \left( \int_0^x F(t) y_2(t) \frac{dt}{P_0(t) W(t)} + \frac{-1 + \alpha}{B_2(y_1)} \right) + y_2(x) \left( \int_0^x F(t) y_1(t) \frac{dt}{P_0(t) W(t)} + \frac{-1 + \alpha}{B_1(y_2)} \right), \quad (3.8)$$

$$-y_\alpha(x) = y_1(x) \left( \int_0^x F(t) y_2(t) \frac{dt}{P_0(t) W(t)} + \frac{1 - \alpha}{B_2(y_1)} \right) + y_2(x) \left( \int_0^x F(t) y_1(t) \frac{dt}{P_0(t) W(t)} + \frac{1 - \alpha}{B_1(y_2)} \right) \quad (3.9)$$

and $[0]^a = [-1 + \alpha, 1 - \alpha]$.

**Proof.** Suppose the solution of $L y = F$ is

$$\begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} u_1(x, \alpha) y_1 + u_2(x, \alpha) y_2, u_3(x, \alpha) y_1 + u_4(x, \alpha) y_2 \end{bmatrix}.$$
Using the method of variation of parameters
\[ u_1'(x,\alpha)y_1 + u_2'(x,\alpha)y_2 = 0 \quad u_3'(x,\alpha)y_1 + u_4'(x,\alpha)y_2 = 0 \]

are obtained. Solving for \( u_1'(x,\alpha), u_2'(x,\alpha), u_3'(x,\alpha) \) and \( u_4'(x,\alpha) \) yields
\[ u_1'(x,\alpha) = \frac{Fy_2}{P_0 W}, \quad u_2'(x,\alpha) = \frac{Fy_1}{P_0 W}, \quad u_3'(x,\alpha) = -\frac{Fy_2}{P_0 W}, \quad u_4'(x,\alpha) = \frac{Fy_1}{P_0 W}. \]

Integrating these yields
\[ u_1(x,\alpha) = \int_{x}^{t} \frac{F(t)y_2(t)}{P_0(t)W(t)} \, dt + u_1(b,\alpha), \quad u_2(x,\alpha) = \int_{x}^{t} \frac{F(t)y_1(t)}{P_0(t)W(t)} \, dt + u_2(a,\alpha), \]
\[ u_3(x,\alpha) = \int_{x}^{t} \frac{F(t)y_2(t)}{P_0(t)W(t)} \, dt + u_3(b,\alpha), \quad u_4(x,\alpha) = \int_{x}^{t} \frac{F(t)y_1(t)}{P_0(t)W(t)} \, dt + u_4(a,\alpha). \]

Using \( B_1(y) = Ay(a) + By'(a) = [-1+\alpha, 1-\alpha] \) and \( B_2(y) = Cy(b) + Dy'(b) = [-1+\alpha, 1-\alpha] \), we solved \( u_1(b,\alpha), u_2(a,\alpha), u_3(b,\alpha), u_4(a,\alpha) \) as
\[ u_1(b,\alpha) = \frac{-1+\alpha}{B_2(y_1)}, \quad u_2(a,\alpha) = \frac{-1+\alpha}{B_1(y_2)}, \quad u_3(b,\alpha) = \frac{1-\alpha}{B_2(y_1)}, \quad u_4(a,\alpha) = \frac{1-\alpha}{B_1(y_2)}. \]

This completes the proof.

**Theorem 3.5.** Suppose the homogeneous boundary value problem (3.3) has a nontrivial solution \( y_1 \) and let \( y_2 \) be any solution of \( Ly = 0 \) that isn't a constant multiple of \( y_1 \). Let \( W = y_1, y_2, y_1, y_2 \). If \( \alpha = 1 \), then the fuzzy boundary value problem (3.6) has infinitely many solutions, all of the form \( y = y_p + c_1 y_1 \), where
\[ y_p = y_1(x) \int_{x}^{t} \frac{F(t)y_2(t)}{P_0(t)W(t)} \, dt + y_2(x) \int_{x}^{t} \frac{F(t)y_1(t)}{P_0(t)W(t)} \, dt \]
and \( c_1 \) is a arbitrary constant. Also \([0]^T = [-1+\alpha, 1-\alpha] \). If \( \alpha \neq 1 \), then (3.6) has no solution.

**Proof.** From the proof of Theorem 3.4, \( [y_p(x)]^T = [\underline{y}_{\alpha_p}(x), \bar{y}_{\alpha_p}(x)] \) is a particular solution of \( Ly = F_1 \), where
\[ \underline{y}_{\alpha_p}(x) = y_1(x) \left( \int_{x}^{t} \frac{F(t)y_2(t)}{P_0(t)W(t)} \, dt + \frac{-1+\alpha}{B_2(y_1)} \right) + y_2(x) \left( \int_{x}^{t} \frac{F(t)y_1(t)}{P_0(t)W(t)} \, dt + \frac{-1+\alpha}{B_1(y_2)} \right), \]
\[ \bar{y}_{\alpha_p}(x) = y_1(x) \left( \int_{x}^{t} \frac{F(t)y_2(t)}{P_0(t)W(t)} \, dt + \frac{1-\alpha}{B_2(y_1)} \right) + y_2(x) \left( \int_{x}^{t} \frac{F(t)y_1(t)}{P_0(t)W(t)} \, dt + \frac{1-\alpha}{B_1(y_2)} \right). \]

Therefore the general solution of (3.6) is of the form...
Example 3.6. Consider the fuzzy boundary value problem

\[ x^2y'' - 2xy' + 2y = 2x^3, \quad y(1) = [3 + \alpha, 5 - \alpha], \quad y'(2) = [2 + \alpha, 4 - \alpha]. \]  

(3.10)

Here \( B_1(y) = y(1) \) and \( B_2(y) = y'(2) \). Let \( \{y_1, y_2\} = \{x^2, x\} \), which is a fundamental set of solutions of homogeneous differential equation in (3.10). Then

\[ B_1(y_1)B_2(y_2) - B_1(y_2)B_2(y_1) = -3. \]

So Theorem 3.3 implies that (3.10) has a unique solution except the \( \alpha \)-level set. The solution of differential equation in (3.10) is

\[ \left[ y(x) \right]^a = \left[ x^2 + c_1(\alpha)x^2 + c_2(\alpha)x, x^2 + c_3(\alpha)x^2 + c_4(\alpha)x \right]. \]

Using boundary conditions, coefficients \( c_1(\alpha), c_2(\alpha), c_3(\alpha), c_4(\alpha) \) are obtained as

\[ c_1(\alpha) = c_2(\alpha) = -4, \quad c_3(\alpha) = 6 + \alpha, \quad c_4(\alpha) = 8 - \alpha. \]

Example 3.7. Consider the fuzzy boundary value problem

\[ y'' + y = 1, \quad y(0) = [-1 + \alpha, 1 - \alpha], \quad y'\left(\frac{\pi}{2}\right) = [-1 + \alpha, 1 - \alpha]. \]  

(3.11)

Here \( B_1(y) = y(0) \) and \( B_2(y) = y'\left(\frac{\pi}{2}\right) \). Let \( \{y_1, y_2\} = \{\sin x, \cos x\} \), which is a fundamental set of solutions of homogeneous differential equation in (3.11). Then

\[ B_1(y_1)B_2(y_2) - B_1(y_2)B_2(y_1) = -1. \]

So Theorem 3.3 implies that (3.11) has a unique solution except the \( \alpha \)-level set. Also, the Wronskian of \( \{y_1, y_2\} \) is

\[ W(x) = \sin x(-\sin x) - \cos x \cos x = -1. \]

Then (3.7), (3.8) and (3.9) yield the solution

\[ \left[ y(x) \right]^a = \left[ (-2 + \alpha)(\cos x + \sin x) + 1, (-\alpha)(\cos x + \sin x) + 1 \right]. \]

Example 3.8. If we take

\[ y(0) = [-1 + \alpha, 1 - \alpha], \quad y(\pi) = [-1 + \alpha, 1 - \alpha] \]  

(3.12)

as the boundary conditions for the fuzzy boundary value problem in (3.11), we have

\[ B_1(y_1)B_2(y_2) - B_1(y_2)B_2(y_1) = 0. \]

The solution of differential equation in (3.11) is

\[ \left[ y(x) \right]^a = \left[ y_1(x), y_2(x) \right]. \]
Where \[ y_\alpha (x) = c_1(\alpha) \cos x + c_2(\alpha) \sin x + 1, \quad \overline{y}_\alpha (x) = c_3(\alpha) \cos x + c_4(\alpha) \sin x + 1. \]

From boundary conditions, we have
\[ y_\alpha (0) = -1 + \alpha \Rightarrow c_1(\alpha) = -2 + \alpha, \quad \overline{y}_\alpha (0) = 1 - \alpha \Rightarrow c_3(\alpha) = -\alpha, \]
\[ y_\alpha (\pi) = -1 + \alpha \Rightarrow c_1(\alpha) = 2 - \alpha, \quad \overline{y}_\alpha (\pi) = 1 - \alpha \Rightarrow c_3(\alpha) = \alpha. \]

Therefore, differential equation in (3.11) with boundary conditions (3.12) has no solution.

**Example 3.9.** Consider the fuzzy boundary value problem
\[ y' + y = \sin 2x, \quad y(0) = [-1 + \alpha, 1 - \alpha], \quad y(\pi) = [-1 + \alpha, 1 - \alpha]. \tag{3.13} \]

The solution of differential equation in (3.13) is
\[ [y(x)]^\alpha = \left[ c_1(\alpha) \cos x + c_2(\alpha) \sin x - \frac{\sin 2x}{3}, c_3(\alpha) \cos x + c_4(\alpha) \sin x - \frac{\sin 2x}{3} \right]. \]

From boundary conditions, we get
\[ y_\alpha (0) = -1 + \alpha \Rightarrow c_1(\alpha) = -1 - \alpha, \quad \overline{y}_\alpha (0) = 1 - \alpha \Rightarrow c_3(\alpha) = 1 - \alpha, \]
\[ y_\alpha (\pi) = -1 + \alpha \Rightarrow c_1(\alpha) = 1 - \alpha, \quad \overline{y}_\alpha (\pi) = 1 - \alpha \Rightarrow c_3(\alpha) = -1 + \alpha. \]

Therefore, if \( \alpha = 1 \), (3.13) has infinitely many solutions, differing by constant multiples of \( \sin x \). If \( \alpha \neq 1 \), (3.13) has no solution.

**Example 3.10.** Consider the fuzzy boundary value problem
\[ y' + y = f(x), \quad y(0) + y'(0) = [-1 + \alpha, 1 - \alpha], \quad y(\pi) + y'(\pi) = [-1 + \alpha, 1 - \alpha]. \tag{3.14} \]

Here \( B_1(y) = y(0) + y'(0) \) and \( B_2(y) = y(\pi) + y'(\pi) \). Let \( \{z_1, z_2\} = \{\cos x, \sin x\} \), which is a fundamental set of solutions of homogeneous differential equation in (3.14). Then
\[ B_1(z_1)B_2(z_2) - B_1(z_2)B_2(z_1) = 2. \]

So Theorem 3.3 implies that (3.14) has a unique solution except the \( \alpha \)-level set. Let
\[ y_1 = B_1(z_1)z_1 - B_1(z_2)z_2 = \cos x - \sin x \quad \text{and} \quad y_2 = B_2(z_2)z_1 - B_2(z_1)z_2 = \cos x + \sin x. \]

Then \( B_1(y_1) = 0 \), \( B_2(y_2) = 0 \) and the Wronskian of \( \{y_1, y_2\} \) is
\[ W(x) = \left( \cos x - \sin x \right) \left( -\sin x + \cos x \right) - \left( \cos x + \sin x \right) \left( -\sin x - \cos x \right) = 2. \]

So (3.7), (3.8) and (3.9) yield the solution
\[ [y(x)]^\alpha = \left[ y_\alpha (x), \overline{y}_\alpha (x) \right], \]

where
\[ y_\alpha (x) = \frac{\cos x - \sin x}{2} \left( \int_0^x f(t) (\cos t + \sin t) \, dt + (1 - \alpha) \right) \]
\[ + \frac{\cos x + \sin x}{2} \left( \int_0^x f(t) (\cos t - \sin t) \, dt + (\alpha - 1) \right), \]
\[ \overline{y}_\alpha (x) = \frac{\cos x - \sin x}{2} \left( \int_0^x f(t) (\cos t - \sin t) \, dt + (1 - \alpha) \right) \]
\[ + \frac{\cos x + \sin x}{2} \left( \int_0^x f(t) (\cos t + \sin t) \, dt + (\alpha - 1) \right). \]
\[ y'_\alpha (x) = \frac{\cos x - \sin x}{2} \left( \int_0^x F(t)(\cos t + \sin t) \, dt + (\alpha - 1) \right) + \frac{\cos x + \sin x}{2} \left( \int_0^x F(t)(\cos t - \sin t) \, dt + (1 - \alpha) \right). \]

References