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Rotations in Minkowski spacetime

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Abstract. With the relation of Olinde Rodrigues-Cartan is obtained an expression for the Lorentz matrix, and it is transformed to a better form for the Newman-Penrose formalism, thus it is possible to realize rotations of the null tetrad of NP.

1. Introduction

Here we employ the notation and conventions of [1]. The Olinde Rodrigues [2]-Cartan [3] expression:

$$\begin{pmatrix} \tilde{x}^0+\tilde{x}^3 & \tilde{x}^1-i\,\tilde{x}^2 \\ \tilde{x}^1+i\,\tilde{x}^2 & \tilde{x}^0-\tilde{x}^3 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x^0+x^3 & x^1-i\,x^2 \\ x^1+i\,x^2 & x^0-x^3 \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}, \tag{1}$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary complex numbers verifying the condition $\alpha\delta-\beta\gamma=1$, implies six degrees of freedom for the Lorentz matrix $L=(L^{\nu}{}_{\mu})$ between the frames of reference $(x^{\mu})=(ct,x,y,z)$ and $(\tilde{x}^{\nu})$:

$$\tilde{x}^{\nu} = L^{\nu}{}_{\mu}\, x^{\mu}\,. \tag{2}$$

From (1) and (2) we obtain the relations [4-8]:

$$L^0{}_0=\tfrac{1}{2}\left(\alpha\bar{\alpha}+\beta\bar{\beta}+\gamma\bar{\gamma}+\delta\bar{\delta}\right),\qquad L^1{}_0=\tfrac{1}{2}\left(\bar{\alpha}\,\gamma+\bar{\beta}\,\delta\right)+cc,\qquad L^2{}_0=\tfrac{i}{2}\left(\alpha\bar{\gamma}-\bar{\beta}\delta\right)+cc,$$

$$L^0{}_1=\tfrac{1}{2}(\bar{\alpha}\beta+\bar{\gamma}\delta)+cc,\qquad L^1{}_1=\tfrac{1}{2}(\bar{\alpha}\delta+\beta\bar{\gamma})+cc,\qquad L^2{}_1=\tfrac{i}{2}\left(\alpha\bar{\delta}+\beta\bar{\gamma}\right)+cc,$$

$$L^0{}_2=\tfrac{i}{2}(\bar{\alpha}\beta+\bar{\gamma}\delta)+cc,\qquad L^1{}_2=\tfrac{i}{2}(\bar{\alpha}\delta+\beta\bar{\gamma})+cc,\qquad L^2{}_2=\tfrac{1}{2}\left(\bar{\alpha}\delta-\bar{\beta}\gamma\right)+cc, \tag{3}$$

$$L^0{}_3=\tfrac{1}{2}\left(\alpha\bar{\alpha}-\beta\bar{\beta}+\gamma\bar{\gamma}-\delta\bar{\delta}\right),\qquad L^1{}_3=\tfrac{1}{2}\left(\bar{\alpha}\gamma-\bar{\beta}\delta\right)+cc,\qquad L^2{}_3=\tfrac{i}{2}\left(\alpha\bar{\gamma}+\bar{\beta}\delta\right)+cc,$$

$$L^3{}_0=\tfrac{1}{2}\left(\alpha\bar{\alpha}+\beta\bar{\beta}-\gamma\bar{\gamma}-\delta\bar{\delta}\right),\qquad L^3{}_1=\tfrac{1}{2}(\bar{\alpha}\beta-\bar{\gamma}\delta)+cc,\qquad L^3{}_2=\tfrac{i}{2}(\bar{\alpha}\beta-\bar{\gamma}\delta)+cc,$$

$$L^3{}_3=\tfrac{1}{2}\left(\alpha\bar{\alpha}-\beta\bar{\beta}-\gamma\bar{\gamma}+\delta\bar{\delta}\right),\qquad \alpha\delta-\beta\gamma=1,$$

where cc means the complex conjugate of all the previous terms.

In Sec. 2, into L we eliminate α, β, γ, δ in favour of another quantities better adapted to the null tetrads of Newman-Penrose (NP) [9-11], thus from (3) we shall obtain the expressions deduced in [12-14].

2. Rotations in special relativity

If in (3) we realize the changes $[\tau=\frac{1}{\sqrt{1-\bar{\Gamma}\bar{\Omega}}}]$:

$$\alpha=-\tau\exp\left(-\tfrac{A+iB}{2}\right),\quad \beta=\tau\exp\left(-\tfrac{A+iB}{2}\right)\bar{\Gamma},\quad \gamma=\tau\exp\left(\tfrac{A+iB}{2}\right)\bar{\Omega},\quad \delta=-\tau\exp\left(\tfrac{A+iB}{2}\right), \tag{4}$$

where A, B are real and Γ, Ω are complex with $\Gamma\Omega \neq 1$ (it is simple to verify that (4) respects the requirement $\alpha\delta - \beta\gamma = 1$), we deduce the relations of [12-14] $\left[Q = \frac{1}{2|1-\Gamma\Omega|}\right]$:

$$L^0{}_0 = Q\,[e^A(1+\Omega\bar{\Omega}) + e^{-A}(1+\Gamma\bar{\Gamma})],\ \ L^1{}_0 = -Q\,[e^{iB}(\Gamma+\bar{\Omega}) + cc],\ \ L^2{}_0 = -iQ\,[e^{-iB}(\Omega+\bar{\Gamma}) - cc],$$

$$L^0{}_1 = -Q\,[e^A(\Omega+\bar{\Omega}) + e^{-A}(\Gamma+\bar{\Gamma})],\ \ L^1{}_1 = Q\,[e^{iB}(1+\bar{\Omega}\Gamma) + cc],\ \ L^2{}_1 = iQ\,[e^{-iB}(1+\Omega\bar{\Gamma}) - cc],$$

$$L^0{}_2 = iQ\,[e^A(\bar{\Omega}-\Omega) + e^{-A}(\Gamma-\bar{\Gamma})],\ \ L^1{}_2 = iQ\,[e^{iB}(1-\bar{\Omega}\Gamma) - cc],\ \ L^2{}_2 = -Q\,[e^{-iB}(\Omega\bar{\Gamma}-1) + cc],$$

$$L^0{}_3 = Q\,[e^A(\Omega\bar{\Omega}-1) + e^{-A}(1-\Gamma\bar{\Gamma})],\ \ L^1{}_3 = Q\,[e^{iB}(\Gamma-\bar{\Omega}) + cc],\ \ L^2{}_3 = iQ\,[e^{-iB}(\bar{\Gamma}-\Omega) - cc],\tag{5}$$

$$L^3{}_0 = Q\,[-e^A(1+\Omega\bar{\Omega}) + e^{-A}(1+\Gamma\bar{\Gamma})],\ \ L^3{}_1 = Q\,[e^A(\Omega+\bar{\Omega}) - e^{-A}(\Gamma+\bar{\Gamma})],$$

$$L^3{}_2 = iQ\,[e^A(\Omega-\bar{\Omega}) + e^{-A}(\Gamma-\bar{\Gamma})],\ \ L^3{}_3 = Q\,[e^A(1-\Omega\bar{\Omega}) + e^{-A}(1-\Gamma\bar{\Gamma})].$$

In [15] we showed that an orthonormal real tetrad experiments a rotation in Minkowski space under a Lorentz matrix:

$$\tilde{e}^{(a)}{}_\mu = L^a{}_b\, e^{(b)}{}_\mu\,,\tag{6}$$

such that $e^{(0)}{}_\mu = e_{(0)\,\mu}$ and $e^{(j)}{}_\nu = -e_{(j)\,\nu}$, $j = 1,2,3$, this due to the metric tensor Diag (1,-1,-1,-1). From (5), (6) and the definitions [9-11]:

$$l^\nu = \frac{1}{\sqrt{2}}\left(e_{(0)}{}^\nu + e_{(3)}{}^\nu\right),\ n^\nu = \frac{1}{\sqrt{2}}\left(e_{(0)}{}^\nu - e_{(3)}{}^\nu\right), m^\nu = \frac{1}{\sqrt{2}}\left(e_{(1)}{}^\nu - ie_{(2)}{}^\nu\right),\ \bar{m}^\nu = \frac{1}{\sqrt{2}}\left(e_{(1)}{}^\nu + ie_{(2)}{}^\nu\right),\tag{7}$$

it is possible to calculate the rotation of this null tetrad of NP $\left[C = \frac{1}{|1-\Gamma\Omega|}\right]$:

$$\tilde{l}^\mu = C\,e^A(l^\mu + \Omega\bar{\Omega}\,n^\mu + \bar{\Omega}\,m^\mu + \Omega\,\bar{m}^\mu),\qquad \tilde{n}^\mu = C\,e^{-A}(n^\mu + \Gamma\bar{\Gamma}\,l^\mu + \Gamma\,m^\mu + \bar{\Gamma}\,\bar{m}^\mu),$$
$$\tilde{m}^\mu = C\,e^{-iB}(\bar{\Gamma}\,l^\mu + \Omega\,n^\mu + m^\mu + \bar{\Gamma}\Omega\,\bar{m}^\mu),\qquad \tilde{\bar{m}}^\mu = C\,e^{iB}(\Gamma\,l^\mu + \bar{\Omega}\,n^\mu + \bar{m}^\mu + \Gamma\bar{\Omega}\,m^\mu),\tag{8}$$

which is very useful for several applications in relativity [9-11, 16, 17]; in (8) it is easy to check that $\tilde{l}^\mu\tilde{n}_\mu = -\tilde{m}^\mu\tilde{\bar{m}}_\mu = 1$.

Thus, in the literature it is natural to employ three types of rotations:

Class I: $\Omega = 0$. l^μ preserves its direction.

$$\tilde{l}^\mu = e^A\,l^\mu,\ \tilde{n}^\mu = e^{-A}(n^\mu + \Gamma\bar{\Gamma}\,l^\mu + \Gamma\,m^\mu + \bar{\Gamma}\,\bar{m}^\mu),\ \tilde{m}^\mu = e^{-iB}(\bar{\Gamma}\,l^\mu + m^\mu),\ \tilde{\bar{m}}^\mu = e^{iB}(\Gamma\,l^\mu + \bar{m}^\mu),\tag{9}$$

Class II: $\Gamma = 0$. n^ν maintains its direction.

$\tilde{l}^\nu = e^{A}(l^\nu + \Omega\bar{\Omega}\, n^\nu + \bar{\Omega}\, m^\nu + \Omega\, \bar{m}^\nu)$, $\tilde{n}^\nu = e^{-A}\, n^\nu$, $\tilde{m}^\nu = e^{-iB}(\Omega\, n^\nu + m^\nu)$, $\tilde{\bar{m}}^\nu = e^{iB}(\bar{\Omega}\, n^\nu + \bar{m}^\nu)$, (10)

Class III: $\Omega = \Gamma = 0.$

$$\tilde{l}^\mu = e^{A}\, l^\mu\,, \qquad \tilde{n}^\mu = e^{-A}\, n^\mu\,, \qquad \tilde{m}^\mu = e^{-iB}\, m^\mu\,, \qquad \tilde{\bar{m}}^\mu = e^{iB}\, \bar{m}^\mu\,, \tag{11}$$

which permit to determine the evolution of, for example, the spin coefficients, the NP components of Weyl, Ricci and Lanczos tensors, etc., under a rotation of the null tetrad. This is important because it is usual to make rotations to align l^μ or/and n^μ with the principal directions of Cartan [18]-Debever [19]-Penrose [20] for the conformal tensor or the Faraday's electromagnetic tensor [21], which gives great simplification in many relativistic calculations.

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