

## Rotations in Minkowski spacetime

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**Abstract.** With the relation of Olinde Rodrigues-Cartan is obtained an expression for the Lorentz matrix, and it is transformed to a better form for the Newman-Penrose formalism, thus it is possible to realize rotations of the null tetrad of NP.

### 1. Introduction

Here we employ the notation and conventions of [1]. The Olinde Rodrigues [2]-Cartan [3] expression:

$$\begin{pmatrix} \tilde{x}^0 + \tilde{x}^3 & \tilde{x}^1 - i \tilde{x}^2 \\ \tilde{x}^1 + i \tilde{x}^2 & \tilde{x}^0 - \tilde{x}^3 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x^0 + x^3 & x^1 - i x^2 \\ x^1 + i x^2 & x^0 - x^3 \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}, \quad (1)$$

where  $\alpha, \beta, \gamma, \delta$  are arbitrary complex numbers verifying the condition  $\alpha\delta - \beta\gamma = 1$ , implies six degrees of freedom for the Lorentz matrix  $L = (L^\nu_\mu)$  between the frames of reference  $(x^\mu) = (ct, x, y, z)$  and  $(\tilde{x}^\nu)$ :

$$\tilde{x}^\nu = L^\nu_\mu x^\mu. \quad (2)$$

From (1) and (2) we obtain the relations [4-8]:

$$\begin{aligned} L^0_0 &= \frac{1}{2}(\alpha\bar{\alpha} + \beta\bar{\beta} + \gamma\bar{\gamma} + \delta\bar{\delta}), & L^1_0 &= \frac{1}{2}(\bar{\alpha}\gamma + \bar{\beta}\delta) + cc, & L^2_0 &= \frac{i}{2}(\alpha\bar{\gamma} - \bar{\beta}\delta) + cc, \\ L^0_1 &= \frac{1}{2}(\bar{\alpha}\beta + \bar{\gamma}\delta) + cc, & L^1_1 &= \frac{1}{2}(\bar{\alpha}\delta + \beta\bar{\gamma}) + cc, & L^2_1 &= \frac{i}{2}(\alpha\bar{\delta} + \beta\bar{\gamma}) + cc, \\ L^0_2 &= \frac{i}{2}(\bar{\alpha}\beta + \bar{\gamma}\delta) + cc, & L^1_2 &= \frac{i}{2}(\bar{\alpha}\delta + \beta\bar{\gamma}) + cc, & L^2_2 &= \frac{1}{2}(\bar{\alpha}\delta - \bar{\beta}\gamma) + cc, \\ L^0_3 &= \frac{1}{2}(\alpha\bar{\alpha} - \beta\bar{\beta} + \gamma\bar{\gamma} - \delta\bar{\delta}), & L^1_3 &= \frac{1}{2}(\bar{\alpha}\gamma - \bar{\beta}\delta) + cc, & L^2_3 &= \frac{i}{2}(\alpha\bar{\gamma} + \bar{\beta}\delta) + cc, \\ L^3_0 &= \frac{1}{2}(\alpha\bar{\alpha} + \beta\bar{\beta} - \gamma\bar{\gamma} - \delta\bar{\delta}), & L^3_1 &= \frac{1}{2}(\bar{\alpha}\beta - \bar{\gamma}\delta) + cc, & L^3_2 &= \frac{i}{2}(\bar{\alpha}\beta - \bar{\gamma}\delta) + cc, \\ L^3_3 &= \frac{1}{2}(\alpha\bar{\alpha} - \beta\bar{\beta} - \gamma\bar{\gamma} + \delta\bar{\delta}), & \alpha\delta - \beta\gamma &= 1, \end{aligned} \quad (3)$$

where  $cc$  means the complex conjugate of all the previous terms.

In Sec. 2, into  $L$  we eliminate  $\alpha, \beta, \gamma, \delta$  in favour of another quantities better adapted to the null tetrads of Newman-Penrose (NP) [9-11], thus from (3) we shall obtain the expressions deduced in [12-14].

### 2. Rotations in special relativity

If in (3) we realize the changes  $[\tau = \frac{1}{\sqrt{1-\tau^2}}]$ :

$$\alpha = -\tau \exp\left(-\frac{A+iB}{2}\right), \quad \beta = \tau \exp\left(-\frac{A+iB}{2}\right) \bar{\Gamma}, \quad \gamma = \tau \exp\left(\frac{A+iB}{2}\right) \bar{\Omega}, \quad \delta = -\tau \exp\left(\frac{A+iB}{2}\right), \quad (4)$$

where  $A, B$  are real and  $\Gamma, \Omega$  are complex with  $\Gamma\Omega \neq 1$  (it is simple to verify that (4) respects the requirement  $\alpha\delta - \beta\gamma = 1$ ), we deduce the relations of [12-14]  $[Q = \frac{1}{2(1-\Gamma\Omega)}$ :

$$L^0_0 = Q [e^A(1 + \Omega\bar{\Omega}) + e^{-A}(1 + \Gamma\bar{\Gamma})], \quad L^1_0 = -Q [e^{iB}(\Gamma + \bar{\Omega}) + cc], \quad L^2_0 = -iQ [e^{-iB}(\Omega + \bar{\Gamma}) - cc],$$

$$L^0_1 = -Q [e^A(\Omega + \bar{\Omega}) + e^{-A}(\Gamma + \bar{\Gamma})], \quad L^1_1 = Q [e^{iB}(1 + \bar{\Omega}\Gamma) + cc], \quad L^2_1 = iQ [e^{-iB}(1 + \Omega\bar{\Gamma}) - cc],$$

$$L^0_2 = iQ [e^A(\bar{\Omega} - \Omega) + e^{-A}(\Gamma - \bar{\Gamma})], \quad L^1_2 = iQ [e^{iB}(1 - \bar{\Omega}\Gamma) - cc], \quad L^2_2 = -Q [e^{-iB}(\Omega\bar{\Gamma} - 1) + cc], \quad (5)$$

$$L^0_3 = Q [e^A(\Omega\bar{\Omega} - 1) + e^{-A}(1 - \Gamma\bar{\Gamma})], \quad L^1_3 = Q [e^{iB}(\Gamma - \bar{\Omega}) + cc], \quad L^2_3 = iQ [e^{-iB}(\bar{\Gamma} - \Omega) - cc],$$

$$L^3_0 = Q [-e^A(1 + \Omega\bar{\Omega}) + e^{-A}(1 + \Gamma\bar{\Gamma})], \quad L^3_1 = Q [e^A(\Omega + \bar{\Omega}) - e^{-A}(\Gamma + \bar{\Gamma})],$$

$$L^3_2 = iQ [e^A(\Omega - \bar{\Omega}) + e^{-A}(\Gamma - \bar{\Gamma})], \quad L^3_3 = Q [e^A(1 - \Omega\bar{\Omega}) + e^{-A}(1 - \Gamma\bar{\Gamma})].$$

In [15] we showed that an orthonormal real tetrad experiments a rotation in Minkowski space under a Lorentz matrix:

$$\tilde{e}^{(a)}_{\mu} = L^a_b e^{(b)}_{\mu}, \quad (6)$$

such that  $e^{(0)}_{\mu} = e_{(0)\mu}$  and  $e^{(j)}_{\nu} = -e_{(j)\nu}$ ,  $j = 1, 2, 3$ , this due to the metric tensor  $\text{Diag}(1, -1, -1, -1)$ . From (5), (6) and the definitions [9-11]:

$$l^{\nu} = \frac{1}{\sqrt{2}}(e_{(0)}^{\nu} + e_{(3)}^{\nu}), \quad n^{\nu} = \frac{1}{\sqrt{2}}(e_{(0)}^{\nu} - e_{(3)}^{\nu}), \quad m^{\nu} = \frac{1}{\sqrt{2}}(e_{(1)}^{\nu} - ie_{(2)}^{\nu}), \quad \bar{m}^{\nu} = \frac{1}{\sqrt{2}}(e_{(1)}^{\nu} + ie_{(2)}^{\nu}), \quad (7)$$

it is possible to calculate the rotation of this null tetrad of NP  $[C = \frac{1}{|1-\Gamma\Omega|}]$ :

$$\tilde{l}^{\mu} = C e^A(l^{\mu} + \Omega\bar{\Omega} n^{\mu} + \bar{\Omega} m^{\mu} + \Omega \bar{m}^{\mu}), \quad \tilde{n}^{\mu} = C e^{-A}(n^{\mu} + \Gamma\bar{\Gamma} l^{\mu} + \Gamma m^{\mu} + \bar{\Gamma} \bar{m}^{\mu}), \quad (8)$$

$$\tilde{m}^{\mu} = C e^{-iB}(\bar{\Gamma} l^{\mu} + \Omega n^{\mu} + m^{\mu} + \bar{\Gamma}\Omega \bar{m}^{\mu}), \quad \tilde{\bar{m}}^{\mu} = C e^{iB}(\Gamma l^{\mu} + \bar{\Omega} n^{\mu} + \bar{m}^{\mu} + \Gamma\bar{\Omega} m^{\mu}),$$

which is very useful for several applications in relativity [9-11, 16, 17]; in (8) it is easy to check that

$$\tilde{l}^{\mu}\tilde{n}_{\mu} = -\tilde{m}^{\mu}\tilde{\bar{m}}_{\mu} = 1.$$

Thus, in the literature it is natural to employ three types of rotations:

Class I:  $\Omega = 0$ .  $l^{\mu}$  preserves its direction.

$$\tilde{l}^{\mu} = e^A l^{\mu}, \quad \tilde{n}^{\mu} = e^{-A}(n^{\mu} + \Gamma\bar{\Gamma} l^{\mu} + \Gamma m^{\mu} + \bar{\Gamma} \bar{m}^{\mu}), \quad \tilde{m}^{\mu} = e^{-iB}(\bar{\Gamma} l^{\mu} + m^{\mu}), \quad \tilde{\bar{m}}^{\mu} = e^{iB}(\Gamma l^{\mu} + \bar{m}^{\mu}), \quad (9)$$

Class II:  $\Gamma = 0$ .  $n^{\nu}$  maintains its direction.

$$\tilde{l}^\nu = e^A(l^\nu + \Omega \bar{\Omega} n^\nu + \bar{\Omega} m^\nu + \Omega \bar{m}^\nu), \quad \tilde{n}^\nu = e^{-A} n^\nu, \quad \tilde{m}^\nu = e^{-iB}(\Omega n^\nu + m^\nu), \quad \tilde{\bar{m}}^\nu = e^{iB}(\bar{\Omega} n^\nu + \bar{m}^\nu), \quad (10)$$

Class III:  $\Omega = \Gamma = 0$ .

$$\tilde{l}^\mu = e^A l^\mu, \quad \tilde{n}^\mu = e^{-A} n^\mu, \quad \tilde{m}^\mu = e^{-iB} m^\mu, \quad \tilde{\bar{m}}^\mu = e^{iB} \bar{m}^\mu, \quad (11)$$

which permit to determine the evolution of, for example, the spin coefficients, the NP components of Weyl, Ricci and Lanczos tensors, etc., under a rotation of the null tetrad. This is important because it is usual to make rotations to align  $l^\mu$  or/and  $n^\mu$  with the principal directions of Cartan [18]-Debever [19]-Penrose [20] for the conformal tensor or the Faraday's electromagnetic tensor [21], which gives great simplification in many relativistic calculations.

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