Rotations in Minkowski spacetime

I. Guerrero-Moreno, G. Leija-Hernández, J. López-Bonilla

ESIME-Zacatenco, IPN, Edif. 5, 1er. Piso, Col. Lindavista CP 07738, México DF;

jlopezb@ipn.mx

Keywords: Lorentz matrix, Null tetrad of Newman-Penrose

Abstract. With the relation of Olinde Rodrigues-Cartan is obtained an expression for the Lorentz matrix, and it is transformed to a better form for the Newman-Penrose formalism, thus it is possible to realize rotations of the null tetrad of NP.

1. Introduction

Here we employ the notation and conventions of [1]. The Olinde Rodrigues [2]-Cartan [3] expression:

$$\begin{pmatrix} \tilde{x}^0 + \tilde{x}^3 & \tilde{x}^1 - i \tilde{x}^2 \\ \tilde{x}^1 + i \tilde{x}^2 & \tilde{x}^0 - \tilde{x}^3 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x^0 + x^3 & x^1 - i x^2 \\ x^1 + i x^2 & x^0 - x^3 \end{pmatrix} \begin{pmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{pmatrix}, \quad (1)$$

where $\alpha, \beta, \gamma, \delta$ are arbitrary complex numbers verifying the condition $\alpha\delta - \beta\gamma = 1$, implies six degrees of freedom for the Lorentz matrix $L = (L^\nu_\mu)$ between the frames of reference $(x^\mu) = (ct, x, y, z)$ and $(\tilde{x}^\nu)$:

$$\tilde{x}^\nu = L^\nu_\mu x^\mu. \quad (2)$$

From (1) and (2) we obtain the relations [4-8]:

$$\begin{aligned} L^0_0 &= \frac{1}{2}(\alpha\bar{\alpha} + \beta\bar{\beta} + \gamma\bar{\gamma} + \delta\bar{\delta}), & L^1_0 &= \frac{1}{2}(\bar{\alpha}\gamma + \bar{\beta}\delta) + cc, & L^2_0 &= \frac{i}{2}(\alpha\bar{\gamma} - \bar{\beta}\delta) + cc, \\ L^0_1 &= \frac{1}{2}(\bar{\alpha}\beta + \bar{\gamma}\delta) + cc, & L^1_1 &= \frac{1}{2}(\bar{\alpha}\delta + \beta\bar{\gamma}) + cc, & L^2_1 &= \frac{i}{2}(\alpha\bar{\delta} + \beta\bar{\gamma}) + cc, \\ L^0_2 &= \frac{i}{2}(\bar{\alpha}\beta + \bar{\gamma}\delta) + cc, & L^1_2 &= \frac{i}{2}(\bar{\alpha}\delta + \beta\bar{\gamma}) + cc, & L^2_2 &= \frac{1}{2}(\bar{\alpha}\delta - \bar{\beta}\gamma) + cc, \\ L^0_3 &= \frac{1}{2}(\alpha\bar{\alpha} - \beta\bar{\beta} + \gamma\bar{\gamma} - \delta\bar{\delta}), & L^1_3 &= \frac{1}{2}(\bar{\alpha}\gamma - \bar{\beta}\delta) + cc, & L^2_3 &= \frac{i}{2}(\alpha\bar{\gamma} + \bar{\beta}\delta) + cc, \\ L^3_0 &= \frac{1}{2}(\alpha\bar{\alpha} + \beta\bar{\beta} - \gamma\bar{\gamma} - \delta\bar{\delta}), & L^3_1 &= \frac{1}{2}(\bar{\alpha}\beta - \bar{\gamma}\delta) + cc, & L^3_2 &= \frac{i}{2}(\bar{\alpha}\beta - \bar{\gamma}\delta) + cc, \\ L^3_3 &= \frac{1}{2}(\alpha\bar{\alpha} - \beta\bar{\beta} - \gamma\bar{\gamma} + \delta\bar{\delta}), & \alpha\delta - \beta\gamma &= 1, \end{aligned} \quad (3)$$

where cc means the complex conjugate of all the previous terms.

In Sec. 2, into L we eliminate $\alpha, \beta, \gamma, \delta$ in favour of another quantities better adapted to the null tetrads of Newman-Penrose (NP) [9-11], thus from (3) we shall obtain the expressions deduced in [12-14].

2. Rotations in special relativity

If in (3) we realize the changes $[\tau = \frac{1}{\sqrt{1-\tau^2}}]$:

$$\alpha = -\tau \exp\left(-\frac{A+iB}{2}\right), \quad \beta = \tau \exp\left(-\frac{A+iB}{2}\right) \bar{\Gamma}, \quad \gamma = \tau \exp\left(\frac{A+iB}{2}\right) \bar{\Omega}, \quad \delta = -\tau \exp\left(\frac{A+iB}{2}\right), \quad (4)$$

where A, B are real and Γ, Ω are complex with $\Gamma\Omega \neq 1$ (it is simple to verify that (4) respects the requirement $\alpha\delta - \beta\gamma = 1$), we deduce the relations of [12-14] $[Q = \frac{1}{2(1-\Gamma\Omega)}]$:

$$\begin{aligned}
 L^0_0 &= Q [e^A(1 + \Omega\bar{\Omega}) + e^{-A}(1 + \Gamma\bar{\Gamma})], \quad L^1_0 = -Q [e^{iB}(\Gamma + \bar{\Omega}) + cc], \quad L^2_0 = -iQ [e^{-iB}(\Omega + \bar{\Gamma}) - cc], \\
 L^0_1 &= -Q [e^A(\Omega + \bar{\Omega}) + e^{-A}(\Gamma + \bar{\Gamma})], \quad L^1_1 = Q [e^{iB}(1 + \bar{\Omega}\Gamma) + cc], \quad L^2_1 = iQ [e^{-iB}(1 + \Omega\bar{\Gamma}) - cc], \\
 L^0_2 &= iQ [e^A(\bar{\Omega} - \Omega) + e^{-A}(\Gamma - \bar{\Gamma})], \quad L^1_2 = iQ [e^{iB}(1 - \bar{\Omega}\Gamma) - cc], \quad L^2_2 = -Q [e^{-iB}(\Omega\bar{\Gamma} - 1) + cc], \\
 L^0_3 &= Q [e^A(\Omega\bar{\Omega} - 1) + e^{-A}(1 - \Gamma\bar{\Gamma})], \quad L^1_3 = Q [e^{iB}(\Gamma - \bar{\Omega}) + cc], \quad L^2_3 = iQ [e^{-iB}(\bar{\Gamma} - \Omega) - cc], \\
 L^3_0 &= Q [-e^A(1 + \Omega\bar{\Omega}) + e^{-A}(1 + \Gamma\bar{\Gamma})], \quad L^3_1 = Q [e^A(\Omega + \bar{\Omega}) - e^{-A}(\Gamma + \bar{\Gamma})], \\
 L^3_2 &= iQ [e^A(\Omega - \bar{\Omega}) + e^{-A}(\Gamma - \bar{\Gamma})], \quad L^3_3 = Q [e^A(1 - \Omega\bar{\Omega}) + e^{-A}(1 - \Gamma\bar{\Gamma})].
 \end{aligned} \tag{5}$$

In [15] we showed that an orthonormal real tetrad experiments a rotation in Minkowski space under a Lorentz matrix:

$$\tilde{e}^{(a)}_{\mu} = L^a_b e^{(b)}_{\mu}, \tag{6}$$

such that $e^{(0)}_{\mu} = e_{(0)\mu}$ and $e^{(j)}_{\nu} = -e_{(j)\nu}$, $j = 1, 2, 3$, this due to the metric tensor $\text{Diag}(1, -1, -1, -1)$. From (5), (6) and the definitions [9-11]:

$$l^{\nu} = \frac{1}{\sqrt{2}}(e_{(0)}^{\nu} + e_{(3)}^{\nu}), \quad n^{\nu} = \frac{1}{\sqrt{2}}(e_{(0)}^{\nu} - e_{(3)}^{\nu}), \quad m^{\nu} = \frac{1}{\sqrt{2}}(e_{(1)}^{\nu} - ie_{(2)}^{\nu}), \quad \bar{m}^{\nu} = \frac{1}{\sqrt{2}}(e_{(1)}^{\nu} + ie_{(2)}^{\nu}), \tag{7}$$

it is possible to calculate the rotation of this null tetrad of NP $[C = \frac{1}{1-\Gamma\Omega}]$:

$$\begin{aligned}
 \tilde{l}^{\mu} &= C e^A (l^{\mu} + \Omega\bar{\Omega} n^{\mu} + \bar{\Omega} m^{\mu} + \Omega \bar{m}^{\mu}), \quad \tilde{n}^{\mu} = C e^{-A} (n^{\mu} + \Gamma\bar{\Gamma} l^{\mu} + \Gamma m^{\mu} + \bar{\Gamma} \bar{m}^{\mu}), \\
 \tilde{m}^{\mu} &= C e^{-iB} (\bar{\Gamma} l^{\mu} + \Omega n^{\mu} + m^{\mu} + \bar{\Gamma}\Omega \bar{m}^{\mu}), \quad \tilde{\bar{m}}^{\mu} = C e^{iB} (\Gamma l^{\mu} + \bar{\Omega} n^{\mu} + \bar{m}^{\mu} + \Gamma\bar{\Omega} m^{\mu}),
 \end{aligned} \tag{8}$$

which is very useful for several applications in relativity [9-11, 16, 17]; in (8) it is easy to check that $\tilde{l}^{\mu}\tilde{n}_{\mu} = -\tilde{m}^{\mu}\tilde{\bar{m}}_{\mu} = 1$.

Thus, in the literature it is natural to employ three types of rotations:

Class I: $\Omega = 0$. l^{μ} preserves its direction.

$$\tilde{l}^{\mu} = e^A l^{\mu}, \quad \tilde{n}^{\mu} = e^{-A} (n^{\mu} + \Gamma\bar{\Gamma} l^{\mu} + \Gamma m^{\mu} + \bar{\Gamma} \bar{m}^{\mu}), \quad \tilde{m}^{\mu} = e^{-iB} (\bar{\Gamma} l^{\mu} + m^{\mu}), \quad \tilde{\bar{m}}^{\mu} = e^{iB} (\Gamma l^{\mu} + \bar{m}^{\mu}), \tag{9}$$

Class II: $\Gamma = 0$. n^{ν} maintains its direction.

$$\tilde{l}^\nu = e^A(l^\nu + \Omega \bar{\Omega} n^\nu + \bar{\Omega} m^\nu + \Omega \bar{m}^\nu), \quad \tilde{n}^\nu = e^{-A} n^\nu, \quad \tilde{m}^\nu = e^{-iB}(\Omega n^\nu + m^\nu), \quad \tilde{\bar{m}}^\nu = e^{iB}(\bar{\Omega} n^\nu + \bar{m}^\nu), \quad (10)$$

Class III: $\Omega = \Gamma = 0$.

$$\tilde{l}^\mu = e^A l^\mu, \quad \tilde{n}^\mu = e^{-A} n^\mu, \quad \tilde{m}^\mu = e^{-iB} m^\mu, \quad \tilde{\bar{m}}^\mu = e^{iB} \bar{m}^\mu, \quad (11)$$

which permit to determine the evolution of, for example, the spin coefficients, the NP components of Weyl, Ricci and Lanczos tensors, etc., under a rotation of the null tetrad. This is important because it is usual to make rotations to align l^μ or/and n^μ with the principal directions of Cartan [18]-Debever [19]-Penrose [20] for the conformal tensor or the Faraday's electromagnetic tensor [21], which gives great simplification in many relativistic calculations.

References

- [1]. A. Hernández G., R. López-Vázquez, J. Rivera R., J. López-Bonilla, Faraday tensor and Maxwell spinor, Journal de Ciencia e Ingeniería 6, No. 1 (2014)
- [2]. B. Olinde Rodrigues, Des lois géométriques qui régissent les déplacements d'un système solide, Journal de Math. (Liouville) 5 (1840) 380-440
- [3]. E. Cartan, Les groupes projectifs qui ne laissent invariante aucune multiplicité plane, Bull. Soc. Math. de France 41 (1913) 53-96
- [4]. Ju. Rumer, Spinorial analysis, Moscow (1936)
- [5]. J. Aharoni, The special theory of relativity, Clarendon Press, Oxford (1959)
- [6]. J. L. Synge, Relativity: the special theory, North-Holland, Amsterdam (1965)
- [7]. J. López-Bonilla, J. Morales, G. Ovando, On the homogeneous Lorentz transformation, Bull. Allahabad Math. Soc. 17 (2002) 53-58
- [8]. M. Acevedo M., J. López-Bonilla, M. Sánchez M., Quaternions, Maxwell equations and Lorentz transformations, Apeiron 12, No. 4 (2005) 371-384
- [9]. E. Newman, R. Penrose, An approach to gravitational radiation by a method of spin coefficients, J. Math. Phys. 3, No. 3 (1962) 566-578
- [10]. S. J. Campbell, J. Wainwright, Algebraic computing and the Newman-Penrose formalism in general relativity, Gen. Rel. Grav. 8, No. 12 (1977) 987-1001
- [11]. D. Kramer, H. Stephani, M. MacCallum, E. Herlt, Exact solutions of Einstein's field equations, Cambridge University Press (1980)
- [12]. R. K. Sachs, Gravitational waves in general relativity VI: The outgoing radiation condition, Proc. Roy. Soc. London A264 (1961) 309-337
- [13]. P. J. Greenberg, J. P. Knauer, Stud. Appl. Math. 53 (1974) 165
- [14]. M. Carmeli, Group theory and general relativity, McGraw-Hill, New York (1977)
- [15]. R. Linares, J. López-Bonilla, J. Quino, Real tetrads and Lorentz transformations, Bol. Soc. Cub. Mat. Comp. 8, No. 2 (2010) 141-153
- [16]. P. J. Greenberg, The algebra of the Riemann curvature tensor in general relativity: Preliminaries, Stud. Appl. Math. 51 (1972) 277
- [17]. S. Chandrasekhar, An introduction to the theory of the Kerr metric and its perturbations, in General Relativity, Eds. S. W. Hawking, W. Israel, Cambridge University Press (1979) Chap. 7
- [18]. E. Cartan, Sur les espaces conformes généralisés et l'Univers optique, C. R. Acad. Sci. Paris 174 (1922) 857-859
- [19]. R. Debever, Le rayonnement gravitationnel: Le tenseur de Riemann en relativité générale, Cah. de Phys 168-169 (1964) 303
- [20]. R. Penrose, A spinor approach to general relativity, Ann. of Phys. 10 (1960) 171-201
- [21]. J. López-Bonilla, R. Meneses, M. Turgut, Faraday tensor: its algebraic structure, J. Vect. Rel. 4, No. 3(2009) 23-32