Rotations in Minkowski spacetime

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Keywords: Lorentz matrix, Null tetrad of Newman-Penrose

Abstract. With the relation of Olinde Rodrigues-Cartan is obtained an expression for the Lorentz matrix, and it is transformed to a better form for the Newman-Penrose formalism, thus it is possible to realize rotations of the null tetrad of NP.

1. Introduction

Here we employ the notation and conventions of [1]. The Olinde Rodrigues [2]-Cartan [3] expression:

\[
\begin{pmatrix}
\tilde{x}^0 + i \tilde{x}^3 \\
\tilde{x}^1 + i \tilde{x}^2 \\
\tilde{x}^0 - i \tilde{x}^3 \\
\end{pmatrix} =
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta \\
\end{pmatrix}
\begin{pmatrix}
x^0 + x^3 \\
x^1 + i x^2 \\
x^0 - i x^3 \\
\end{pmatrix}
\begin{pmatrix}
\tilde{\alpha} & \tilde{\beta} \\
\tilde{\gamma} & \tilde{\delta} \\
\end{pmatrix},
\]

where \(\alpha, \beta, \gamma, \delta\) are arbitrary complex numbers verifying the condition \(\alpha \delta - \beta \gamma = 1\), implies six degrees of freedom for the Lorentz matrix \(L = (L^\nu_\mu)\) between the frames of reference \((x^\mu) = (ct,x,y,z)\) and \((\tilde{x}^\nu)\):

\[
\tilde{x}^\nu = L^\nu_\mu x^\mu.
\]

From (1) and (2) we obtain the relations [4-8]:

\[
\begin{align*}
L^0_0 &= \frac{1}{2}(\alpha \tilde{\alpha} + \beta \tilde{\beta} + \gamma \tilde{\gamma} + \delta \tilde{\delta}), \\
L^1_0 &= \frac{1}{2}((\alpha \gamma + \beta \delta) + cc), \\
L^2_0 &= \frac{1}{2}((\alpha \delta - \beta \gamma) + cc), \\
L^0_1 &= \frac{1}{2}(\alpha \beta + \gamma \delta) + cc, \\
L^1_1 &= \frac{1}{2}((\alpha \delta - \beta \gamma) + cc), \\
L^2_1 &= \frac{1}{2}(\alpha \beta + \gamma \delta) + cc, \\
L^0_2 &= \frac{1}{2}(\alpha \beta - \gamma \delta) + cc, \\
L^1_2 &= \frac{1}{2}((\alpha \delta - \beta \gamma) + cc), \\
L^2_2 &= \frac{1}{2}(\alpha \beta + \gamma \delta) + cc, \\
L^0_3 &= \frac{1}{2}(\alpha \beta - \gamma \delta) + cc, \\
L^1_3 &= \frac{1}{2}((\alpha \delta - \beta \gamma) + cc), \\
L^2_3 &= \frac{1}{2}(\alpha \beta + \gamma \delta) + cc,
\end{align*}
\]

where \(cc\) means the complex conjugate of all the previous terms.

In Sec. 2, into \(L\) we eliminate \(\alpha, \beta, \gamma, \delta\) in favour of another quantities better adapted to the null tetrads of Newman-Penrose (NP) [9-11], thus from (3) we shall obtain the expressions deduced in [12-14].

2. Rotations in special relativity

In (3) we realize the changes \(\tau = \frac{1}{\sqrt{1 - \gamma}}\):

\[
\alpha = -\tau \exp\left(-\frac{A + iB}{2}\right), \quad \beta = \tau \exp\left(-\frac{A + iB}{2}\right) \Gamma, \quad \gamma = \tau \exp\left(\frac{A + iB}{2}\right) \Omega, \quad \delta = \tau \exp\left(-\frac{A + iB}{2}\right),
\]

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where $A$, $B$ are real and $\Gamma$, $\Omega$ are complex with $\Gamma\Omega \neq 1$ (it is simple to verify that (4) respects the requirement $\alpha\delta - \beta\gamma = 1$), we deduce the relations of $[12-14]$:

\[
L^0_0 = Q [e^A(1 + \Omega\Omega) + e^{-A}(1 + \Gamma\Gamma)], \quad L^1_0 = -Q [e^{iB}(\Gamma + \Omega) + cc], \quad L^2_0 = -iQ [e^{-iB}(\Omega + \Gamma) - cc],
\]

\[
L^0_1 = -Q [e^A(\Omega + \bar{\Omega}) + e^{-A}(\Gamma + \bar{\Gamma})], \quad L^1_1 = Q [e^{iB}(1 + \bar{\Omega}\Omega) + cc], \quad L^2_1 = iQ [e^{-iB}(1 + \Omega\bar{\Gamma}) - cc],
\]

\[
L^0_2 = iQ [e^A(\bar{\Omega} - \Omega) + e^{-A}(\Gamma - \bar{\Gamma})], \quad L^1_2 = iQ [e^{iB}(1 - \bar{\Omega}\Omega) - cc], \quad L^2_2 = -Q [e^{-iB}(\Omega\bar{\Gamma} - 1) + cc],
\]

\[
L^0_3 = Q [e^A(\Omega\bar{\Omega} - 1) + e^{-A}(1 - \Gamma\Gamma)], \quad L^1_3 = Q [e^{iB}(\Gamma - \bar{\Omega}) + cc], \quad L^2_3 = iQ [e^{-iB}(\Gamma - \Omega) - cc],
\]

\[
L^3_0 = Q [-e^A(1 + \Omega\bar{\Omega}) + e^{-A}(1 + \Gamma\bar{\Gamma})], \quad L^3_1 = Q [e^A(\Omega + \bar{\Omega}) - e^{-A}(\Gamma + \bar{\Gamma})],
\]

\[
L^3_2 = iQ [e^A(\bar{\Omega} - \Omega) + e^{-A}(\Gamma - \bar{\Gamma})], \quad L^3_3 = Q [e^A(1 - \Omega\bar{\Omega}) + e^{-A}(1 - \Gamma\bar{\Gamma})].
\]

In [15] we showed that an orthonormal real tetrad experiments a rotation in Minkowski space under a Lorentz matrix:

\[
\hat{e}^{(a)}_{\mu} = L^a_b e^{(b)}_{\mu},
\]

such that $e^{(a)}_{\mu} = e^{(a)}_{\mu}$ and $e^{(j)}_{\nu} = -e^{(j)}_{\nu}$, $j = 1, 2, 3$, this due to the metric tensor Diag $(1, -1, -1, -1)$. From (5), (6) and the definitions $[9-11]$:

\[
l^\nu = \frac{1}{\sqrt{2}} (e^{(1)}_\nu + e^{(3)}_\nu), \quad n^\nu = \frac{1}{\sqrt{2}} (e^{(0)}_\nu - e^{(1)}_\nu), \quad m^\nu = \frac{1}{\sqrt{2}} (e^{(1)}_\nu - i e^{(2)}_\nu), \quad \bar{m}^\nu = \frac{1}{\sqrt{2}} (e^{(1)}_\nu + i e^{(2)}_\nu),
\]

it is possible to calculate the rotation of this null tetrad of NP $[C = \frac{1}{1 - i\Gamma\bar{\Omega}}]$:

\[
\hat{\mu} = C e^A (l^\mu + \Omega \bar{\Omega} n^\mu + \bar{\Omega} m^\mu + \Omega \bar{m}^\mu), \quad \hat{n}^\mu = C e^{-A} (n^\mu + \Gamma \bar{\Gamma} l^\mu + \Gamma m^\mu + \bar{\Gamma} \bar{m}^\mu),
\]

\[
\hat{m}^\mu = C e^{-iB} (\Gamma l^\mu + \Omega n^\mu + m^\mu + \bar{\Omega} \bar{m}^\mu), \quad \hat{\bar{m}}^\mu = C e^{iB} (\bar{\Gamma} l^\mu + \bar{\Omega} n^\mu + \bar{m}^\mu + \Gamma \bar{m}^\mu),
\]

which is very useful for several applications in relativity $[9-11, 16, 17]$; in (8) it is easy to check that

\[
\hat{l}^\mu \hat{n}_{\mu} = -\hat{m}^\mu \hat{\bar{m}}_{\mu} = 1.
\]

Thus, in the literature it is natural to employ three types of rotations:

- **Class I**: $\Omega = 0$. $l^\mu$ preserves its direction.

- **Class II**: $\Gamma = 0$. $n^\nu$ maintains its direction.
\[ \tilde{v} = e^A (l^\nu + \Omega n^\nu + \bar{n} m^\nu + \bar{m} \bar{n}^\nu), \quad \tilde{n} = e^{-A} n^\nu, \quad \tilde{m} = e^{-iB} (\Omega n^\nu + m^\nu), \quad \tilde{m} = e^{iB} (\bar{n} n^\nu + \bar{m} \bar{n}^\nu), \]

Class III: \( \Omega = \Gamma = 0 \).

\[ \tilde{n} = e^A l^\mu, \quad \tilde{n} = e^{-A} n^\mu, \quad \tilde{m} = e^{-iB} m^\mu, \quad \tilde{m} = e^{iB} \bar{m}^\mu, \]

which permit to determine the evolution of, for example, the spin coefficients, the NP components of Weyl, Ricci and Lanczos tensors, etc., under a rotation of the null tetrad. This is important because it is usual to make rotations to align \( l^\mu \) or/and \( n^\mu \) with the principal directions of Cartan [18]-Debever [19]-Penrose [20] for the conformal tensor or the Faraday’s electromagnetic tensor [21], which gives great simplification in many relativistic calculations.

References

[4]. Ju. Rumer, Spinorial analysis, Moscow (1936)