

## Singular Value Decomposition

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**Abstract.** We study the SVD of an arbitrary matrix  $A_{n \times m}$ , especially its subspaces of activation, which leads in natural manner to pseudoinverse of Moore-Bjehnammar-Penrose. Besides, we analyze the compatibility of linear systems and the uniqueness of the corresponding solution, and our approach gives the Lanczos classification for these systems.

### 1. Introduction

For any real matrix  $A_{n \times m}$ , Lanczos [1] constructs the matrix:

$$S_{(n+m) \times (n+m)} = \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}, \quad (1)$$

and he studies the eigenvalue problem:

$$S\vec{\omega} = \lambda\vec{\omega}, \quad (2)$$

where the proper values are real because  $S$  is a real symmetric matrix. Besides:

$$\text{rank } A \equiv p = \text{Number of positive eigenvalues of } S, \quad (3)$$

such that  $1 \leq p \leq \min(n, m)$ . Then the singular values or canonical multipliers, thus called by Picard [2] and Sylvester [3], respectively, follow the scheme:

$$\lambda_1, \lambda_2, \dots, \lambda_p, -\lambda_1, -\lambda_2, \dots, -\lambda_p, 0, 0, \dots, 0, \quad (4)$$

that is,  $\lambda = 0$  has the multiplicity  $n + m - 2p$ . Only in the case  $p = n = m$  can occur the absence of the null eigenvalue.

The proper vectors of  $S$ , named „essential axes“ by Lanczos, can be written in the form:

$$\vec{\omega}_{(n+m) \times 1} = \begin{pmatrix} \vec{u} \\ \vec{v} \end{pmatrix} \begin{matrix} n \\ m \end{matrix} \quad (5)$$

then (1) and (2) imply the Modified Eigenvalue Problem:

$$A_{n \times m} \vec{v}_{m \times 1} = \lambda \vec{u}_{n \times 1}, \quad A^T_{m \times n} \vec{u}_{n \times 1} = \lambda \vec{v}_{m \times 1}, \quad (6)$$

Thus

$$A^T A \vec{v} = \lambda^2 \vec{v}, \quad A A^T \vec{u} = \lambda^2 \vec{u}, \quad (7)$$

with special interest in the associated vectors with the positive eigenvalues because they permit to introduce the matrices:

$$U_{n \times p} = (\vec{u}_1, \vec{u}_2, \dots, \vec{u}_p), \quad V_{m \times p} = (\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p), \quad (8)$$

verifying  $U^T U = V^T V = I_{p \times p}$  because:

$$\vec{u}_j \cdot \vec{u}_k = \vec{v}_j \cdot \vec{v}_k = \delta_{jk}, \quad (9)$$

therefore  $\vec{\omega}_j \cdot \vec{\omega}_k = 2\delta_{jk}$ ,  $j, k = 1, 2, \dots, p$ . Thus, the SVD (Singular Value Decomposition) express [1, 4-6] that  $A$  is the product of three matrices:

$$A_{n \times m} = U_{n \times p} \Lambda_{p \times p} V^T_{p \times m}, \quad \Lambda = \text{Diag}(\lambda_1, \lambda_2, \dots, \lambda_p). \quad (10)$$

This relation tells that in the construction of  $A$  we do not need information about the null proper value; the information from  $\lambda = 0$  is important to study the existence and uniqueness of the solutions for a linear system associated to  $A$ . This approach of Lanczos is similar [7] to Schmidt [8] and Jordan [9, 10] methods; we can consider that Jordan, Sylvester [3] and Beltrami [11] are the founders of the SVD [12], and there is abundant literature [13-25] on this matrix factorization and its applications.

In Sec. 2, we realize an analysis of the proper vectors  $\vec{\omega}_j, j = 1, \dots, n + m$ , associated to the eigenvalues (4), which leads to the subspaces of activation of  $A$  with the pseudoinverse of Moore [26]-Bjerrhammar [27]-Penrose [28]. In Sec. 3, we study the compatibility of linear systems, with special emphasis in the important participation of the null singular value and its corresponding eigenvectors.

## 2. Subspaces of activation and natural inverse matrix

From (6), the proper vectors associated with the positive eigenvalues verify:

$$A\vec{v}_j = \lambda_j \vec{u}_j, \quad A^T \vec{u}_j = \lambda_j \vec{v}_j, \quad j = 1, \dots, p \quad (11)$$

then

$$A(-\vec{v}_j) = (-\lambda_j) \vec{u}_j, \quad A^T \vec{u}_j = (-\lambda_j)(-\vec{v}_j), \quad (12)$$

that is

$$S \begin{pmatrix} \vec{u}_k \\ \vec{v}_k \end{pmatrix} = \lambda_k \begin{pmatrix} \vec{u}_k \\ \vec{v}_k \end{pmatrix} \quad \text{implies} \quad S \begin{pmatrix} \vec{u}_k \\ -\vec{v}_k \end{pmatrix} = (-\lambda_k) \begin{pmatrix} \vec{u}_k \\ -\vec{v}_k \end{pmatrix}. \quad (13)$$

Therefore, the eigenvectors  $\begin{pmatrix} \vec{u}_j \\ \vec{v}_j \end{pmatrix}$  and  $\begin{pmatrix} \vec{u}_j \\ -\vec{v}_j \end{pmatrix}$  correspond to the proper values  $\lambda_1, \dots, \lambda_p$  and  $-\lambda_1, \dots, -\lambda_p$ , respectively. Thus we must have  $n + m - 2p$  eigenvectors connected to  $\lambda = 0$ , which we denote by  $\vec{\omega}_r^{(0)}$ , and from (6):

$$\vec{\omega}_j^{(0)} = \begin{pmatrix} \vec{u}_j^{(0)} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} n \\ \\ \\ m \end{matrix}, \quad A^T \vec{u}_j^{(0)} = \vec{0}, \quad j = 1, \dots, n - p, \quad (14)$$

$$\vec{\omega}_{(n-p)+k}^{(0)} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vec{v}_k^{(0)} \end{pmatrix} \begin{matrix} n \\ \\ \\ m \end{matrix}, \quad A \vec{v}_k^{(0)} = \vec{0}, \quad k = 1, \dots, m - p. \quad (15)$$

The conditions (14) and (15) can be multiplied by  $A$  and  $A^T$ , then  $\vec{u}_j^{(0)}$  and  $\vec{v}_k^{(0)}$  are eigenvectors of the Gram matrices  $AA^T$  and  $A^T A$ :

$$(AA^T)_{n \times n} \vec{u}_j^{(0)} = \vec{0}, \quad (A^T A)_{m \times m} \vec{v}_k^{(0)} = \vec{0}, \tag{16}$$

but by (7) these matrices have  $p$  proper vectors for  $\lambda_1, \dots, \lambda_p$ , therefore only there are  $n - p$  and  $m - p$  vectors  $\vec{u}_j^{(0)}$  and  $\vec{v}_k^{(0)}$  that can be selected with orthonormality:

$$\vec{u}_j^{(0)} \cdot \vec{u}_r^{(0)} = \delta_{jr}, \quad \vec{v}_k^{(0)} \cdot \vec{v}_q^{(0)} = \delta_{kq}, \tag{17}$$

that is,  $\vec{\omega}_j^{(0)} \cdot \vec{\omega}_k^{(0)} = \delta_{jk}$ , then  $\{\vec{u}_j^{(0)}\}$  and  $\{\vec{v}_k^{(0)}\}$  are bases for the Kernel  $A^T$  and Kernel  $A$ , respectively.

If into (14) we employ (10) [SVD of  $A$ ] results  $V\Lambda U^T \vec{u}_j^{(0)} = \vec{0}$ , whose multiplication by the left with  $\Lambda^{-1} V^T$  [remembering that  $U^T U = V^T V = I$ ], gives the compatibility condition:

$$U^T \vec{u}_j^{(0)} = \vec{0} \Rightarrow \vec{u}_r \cdot \vec{u}_j^{(0)} = 0, \quad r = 1, \dots, p; \quad j = 1, \dots, n - p, \tag{18}$$

Equivalently

$$\text{Col } U \perp \vec{u}_k^{(0)}, \quad k = 1, \dots, n - p. \tag{19}$$

Similarly, if we use SVD into (15) and we multiply by  $\Lambda^{-1} U^T$ :

$$V^T \vec{v}_k^{(0)} = \vec{0}, \quad \vec{v}_r \cdot \vec{v}_k^{(0)} = 0, \quad r = 1, \dots, p; \quad k = 1, \dots, m - p \tag{20}$$

$$\therefore \text{Col } V \perp \vec{v}_j^{(0)}, \quad j = 1, \dots, m - p. \tag{21}$$

It is convenient to make two remarks:

a).- From  $A = U\Lambda V^T$  is evident that the matrices  $U, \Lambda$  and  $V$  permit to construct  $A$ , but is useful to know more about the structure of  $A$  and its transpose:

$$A = (\vec{a}_1 \dots \vec{a}_m), \quad A^T = (\vec{c}_1 \dots \vec{c}_n), \tag{22}$$

where  $(\vec{a}_j)_{n \times 1}$  and  $(\vec{c}_k)_{m \times 1}$  are the corresponding columns. Then from (10) we obtain the expressions:

$$\vec{a}_j = \lambda_1 v_1^{(j)} \vec{u}_1 + \dots + \lambda_p v_p^{(j)} \vec{u}_p, \quad j = 1, \dots, m, \quad \vec{c}_k = \lambda_1 u_1^{(k)} \vec{v}_1 + \dots + \lambda_p u_p^{(k)} \vec{v}_p, \quad k = 1, \dots, n \tag{23}$$

with the notation:

$$v_r^{(j)} = j \text{ th - component of } \vec{v}_r, \tag{24}$$

and similar for  $u_r^{(k)}$ ; we observe that  $\vec{c}_k^T$  are the rows of  $A$ .

and similar for  $u_r^{(k)}$ ; we observe that  $\vec{c}_k^T$  are the rows of  $A$ .

From (23) are immediate the equalities of subspaces:

$$\text{Col } A = \text{Col } U, \quad \text{Row } A = \text{Col } V, \quad (25)$$

but  $\dim \text{Col } U = \dim \text{Col } V = p$ , then:

$$\text{rank } A = \dim \text{Col } A = \dim \text{Row } A = p \quad (26)$$

in according with (3).

).- We have the rank-nullity theorem [29-31]:

$$\dim (\text{Kernel } A) + \text{rank } A = m, \quad (27)$$

$$\dim (\text{Kernel } A^T) + \text{rank } A^T = n, \quad (28)$$

but  $\text{rank } A^T = \text{rank } A = p$ , then  $\dim (\text{Kernel } A^T) = n - p$  in harmony with the  $(n - p)$  vectors  $\vec{u}_j^{(0)}$  verifying (14).

If  $A_{n \times m}$  acts on an arbitrary vector  $\vec{x} \in E_m$  produces a vector  $\vec{y} \in E_n$ , with the decompositions:

$$\begin{aligned} \vec{x} &= \vec{x}^{(0)} + \vec{x}_{\text{CV}}, & \vec{y} &= \vec{y}^{(0)} + \vec{y}_{\text{CU}}, \\ \vec{x} &= \vec{x}^{(0)} + \vec{x}_{\text{CV}}, & \vec{y} &= \vec{y}^{(0)} + \vec{y}_{\text{CU}}, \end{aligned}$$

$$\vec{x}^{(0)} \in \text{Kernel } A, \quad \vec{x}_{\text{CV}} \in \text{Col } V, \quad A\vec{x}^{(0)} = \vec{0}, \quad \vec{x}^{(0)} \cdot \vec{x}_{\text{CV}} = 0, \quad (29)$$

where

$$\vec{y}^{(0)} \in \text{Kernel } A^T, \quad \vec{y}_{\text{CU}} \in \text{Col } U, \quad A^T\vec{y}^{(0)} = \vec{0}, \quad \vec{y}^{(0)} \cdot \vec{y}_{\text{CU}} = 0, \quad (30)$$

Therefore,  $A\vec{x} = A\vec{x}_{\text{CV}} = \vec{y}$  and in the construction of  $\vec{y}$  we lost the information about  $\vec{x}^{(0)}$ , then it is not possible to recover  $\vec{x}$  from  $\vec{y}$ , that is, it is utopian to search for an ‘inverse matrix’ acting on  $\vec{y}$  to give  $\vec{x}$ . However, when  $\vec{x}^{(0)} = \vec{0}$  and  $\vec{y}^{(0)} = \vec{0}$  we can introduce a ‘natural inverse matrix’, thus named it by Lanczos, which coincides with the pseudoinverse of Moore [16]-Bjerhammar [27]-Penrose [28]:

“Any matrix  $A_{n \times m}$ , restricted to its subspaces of activation, always can be inverted”.

(31)

In fact, if  $\vec{x} \in \text{Col } V$  is an arbitrary vector,  $\vec{x} = q_1\vec{v}_1 + \dots + q_p\vec{v}_p$ , then from (6):

$$A\vec{x} = \lambda_1 q_1 \vec{u}_1 + \dots + \lambda_p q_p \vec{u}_p = \vec{y} \in \text{Col } U, \quad (32)$$

and now we search the inverse natural  $A_N^{-1}$  such that:

$$A_N^{-1} \vec{y} = \vec{x}, \quad (33)$$

or more general:

$$A_N^{-1}A\vec{x} = \vec{x}, \quad \forall \vec{x} \in \text{Col } V, \quad AA_N^{-1}\vec{y} = \vec{y}, \quad \forall \vec{y} \in \text{Col } U. \tag{34}$$

If the decomposition (10) is applied to (32) we deduce the natural inverse matrix:

$$A_N^{-1}{}_{m \times n} = V_{m \times p} \Lambda_{p \times p}^{-1} U^T_{p \times n}, \tag{35}$$

satisfying (33) and (34). With (35) is easy to prove the properties [24, 32]:

$$AA_N^{-1}A = A, \quad A_N^{-1}AA_N^{-1} = A_N^{-1}, \quad (AA_N^{-1})^T = AA_N^{-1}, \quad (A_N^{-1}A)^T = A_N^{-1}A, \tag{36}$$

which characterize the pseudoinverse of Moore-Bjerhammar-Penrose, that is, the inverse matrix [32, 33] of these authors coincides with the natural inverse (35) deduced by Lanczos [1, 4-6]. which characterize the pseudoinverse of Moore-Bjerhammar-Penrose, that is, the inverse matrix [32, 33] of these authors coincides with the natural inverse (35) deduced by Lanczos [1, 4-6].

In the SVD only participate the positive proper values of  $S$ , without the explicit presence of the vectors  $\vec{u}_j^{(0)}$  and  $\vec{v}_k^{(0)}$  associated with the null eigenvalue, then it is natural to investigate the role performed by the information related with  $\lambda = 0$ . In Sec. 3, we study linear systems where  $A$  is the corresponding matrix of coefficients, and we exhibit that the  $\vec{u}_j^{(0)}$  permit to analyze the compatibility of such systems; besides, when they are compatibles then with the  $\vec{v}_k^{(0)}$  we search if the solution is unique. In other words, the null eigenvalue does not participates when we consider to  $A$  as an algebraic operator and we construct its factorization (10), but  $\lambda = 0$  is important if  $A$  acts as the matrix of coefficients of a linear system.

### 3. Compatibility of linear systems

A linear system of  $n$  equations with  $m$  unknowns can be written in the matrix form:

$$A_{n \times m} \vec{x}_{m \times 1} = \vec{b}_{n \times 1} \tag{37}$$

where (10) implies that  $U\Lambda V^T \vec{x} = \vec{b}$  whose multiplication by  $\vec{u}_j^{(0)T}$  gives the compatibility conditions:

$$\vec{u}_j^{(0)} \cdot \vec{b} = 0, \quad j = 1, \dots, n - p \tag{38}$$

due to (19). Then the system (37) is compatible if  $\vec{b}$  is orthogonal to all independent solutions of the adjoint system  $A^T \vec{u} = \vec{0}$ , therefore:

$$"A\vec{x} = \vec{b} \quad \text{has solution if} \quad \vec{b} \in \text{Col } U", \tag{39}$$

which is the traditional formulation [6] of the compatibility condition for a linear system given. From (25) and (39) is clear that  $A$  and the augmented matrix  $(A \vec{b})$  have the same column space:

$$\text{Col } A = \text{Col } (A \vec{b}) = \text{Col } U, \tag{40}$$

thus at the books [32] we find the result:

$$"A\vec{x} = \vec{b} \quad \text{is compatible if} \quad \text{rank } A = \text{rank } (A \vec{b})". \tag{41}$$

If  $\vec{b} \in \text{Col } U$ , then from (11):

$$\vec{b} = b^{(1)}\vec{u}_1 + \dots + b^{(p)}\vec{u}_p = A\vec{Q}, \quad \vec{Q} = \frac{b^{(1)}}{\lambda_1}\vec{v}_1 + \dots + \frac{b^{(p)}}{\lambda_p}\vec{v}_p, \quad (42)$$

and (37) leads to:

$$A(\vec{x} - \vec{Q}) = \vec{0}. \quad (43)$$

The set of solutions of (43) is the Kernel  $A$  with dimension  $(m - p)$  due to (27), therefore (43) has the unique solution  $\vec{x} - \vec{Q} = \vec{0}$  when  $p = m$ , that is, when rank  $A$  coincides with the number of unknowns we have not vectors  $\vec{v}_k^{(0)} \neq \vec{0}$  verifying  $A\vec{v}_k^{(0)} = \vec{0}$ . Then:

$$\text{"The compatible system } A\vec{x} = \vec{b} \text{ has unique solution only when } p = m\text{"}, \quad (44)$$

$$x_r = Q^{(r)} = \frac{b^{(1)}}{\lambda_1}v_1^{(r)} + \dots + \frac{b^{(p)}}{\lambda_p}v_p^{(r)} = \vec{b} \cdot \vec{t}_r, \quad r = 1, \dots, m \quad (45)$$

where

$$\vec{t}_r = \frac{v_1^{(r)}}{\lambda_1}\vec{u}_1 + \dots + \frac{v_p^{(r)}}{\lambda_p}\vec{u}_p \in \text{Col } U, \quad (46)$$

thus the value of each unknown is the projection of  $\vec{b}$  onto each vector (46). In consequence,  $\vec{b} \in \text{Col } U$  guarantees the solution of (37), and it is unique only if  $p = m$ .

Besides, from (42) we see that the solution  $\vec{x} = \vec{Q}$  implies that  $\vec{x} \in \text{Col } V$ , then we have the system  $A\vec{x} = \vec{b}$  where  $\vec{x}$  and  $\vec{b}$  are totally embedded into  $\text{Col } V$  and  $\text{Col } U$ , respectively, that is,  $\vec{x}$  and  $\vec{b}$  are into the subspaces of activation of  $A$ , thus from (32) and (33) there is the natural inverse  $A_N^{-1}$  such that:

$$\vec{x} = A_N^{-1}\vec{b} = V_{m \times m} \Lambda_{m \times m}^{-1} U_{m \times n}^T \vec{b} = V \Lambda^{-1} \begin{pmatrix} b^{(1)} \\ \vdots \\ b^{(m)} \end{pmatrix} = V \begin{pmatrix} \frac{b^{(1)}}{\lambda_1} \\ \vdots \\ \frac{b^{(m)}}{\lambda_m} \end{pmatrix} = \begin{pmatrix} \frac{b^{(1)}}{\lambda_1}v_1^{(1)} + \dots + \frac{b^{(m)}}{\lambda_m}v_m^{(1)} \\ \vdots \\ \frac{b^{(1)}}{\lambda_1}v_1^{(m)} + \dots + \frac{b^{(m)}}{\lambda_m}v_m^{(m)} \end{pmatrix}, \quad p = m, \quad (47)$$

in according with (45). The vectors (46) are important because their inner products with  $\vec{b}$  give the solution of (37) via (45), and they also are remarkable because permit to construct the natural inverse:

$$A_N^{-1}{}_{m \times n} = (\vec{t}_1 \vec{t}_2 \dots \vec{t}_m)^T, \quad p = m. \quad (48)$$

Lanczos [6] considers three situations:

- i).  $n < m$ : The linear system is under-determined because it has more unknowns than equations, and from  $1 \leq p \leq \min(n, m)$  is impossible the case  $p = m$ , therefore, if (37) is compatible then its solution cannot be unique.
- ii).  $n = m$ : The system is even-determined with unique solution when  $p = m$ , that is, if  $\det A \neq 0$ . In this case also  $p = n$ , we have not vectors  $\vec{u}_j^{(0)} \neq \vec{0}$ , thus  $\vec{b} \in \text{Col } U$  and automatically the system is compatible.
- iii).  $n > m$ : The linear system is over-determined, and by  $1 \leq p \leq \min(n, m)$  can occur the case  $p = m$  for unique solution if the system is compatible.

Thus it is immediate the classification of linear systems introduced by Lanczos [6]:

$$\begin{array}{ll} \text{Free and complete:} & p = n = m, \quad \text{unique solution,} \\ \text{Restricted and complete:} & p = m < n, \quad \text{over-determined, unique solution,} \end{array} \quad (49)$$

Free and incomplete:  $p = n < m$ , under-determined, non-unique solution,

Restricted and incomplete:  $p < n$  and  $p < m$ , solution without uniqueness,

with the meanings:

Free: The conditions (30) are satisfied trivially.

Restricted: It is necessary to verify that  $\vec{b} \in \text{Col } U$ . (50)

When  $p \neq m$ , the homogeneous system  $A\vec{v} = \vec{0}$  has the non-trivial solutions  $\vec{v}_k^{(0)}$ , then from (27) we conclude that the general solution of (37) is:

$$\vec{x} = \vec{Q} + c_1 \vec{v}_1^{(0)} + \dots + c_{m-p} \vec{v}_{m-p}^{(0)}, \quad (51)$$

where the  $c_k$  are arbitrary constants.

#### 4. Conclusions

With the SVD we can find the subspaces of activation of , and it leads to natural inverse [6, 26-28] of any matrix, known it in the literature as the Moore-Penrose pseudoinverse. Besides, the SVD gives a better understanding of the compatibility of linear systems. On the other hand, Lanczos [6] showed that the Singular Value Decomposition provides a universal platform to study linear differential and integral operators for arbitrary boundary conditions.

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