

Lanczos approach to Noether's theorem

P. Lam-Estrada ¹, J. López-Bonilla ², R. López-Vázquez ²

¹ Dept. of Maths., National Polytechnic Institute (IPN), Edif. 9, Col. Lindavista CP 07738, México

² ESIME-Zacatenco, IPN, Edif. 5, 1er. Piso, Col. Lindavista CP 07738, Mexico city;

jlopezb@ipn.mx

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Abstract. If the action $A = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt$ is invariant under the infinitesimal transformation $\tilde{t} = t + \varepsilon \tau(q, t)$, $\tilde{q}_r = q_r + \varepsilon \zeta_r(q, t)$, $r = 1, \dots, n$, with $\varepsilon = \text{constant} \ll 1$, then the Noether's theorem permits to construct the corresponding conserved quantity. The Lanczos method accepts that $\varepsilon = q_{n+1}$ is a new degree of freedom, thus the Euler-Lagrange equation for this new variable gives the Noether's constant of motion.

If in the functional (the concept of action was proposed by Leibnitz [1]) $A = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt$ we apply the infinitesimal transformation ($\varepsilon = \text{constant} \ll 1$):

$$\tilde{t} = t + \varepsilon \tau(q, t), \quad \tilde{q}_r = q_r + \varepsilon \zeta_r(q, t), \quad r = 1, \dots, n \quad (1)$$

that is

$$\tilde{A} = \int_{\tilde{t}_1}^{\tilde{t}_2} L(\tilde{q}, \frac{d\tilde{q}}{d\tilde{t}}, \tilde{t}) d\tilde{t} \quad (2)$$

then we say that the action is invariant if:

$$\tilde{A} = A + \varepsilon \int_{t_1}^{t_2} \frac{d}{dt} F(q, t) dt, \quad (3)$$

thus the Euler-Lagrange equations (Lagrangian expressions [2, 3]) corresponding to the variational principle $\delta A = 0$:

$$E_r \equiv \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_r} \right) - \frac{\partial L}{\partial q_r} = 0, \quad r = 1, \dots, n \quad (4)$$

remain intact. Noether [2] studied the case $F = 0$, and she suggested [3, 4] to Bessel-Hagen [5] the analysis of (4) with $F \neq 0$.

Therefore, we have a symmetry up to divergence and Noether [2, 5-8] proved the existence of the Rund-Trautman identity [6, 7, 9, 10]:

$$\frac{\partial L}{\partial q_r} \zeta_r + \frac{\partial L}{\partial \dot{q}_r} \dot{\zeta}_r + \frac{\partial L}{\partial t} \tau - \left(\frac{\partial L}{\partial \dot{q}_r} \dot{q}_r - L \right) \dot{\tau} - \frac{dF}{dt} = 0, \quad (5)$$

which can be written in the form:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_r} \zeta_r - H\tau - F \right) = (\zeta_r - \dot{q}_r \tau) E_r, \quad H = \frac{\partial L}{\partial \dot{q}_c} \dot{q}_c - L. \quad (6)$$

In (5) and (6) we employ the convention of Dedekind [11, 12]-Einstein because we sum over repeated indices.

If in (6) we use the Euler-Lagrange equations (4) we deduce the constant of motion associated to (1):

$$\frac{\partial L}{\partial \dot{q}_r} \zeta_r - H\tau - F = \text{Constant}, \quad (7)$$

thus we have a connection between symmetries and conservation laws [3, 6, 7, 9, 13-15].

In the next Section we show the Lanczos technique [16-18] to obtain the Noether's conserved quantity (7).

Lanczos method to construct conservation laws

Lanczos [16, 17] applies the infinitesimal transformation (1) (with $\varepsilon = \text{constant}$) to the action (2) and uses expansion of Taylor up to first order in ε , thus:

$$\tilde{A} = A + \varepsilon \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_r} \zeta_r + \frac{\partial L}{\partial \dot{q}_r} \dot{\zeta}_r + \frac{\partial L}{\partial t} \tau - H\tau \right) dt,$$

then this integrand is equal to $\frac{dF}{dt}$ in harmony with the Rund-Trautman identity (5).

Now Lanczos proposes to employ (1) into (2) but considering that ε is a function, therefore up to 1th order in ε :

$$\tilde{A} = \int_{t_1}^{t_2} \left[L + \varepsilon \frac{dF}{dt} + \dot{\varepsilon} \left(\frac{\partial L}{\partial \dot{q}_r} \zeta_r - H\tau \right) \right] dt = \int_{t_1}^{t_2} \bar{L} dt, \quad (8)$$

$$\frac{d}{dt} \left(\frac{\partial \bar{L}}{\partial \dot{\varepsilon}} \right) - \frac{\partial \bar{L}}{\partial \varepsilon} = 0. \quad (9)$$

It is clear that:

$$\frac{\partial \bar{L}}{\partial \dot{\varepsilon}} = \frac{\partial L}{\partial \dot{q}_r} \zeta_r - H\tau, \quad \frac{\partial \bar{L}}{\partial \varepsilon} = \frac{dF}{dt},$$

therefore (9) implies (7). In other words, if the parameter of the symmetry is considered as an additional degree of freedom of the variational principle, then its Euler-Lagrange equation gives the Noether's constant of motion.

The work [18] has applications of this Lanczos technique to some Lagrangians employed in [8, 19-21].

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