Lanczos approach to Noether’s theorem

P. Lam-Estrada 1, J. López-Bonilla 2, R. López-Vázquez 2

1 Dept. of Maths., National Polytechnic Institute (IPN), Edif. 9, Col. Lindavista CP 07738, México
2 ESIME-Zacatenco, IPN, Edif. 5, 1er. Piso, Col. Lindavista CP 07738, Mexico city;
   jlopezb@ipn.mx

Keywords: Noether’s theorem; Invariance of the action; Variational Lanczos technique.

Abstract. If the action is invariant under the infinitesimal transformation $\tilde{t} = t + \varepsilon \tau(q,t)$, $\tilde{q}_r = q_r + \varepsilon \zeta_r(q,t)$, $r = 1, \ldots, n$, with $\varepsilon = \text{constant} \ll 1$, then the Noether’s theorem permits to construct the corresponding conserved quantity. The Lanczos method accepts that $\varepsilon = \varepsilon_{n+1}$ is a new degree of freedom, thus the Euler-Lagrange equation for this new variable gives the Noether’s constant of motion.

If in the functional (the concept of action was proposed by Leibnitz [1])

$$A = \int_{t_1}^{t_2} L(q, \dot{q}, t) \, dt$$

we apply the infinitesimal transformation

$$\varepsilon = \text{constant} \ll 1):$$

$$\tilde{t} = t + \varepsilon \tau(q,t), \quad \tilde{q}_r = q_r + \varepsilon \zeta_r(q,t), \quad r = 1, \ldots, n$$

that is

$$\tilde{A} = \int_{\tilde{t}_1}^{\tilde{t}_2} L(\tilde{q}, \frac{d}{dt} \tilde{t}) \, d\tilde{t}$$

then we say that the action is invariant if:

$$\tilde{A} = A + \varepsilon \int_{t_1}^{t_2} \frac{d}{dt} F(q,t) \, dt,$$

thus the Euler-Lagrange equations (Lagrangian expressions [2, 3]) corresponding to the variational principle $\delta A = 0$:

$$E_r = \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{q}_r} \right) - \frac{\partial L}{\partial q_r} = 0, \quad r = 1, \ldots, n$$


Therefore, we have a symmetry up to divergence and Noether [2, 5-8] proved the existence of the Rund-Trautman identity [6, 7, 9, 10]:

$$\frac{\partial L}{\partial q_r} \zeta_r + \frac{\partial L}{\partial \dot{q}_r} \dot{\zeta}_r + \frac{\partial L}{\partial \tau} \tau - \left( \frac{\partial L}{\partial \dot{q}_r} \dot{q}_r - L \right) \dot{t} - \frac{dF}{dt} = 0,$$

which can be written in the form:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial q_r} \zeta_r - H \tau - F \right) = (\zeta_r - \dot{q}_r \tau) E_r, \quad H = \frac{\partial L}{\partial \dot{q}_r} \dot{q}_r - L.$$

In (5) and (6) we employ the convention of Dedekind [11, 12]-Einstein because we sum over repeated indices.

If in (6) we use the Euler-Lagrange equations (4) we deduce the constant of motion associated to (1):

$$\frac{\partial L}{\partial q_r} \zeta_r - H \tau - F = \text{Constant},$$

(7)
thus we have a connection between symmetries and conservation laws [3, 6, 7, 9, 13-15].

In the next Section we show the Lanczos technique [16-18] to obtain the Noether’s conserved quantity (7).

**Lanczos method to construct conservation laws**

Lanczos [16, 17] applies the infinitesimal transformation (1) \((\text{with } \varepsilon = \text{constant})\) to the action (2) and uses expansion of Taylor up to first order in \(\varepsilon\), thus:

\[
\tilde{A} = A + \varepsilon \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q_r} \xi_r + \frac{\partial L}{\partial \dot{q}_r} \dot{\xi}_r + \frac{\partial L}{\partial \tau} \tau - H t \right) dt,
\]

then this integrand is equal to \(\frac{dF}{dt}\) in harmony with the Rund-Trautman identity (5).

Now Lanczos proposes to employ (1) into (2) but considering that \(F\) is a function, therefore up to 1st order in \(\varepsilon\):

\[
\tilde{A} = \int_{t_1}^{t_2} \left[ L + \varepsilon \frac{dF}{dt} + \dot{\varepsilon} \left( \frac{\partial L}{\partial \dot{q}_r} \dot{\xi}_r - H t \right) \right] dt = \int_{t_1}^{t_2} \tilde{L} dt,
\]

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_r} \right) - \frac{\partial L}{\partial q_r} = 0,
\]

It is clear that:

\[
\frac{\partial L}{\partial q_r} = \frac{\partial L}{\partial q_r} \xi_r - H t, \quad \frac{\partial L}{\partial \dot{q}_r} = \frac{dF}{d\tau},
\]

therefore (9) implies (7). In other words, if the parameter of the symmetry is considered as an additional degree of freedom of the variational principle, then its Euler-Lagrange equation gives the Noether’s constant of motion.

The work [18] has applications of this Lanczos technique to some Lagrangians employed in [8, 19-21].

**References**

[1]. G. Leibnitz, Dynamica de potentia et leqibus nature corporae (1669) (published in 1890)


[5]. E. Bessel-Hagen, Über die erhaltungssätze der elektrodynamik, Mathematische Annalen 84 (1921) 258-276


[8]. M. Havelková, Symmetries of a dynamical system represented by singular Lagrangians, Comm. in Maths 20, No. 1 (2012) 23-32


[10]. H. Rund, A direct approach to Noether’s theorem in the calculus of variations, Utilitas Math. 2 (1972) 205-214


