

ON $\hat{\alpha}g$ CONNECTED SPACES

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Abstract. In this paper, we define $\hat{\alpha}g$ connected spaces and derive some of its properties.

1.Introduction

Throughout this paper, a space means a topological space on which no separation axioms are assumed unless explicitly stated. In 1963, Levine [4] initiated semiopen sets and studied their properties. Mathematicians gave several papers containing interesting results about new types of sets. Njastad [5] defined α open sets. Senthilkumaran etal [6] defined a new type of sets called αg closed sets and studied their properties. In this paper we define αg connected spaces and derive some of its properties.

2. Preliminaries

Definition 2.1: A subset A of a topological space X is said to be

- 1)A preopen set if $A \subset \text{int cl } A$ and a preclosed set if $\text{cl int } A \subset A$
- 2)A regular open set if $A = \text{int cl } A$ and a regular closed set if $A = \text{cl int } A$.
- 3)A semiopen set if $A \subset \text{cl int } A$ and a semi closed set if $\text{int cl } A \subset A$
- 4)A α - open set if $A \subset \text{int cl int } A$ and a α - closed set if $\text{cl int cl } A \subset A$.

Definition 2.2: A subset A of a topological space X is called $\hat{\alpha}g$ closed set if $\text{int cl int } A \subset U$ whenever $A \subset U$ and U is open in X.

The complement of $\hat{\alpha}g$ closed set in X is called $\hat{\alpha}g$ open set in X.

The intersection of all $\hat{\alpha}g$ closed sets containing A is called $\hat{\alpha}g$ closure of A and shall be denoted by $\hat{\alpha}g \text{ cl } A$. In general $\hat{\alpha}g \text{ cl } A$ is not $\hat{\alpha}g$ closed in X.

The union of all $\hat{\alpha}g$ open sets that are contained in A is called $\hat{\alpha}g$ interior of A and shall be denoted by $\hat{\alpha}g \text{ int } A$. In general, $\hat{\alpha}g \text{ int } A$ is not $\hat{\alpha}g$ open in X.

In what follows, let us assume arbitrary intersection of $\hat{\alpha}g$ closed sets in X is $\hat{\alpha}g$ closed in X. Then $\hat{\alpha}g \text{ cl } A$ will be $\hat{\alpha}g$ closed in X and $\hat{\alpha}g \text{ int } A$ is $\hat{\alpha}g$ open in X.

3. $\hat{\alpha}g$ Separated sets

Definition 3.1 : Let X be a topological space . Two nonempty subsets A and B of X are said to be $\hat{\alpha}g$ separated if and only if $A \cap \hat{\alpha}g \text{ cl } B = \phi$ and $\hat{\alpha}g \text{ cl } A \cap B = \phi$

These two conditions are equivalent to a single condition $(A \cap \hat{\alpha}g \text{ cl } B) \cup (\hat{\alpha}g \text{ cl } A \cap B) = \phi$

Remark 3.2 : A and B are $\hat{\alpha}g$ separated if and only if A and B are disjoint and neither of them contains the $\hat{\alpha}g$ limit point of the other.

Theorem 3.3: If A and B are $\hat{\alpha}g$ separated subsets of X and $C \subset A$ and $D \subset B$, then C and D are $\hat{\alpha}g$ separated.

Proof : $A \cap \hat{\alpha}g \text{ cl } B = \phi$ and $\hat{\alpha}g \text{ cl } A \cap B = \phi$.

$C \subset A \Rightarrow \hat{\alpha}g \text{ cl } C \subset \hat{\alpha}g \text{ cl } A$ and $D \subset B \Rightarrow \hat{\alpha}g \text{ cl } D \subset \hat{\alpha}g \text{ cl } B$

$C \cap \hat{\alpha}g \text{ cl } D \subset A \cap \hat{\alpha}g \text{ cl } B = \phi$

Similarly, $\hat{\alpha} g \text{ cl } A \cap B = \phi$. Hence C and D are $\hat{\alpha} g$ separated.

Theorem 3.4 : Two $\hat{\alpha} g$ closed ($\hat{\alpha} g$ open) subset of the topological space X are $\hat{\alpha} g$ separated if and only if they are disjoint.

Proof : Since any two $\hat{\alpha} g$ separated sets are disjoint, we need only to show that two disjoint $\hat{\alpha} g$ closed ($\hat{\alpha} g$ open) sets are $\hat{\alpha} g$ separated. Let A and B be disjoint and $\hat{\alpha} g$ closed. Then $A \cap B = \phi$, $\hat{\alpha} g \text{ cl } A = A$, $\hat{\alpha} g \text{ cl } B = B$. So, $\hat{\alpha} g \text{ cl } A \cap B = \phi$ and $A \cap \hat{\alpha} g \text{ cl } B = \phi$.

Hence A and B are $\hat{\alpha} g$ separated. Let A and B be disjoint and $\hat{\alpha} g$ open. Then $\hat{\alpha} g \text{ cl } A' = A'$ and $\hat{\alpha} g \text{ cl } B' = B'$. $A \cap B = \phi \Rightarrow A \subset B'$ and $B \subset A'$

$\Rightarrow \hat{\alpha} g \text{ cl } A \subset \hat{\alpha} g \text{ cl } B' = B'$ and

$\hat{\alpha} g \text{ cl } B \subset \hat{\alpha} g \text{ cl } A' = A'$

$\Rightarrow \hat{\alpha} g \text{ cl } A \cap B = \phi$ and $A \cap \hat{\alpha} g \text{ cl } B = \phi$

Hence A and B are $\hat{\alpha} g$ separated.

Theorem 3.5 : If A and B are $\hat{\alpha} g$ separated sets of a topological space X, then

i) $A \cup B$ is $\hat{\alpha} g$ closed \Rightarrow A and B are $\hat{\alpha} g$ closed

ii) $A \cup B$ is $\hat{\alpha} g$ open \Rightarrow A and B are $\hat{\alpha} g$ open

Proof : Let A and B be $\hat{\alpha} g$ separated sets of a topological space X so that $A \neq \phi$ and $B \neq \phi$

$\hat{\alpha} g \text{ cl } A \cap B = \phi$ and $A \cap \hat{\alpha} g \text{ cl } B = \phi$

Let $A \cup B$ be $\hat{\alpha} g$ closed

$\hat{\alpha} g \text{ cl } (A \cup B) = A \cup B$. That is $\hat{\alpha} g \text{ cl } A \cup \hat{\alpha} g \text{ cl } B = A \cup B$

$\hat{\alpha} g \text{ cl } A = \hat{\alpha} g \text{ cl } A \cap (\hat{\alpha} g \text{ cl } A \cup \hat{\alpha} g \text{ cl } B)$

$= \hat{\alpha} g \text{ cl } A \cap (A \cup B)$

$= (\hat{\alpha} g \text{ cl } A \cap A) \cup (\hat{\alpha} g \text{ cl } A \cap B)$

$= A \cup \phi = A$.

Hence A is $\hat{\alpha} g$ closed. Similarly, B is $\hat{\alpha} g$ closed.

Let $A \cup B$ be $\hat{\alpha} g$ open.

$\hat{\alpha} g \text{ cl } B$ is $\hat{\alpha} g$ closed. Hence $X - \hat{\alpha} g \text{ cl } B$ is $\hat{\alpha} g$ open

$(A \cup B) \cap (X - \hat{\alpha} g \text{ cl } B) = (A \cap (X - \hat{\alpha} g \text{ cl } B)) \cup (B \cap (X - \hat{\alpha} g \text{ cl } B))$

$= A \cup \phi = A$.

Hence A is $\hat{\alpha} g$ open. Similarly B is $\hat{\alpha} g$ open.

Theorem 3.6 : Two disjoint sets A and B are $\hat{\alpha} g$ separated in a topological space $X = A \cup B$ if and only if they are both $\hat{\alpha} g$ open and αg closed in X.

Proof : Let disjoint sets A and B be $\hat{\alpha} g$ separated in X. $A \cap \hat{\alpha} g \text{ cl } B = \phi$ and $\hat{\alpha} g \text{ cl } A \cap B = \phi$

$X = A \cup B$. $\hat{\alpha} g \text{ cl } A = \hat{\alpha} g \text{ cl } A \cap X = \hat{\alpha} g \text{ cl } A \cap (A \cup B)$

$= (\hat{\alpha} g \text{ cl } A \cap A) \cup (\hat{\alpha} g \text{ cl } A \cap B) = A$.

Hence A is $\hat{\alpha} g$ closed. Similarly B is $\hat{\alpha} g$ closed.

Since A and B are disjoint and $A \cup B = X$, B is $\hat{\alpha} g$ open. Similarly A is $\hat{\alpha} g$ open.

Conversely, let disjoint sets A and B be both $\hat{\alpha} g$ open and $\hat{\alpha} g$ closed in $X = A \cup B$.

A is $\hat{\alpha} g$ closed in X. So, $A = \hat{\alpha} g \text{ cl } A = \hat{\alpha} g \text{ cl } A \cap X$

$= \hat{\alpha} g \text{ cl } A \cap (A \cup B) = (\hat{\alpha} g \text{ cl } A \cap A) \cup (\hat{\alpha} g \text{ cl } A \cap B)$

$= A \cup (\hat{\alpha} g \text{ cl } A \cap B)$

$A \cup B = \phi \Rightarrow A \cup (\hat{\alpha} g \text{ cl } A \cap B) = \phi \Rightarrow \hat{\alpha} g \text{ cl } A \cap B = \phi$

Similarly, $A \cap \hat{\alpha} g \text{ cl } B = \phi$

So, A and B are $\hat{\alpha} g$ separated.

Theorem 3.7 : If A and B are proper subsets of a space X and both A and B are $\hat{\alpha} g$ closed and $\hat{\alpha} g$ open, then $A - B$ is $\hat{\alpha} g$ separated from $B - A$.

Proof : $(A - B) \cap (B - A) = \phi$ and $A - B$ and $B - A$ are non empty.

$(A - B) \cap \hat{\alpha} g \text{ cl } (B - A) = (A \cap B') \cap \hat{\alpha} g \text{ cl } (B \cap A') \subset (A \cap B') \cap (\hat{\alpha} g \text{ cl } B \cap \hat{\alpha} g \text{ cl } A')$

$= (A \cap B' \cap \hat{\alpha} g \text{ cl } B) \cap (A \cap B' \cap \hat{\alpha} g \text{ cl } A')$

$$=(A \cap B' \cap B) \cap (A \cap B' \cap A') = \phi$$

Similarly, $\hat{\alpha}g \text{ cl } (A - B) \cap (B - A) = \phi$

Hence $A - B$ and $B - A$ are $\hat{\alpha}g$ separated.

4. $\hat{\alpha}g$ Connected and $\hat{\alpha}g$ Disconnected sets

Definition 4.1 : Let X be a topological space. A subset A of X is said to be $\hat{\alpha}g$ disconnected if and only if it is the union of two non empty $\hat{\alpha}g$ separated sets. That is, if and only if there exists non empty sets C and D such that $C \cap \hat{\alpha}g \text{ cl } D = \phi$, $\hat{\alpha}g \text{ cl } C \cap D = \phi$ and $A = C \cup D$. A is said to be $\hat{\alpha}g$ connected if and only if it is not $\hat{\alpha}g$ disconnected.

Remark 4.2 : The empty set is trivially $\hat{\alpha}g$ connected.

Also, every singleton set is $\hat{\alpha}g$ connected.

Definition 4.3 : Two points a and b of a topological space X is $\hat{\alpha}g$ connected if and only if they are contained in a $\hat{\alpha}g$ connected subset of X .

Theorem 4.4 : A topological space X is $\hat{\alpha}g$ disconnected if and only if there exists a non empty proper subset of X which is both $\hat{\alpha}g$ open and $\hat{\alpha}g$ closed in X .

Proof : Let A be a non empty proper subset of X which is both $\hat{\alpha}g$ open and $\hat{\alpha}g$ closed. Let us show that X is $\hat{\alpha}g$ disconnected. . Let $B = A'$. As A is a proper subset, B is nonempty. Moreover, $A \cup B = X$ and $A \cap B = \phi$. As A is both $\hat{\alpha}g$ open and $\hat{\alpha}g$ closed, B is also both $\hat{\alpha}g$ open and $\hat{\alpha}g$ closed. Therefore, by theorem 3.6, A and B are $\hat{\alpha}g$ separated. Hence $X = A \cup B$, is $\hat{\alpha}g$ disconnected. Conversely, let X be $\hat{\alpha}g$ disconnected. Then, there exists nonempty sets A and B such that $A \cap \hat{\alpha}g \text{ cl } B = \phi$, $\hat{\alpha}g \text{ cl } A \cap B = \phi$ and $X = A \cup B$ Now $A \subset \hat{\alpha}g \text{ cl } A$. So, $A \cap B = \phi$
Hence $A = B'$

Since B is nonempty and $B \cup X = X$, it follows $B = A'$ is a proper subset of X ,

Now $A \cup \hat{\alpha}g \text{ cl } B = X$ as $A \cup B = X \Rightarrow A \cup \hat{\alpha}g \text{ cl } B = X$.

Also $A \cap \hat{\alpha}g \text{ cl } B = \phi \Rightarrow A = (\hat{\alpha}g \text{ cl } B)'$ and similarly, $B = (\hat{\alpha}g \text{ cl } A)'$. Since $\hat{\alpha}g \text{ cl } A$ and $\hat{\alpha}g \text{ cl } B$ are $\hat{\alpha}g$ closed sets, B and A are $\hat{\alpha}g$ open sets. Since $A = B'$, A is also $\hat{\alpha}g$ closed.

This completes the proof.

Remark 4.5 : In the above theorem, we have also shown that B is also a proper subset of X which is both $\hat{\alpha}g$ open and $\hat{\alpha}g$ closed.

Corollary 4.6 : A topological space X is $\hat{\alpha}g$ connected if and only if the only nonempty subset of X which is both $\hat{\alpha}g$ open and $\hat{\alpha}g$ closed in X is X itself.

Theorem 4.7 : A topological space X is $\hat{\alpha}g$ disconnected if and only if any one of the following statement holds.

i) X is the union of two disjoint nonempty $\hat{\alpha}g$ open sets.

ii) X is the union of two disjoint non empty $\hat{\alpha}g$ closed sets.

Proof : Let X be $\hat{\alpha}g$ disconnected. Then, there exists a nonempty proper subset A of X which is both $\hat{\alpha}g$ open and $\hat{\alpha}g$ closed. Then A' is also both $\hat{\alpha}g$ open and $\hat{\alpha}g$ closed. Also $A \cup A' = X$. Hence the sets A and A' satisfy the requirement of (i) and (ii).

Conversely, let $X = A \cup B$ and $A \cap B = \phi$ where A and B are nonempty $\hat{\alpha}g$ open sets.

It follows that $A = B'$, so that A is $\hat{\alpha}g$ closed. Since B is nonempty, A is a proper subset of X . Then A is a nonempty proper subset of X . Then A is a nonempty proper subset of X which is both $\hat{\alpha}g$ open and $\hat{\alpha}g$ closed. Hence by theorem 4.4, X is $\hat{\alpha}g$ disconnected. Similarly, assuming (ii) we can prove X is $\hat{\alpha}g$ disconnected.

Remark 4.8 : $A \cup B$ is called a $\hat{\alpha}g$ disconnection of X .

Definition 4.9 : Let A be a subset of a topological space X . A point x of X is said to be $\hat{\alpha}g$ exterior point of A if it is $\hat{\alpha}g$ interior point of the complement A' of A . The set of all $\hat{\alpha}g$ exterior points of A is called the $\hat{\alpha}g$ exterior of A and shall be denoted by $\hat{\alpha}g \text{ ext } (A)$.

Definition 4.10 : A point x of a topological space X is said to be a $\hat{a}g$ frontier point or $\hat{a}g$ boundary point of a subset A of X if it is neither a $\hat{a}g$ interior point nor $\hat{a}g$ exterior point of A . The set of all $\hat{a}g$ frontierpoints of A is called the $\hat{a}g$ frontier of A and shall be denoted by $\hat{a}g Fr(A)$.

Theorem 4.11 : A topological space X is $\hat{a}g$ connected if and only if every nonempty proper subset of X has a nonempty $\hat{a}g$ frontier.

Proof : Let every nonempty proper subset of X have a nonempty $\hat{a}g$ frontier. Let us show X is $\hat{a}g$ connected. Let X be $\hat{a}g$ disconnected. Then, there exists nonempty disjoint sets G and H both $\hat{a}g$ open and $\hat{a}g$ closed in X such that $X = G \cup H$. Therefore $G = \hat{a}g \text{ int } G = \hat{a}g \text{ cl } G$. But $\hat{a}g Fr(G) = \emptyset$, which is a contradiction to our hypothesis. Hence X must be $\hat{a}g$ connected.

Conversely, let X be $\hat{a}g$ connected. Let, if possible, there exists a nonempty proper subset of A of X such that $\hat{a}g Fr(A) = \emptyset$ Now $\hat{a}g \text{ cl } A = \hat{a}g \text{ int } A \cup \hat{a}g Fr(A)$ by theorem 6.2[7]

$$= A \cup \hat{a}g Fr(A) \text{ by theorem 6.4[7]}$$

Hence $A = \hat{a}g \text{ int } A = \hat{a}g \text{ cl } A$, showing that A is both $\hat{a}g$ open and $\hat{a}g$ closed. Therefore X is $\hat{a}g$ disconnected, which is a contradiction. This completes the proof.

Theorem 4.12 : i) Every indiscrete topological space X , where X is non empty is $\hat{a}g$ connected.

ii) Every discrete topological space X , where X contains more than one point is $\hat{a}g$ connected.

Proof i) : There is no proper subset of X , which is both $\hat{a}g$ open and $\hat{a}g$ closed. So, X is $\hat{a}g$ connected.

ii) X contains more than one point. Every singleton subset of X is a nonempty proper subset of X which is both $\hat{a}g$ open and $\hat{a}g$ closed. Hence X is $\hat{a}g$ disconnected.

Theorem 4.13 : Let X be a topological space and let E be $\hat{a}g$ connected subset of X such that $E \subset A \cup B$ where A and B are $\hat{a}g$ separated sets. Then $E \subset A$ or $E \subset B$, that is E cannot intersect both A and B .

Proof : Since A and B are $\hat{a}g$ separated, $A \cap \hat{a}g \text{ cl } B = \emptyset$ and $\hat{a}g \text{ cl } A \cap B = \emptyset$

$$\text{Now, } E \subset A \cup B \Rightarrow E = E \cap (A \cup B) = (E \cap A) \cup (E \cap B)$$

Let us prove one of the sets $E \cap A$ or $E \cap B$ is empty. For, if possible, none of these sets is empty

That is, let $E \cap A \neq \emptyset$ and $E \cap B \neq \emptyset$

$$\text{Then } (E \cap A) \cap \hat{a}g \text{ cl } (E \cap B) \subset (E \cap A) \cap (\hat{a}g \text{ cl } E \cap \hat{a}g \text{ cl } B) = (E \cap (\hat{a}g \text{ cl } E \cap \hat{a}g \text{ cl } B)) \cap (A \cap (\hat{a}g \text{ cl } E \cap \hat{a}g \text{ cl } B)) = (E \cap \hat{a}g \text{ cl } B) \cap \emptyset = \emptyset$$

$$\text{Similarly, } \hat{a}g \text{ cl } (E \cap A) \cap (E \cap B) = \emptyset$$

Hence $E \cap A$ and $E \cap B$ are $\hat{a}g$ separated sets. So, E is $\hat{a}g$ disconnected, which is a contradiction. Hence one of the sets $E \cap A$ or $E \cap B$ is empty.

If $E \cap A = \emptyset$, then $E = E \cap B$, which implies $E \subset B$.

Corollary 4.14 : If E is a $\hat{a}g$ connected subset of a space X such that $E \subset A \cup B$, where A and B are disjoint $\hat{a}g$ open and $\hat{a}g$ closed subsets of X , then A and B are $\hat{a}g$ separated.

Proof : A and B are $\hat{a}g$ open with $A \cap B = \emptyset$

$$\text{Then } A \subset B' \Rightarrow \hat{a}g \text{ cl } A \subset \hat{a}g \text{ cl } B' = B'$$

$$\Rightarrow \hat{a}g \text{ cl } A \cap B = \emptyset$$

$$\text{Similarly } A \cap \hat{a}g \text{ cl } B = \emptyset$$

Hence A and B are $\hat{a}g$ separated.

Theorem 4.15 : Let E be a $\hat{a}g$ connected subset of a space X . If F is a subset of X such that $E \subset F \subset \hat{a}g \text{ cl } E$, then F is $\hat{a}g$ connected. In particular $\hat{a}g \text{ cl } E$ is $\hat{a}g$ connected.

Proof : Let F be $\hat{a}g$ disconnected. Then there exist nonempty sets A and B such that

$$A \cap \hat{a}g \text{ cl } B = \emptyset, \hat{a}g \text{ cl } A \cap B = \emptyset \text{ and } A \cup B = F.$$

Since $E \subset F = A \cup B$, it follows from theorem 4.13 that $E \subset A$ or $E \subset B$.

Let $E \subset A$. This implies $\hat{a}g \text{ cl } E \subset \hat{a}g \text{ cl } A$.

$$\text{This implies } \hat{a}g \text{ cl } E \cap B \subset \hat{a}g \text{ cl } A \cap B = \emptyset$$

$$\text{That is } \hat{a}g \text{ cl } E \cap B = \emptyset. \text{ Also } A \cup F \subset \hat{a}g \text{ cl } E$$

$$\Rightarrow B \subset F \subset \hat{a}g \text{ cl } E \Rightarrow \hat{a}g \text{ cl } E \cap B = B$$

Hence $B = \emptyset$, a contradiction. So, F must be $\hat{a}g$ connected. Again $E \subset \hat{a}g \text{ cl } E \subset \hat{a}g \text{ cl } E$, $\hat{a}g \text{ cl } E$ is $\hat{a}g$ connected.

Theorem 4.16 : If every two points of a subset E of a topological space X are contained in some $\hat{a}g$ connected subset of E , then E is $\hat{a}g$ connected subset of X .

Proof : Let E be not $\hat{a}g$ connected. Then there exist non empty subsets A and B of X such that $A \cap \hat{a}g \text{ cl } B = \emptyset$, $\hat{a}g \text{ cl } A \cap B = \emptyset$ and $E = A \cup B$.

Since A and B are non empty, there exists a point $a \in A$ and a point $b \in B$. By hypothesis a and b must be contained in some $\hat{a}g$ connected subset F of E . Since $F \subset A \cup B$, by theorem 4.13, $F \subset A$ or $F \subset B$. It follows a and b are both in A or B . Let $a, b \in A$. As $b \in B$, we have $A \cap B \neq \emptyset$, a contradiction. Hence E must be $\hat{a}g$ connected.

Theorem 4.17 : A is a $\hat{a}g$ connected subset of a $\hat{a}g$ connected topological space X such that A' is the union of two $\hat{a}g$ separated sets B and C . Then $A \cup B$ and $A \cup C$ are $\hat{a}g$ connected.

Proof : Let $A \cup B$ be $\hat{a}g$ disconnected. Then, there exist two nonempty $\hat{a}g$ separated sets G and H , whose union is $A \cup B$. A is a $\hat{a}g$ connected subset of $\hat{a}g$ disconnected set $A \cup B$ having the $\hat{a}g$ separation

$A \cup B = G \cup H$. Then by theorem 4.13, either $A \subset G$ or $A \subset H$.

Suppose that $A \subset G$.

Now, since $A \cup B = G \cup H$

$A \subset G \Rightarrow A \cup B \subset G \cup B \Rightarrow G \cup H \subset G \cup B \Rightarrow H \subset B$

Since B and C are $\hat{a}g$ separated sets and $H \subset B$, $C \subset C$, it follows from theorem 3.3, that H and C are also $\hat{a}g$ separated. Thus the set H is $\hat{a}g$ separated from G as well as C and therefore H is $\hat{a}g$ separated from $G \cup C$

But $A' = B \cup C \Rightarrow X = A \cup A' = A \cup (B \cup C)$

$\Rightarrow X = (A \cup B) \cup C \Rightarrow X = (G \cup H) \cup C = (G \cup C) \cup H$

Consequently, X has been expressed as the union of nonempty $\hat{a}g$ separated sets. This implies X is $\hat{a}g$ disconnected, a contradiction.

Similar contradiction will arise if $A \subset H$.

Hence $A \cup B$ is $\hat{a}g$ connected. Similarly $A \cup C$ is $\hat{a}g$ connected.

$\hat{a}g$ continuity and $\hat{a}g$ connectedness

Definition 5.1 : Let X and Y be topological spaces. A mapping $f ; X \rightarrow Y$ is said to be $\hat{a}g$ continuous if and only if the inverse image of every open set of Y is $\hat{a}g$ open in X .

Theorem 5.2 : If f is a $\hat{a}g$ continuous mapping of a $\hat{a}g$ connected space X onto an arbitrary topological space Y , then Y is connected.

Proof : Let Y be disconnected. Then there exists a nonempty proper subset G of Y which is both open and closed in Y . Since f is $\hat{a}g$ continuous and onto Y , $f^{-1}(G)$ is a nonempty proper subset of X which is both $\hat{a}g$ open and $\hat{a}g$ closed in X . Therefore X is $\hat{a}g$ disconnected, a contradiction. Hence Y must be connected.

Theorem 5.3 : A topological space X is $\hat{a}g$ disconnected if and only if there exists a $\hat{a}g$ continuous mapping of X onto the discrete two point space $\{0,1\}$

Proof : Let X be $\hat{a}g$ disconnected. Then, there exist two nonempty disjoint $\hat{a}g$ open subsets G_1 and G_2 of X such that $X = G_1 \cup G_2$. Define a mapping f of X onto $\{0,1\}$ by setting $f(x) = 0$ if $x \in G_1$ and $f(x) = 1$ if $x \in G_2$. Since $\{0,1\}$ is discrete, its open sets are \emptyset , $\{0\}$, $\{1\}$ and $\{0,1\}$

$f^{-1}(\{0\}) = G_1$, $f^{-1}(\{1\}) = G_2$, $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(\{0,1\}) = X$

Hence f is $\hat{a}g$ continuous.

Conversely, let there exists a $\hat{a}g$ continuous mapping of X onto the discrete space $\{0,1\}$.

If X is $\hat{a}g$ connected, then $\{0,1\}$ is connected by the above theorem. But this is impossible, since every discrete space is disconnected.

Theorem 5.4 : A space X is $\hat{a}g$ connected if and only if every $\hat{a}g$ continuous function f from X into the discrete two points space $\{0,1\}$ is constant.

Proof : Let X be $\hat{a}g$ connected and let $y \in f(x) \subset \{0,1\}$, where $f : X \rightarrow \{0,1\}$ is any $\hat{a}g$ continuous function. Then $\{y\}$ is both open and closed in the discrete space $\{0,1\}$. f is $\hat{a}g$ continuous. Therefore $f^{-1}(\{y\})$ is $\hat{a}g$ open and $\hat{a}g$ closed and nonempty. Since X is $\hat{a}g$ connected, $f^{-1}(\{y\}) = X$. Therefore f is a constant function.

Conversely, let every $\hat{a}g$ continuous function $f : X \rightarrow \{0,1\}$ be constant.

Suppose, if possible, X be $\hat{a}g$ disconnected. Then, there exists a nonempty proper subset A of X which is both $\hat{a}g$ open and $\hat{a}g$ closed in X . Consider the characteristic function K_A of A defined by

$$K_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \in A' \end{cases}$$

Then $K_A^{-1}(\phi) = \phi$, $K_A^{-1}(\{1\}) = A$, $K_A^{-1}(\{0\}) = A'$,

$K_A^{-1}(\{0,1\}) = X$. Therefore K_A is $\hat{a}g$ continuous.

But K_A is non constant, a contradiction.

Hence X must be $\hat{a}g$ connected.

Definition 5.5 : A map $f : X \rightarrow Y$ is called $\hat{a}g$ irresolute if $f^{-1}(V)$ is $\hat{a}g$ closed in X for every $\hat{a}g$ closed set V of Y . It is called irresolute if $f^{-1}(V)$ is open in X for every open set V of Y .

Theorem 5.6 : A map $f : X \rightarrow Y$ is $\hat{a}g$ irresolute if for every $\hat{a}g$ open set A of Y , $f^{-1}(A)$ is $\hat{a}g$ open in X .

Proof : obvious

Theorem 5.7 : Let $f : X \rightarrow Y$ be a map. Then, if X is $\hat{a}g$ connected and f is $\hat{a}g$ irresolute surjective, then Y is $\hat{a}g$ connected.

Proof : Let Y be $\hat{a}g$ disconnected. Then $Y = A \cup B$, where A and B are disjoint nonempty $\hat{a}g$ open subsets of Y . Since f is $\hat{a}g$ irresolute surjective, $X = f^{-1}(A) \cup f^{-1}(B)$. $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint nonempty $\hat{a}g$ open subset of X . So, X is $\hat{a}g$ disconnected, a contradiction. This completes the proof.

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