ON $\alpha g$ CONNECTED SPACES

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Keywords: $\alpha g$ separated sets, $\alpha g$ disconnected spaces, $\alpha g$ connected spaces.

Abstract. In this paper, we define $\alpha g$ connected spaces and derive some of its properties.

1. Introduction

Throughout this paper, a space means a topological space on which no separation axioms are assumed unless explicitly stated. In 1963, Levine [4] initiated semiopen sets and studied their properties. Mathematicians gave several papers containing interesting results about new types of sets. Njastad [5] defined $\alpha$ open sets. Senthilkumaran et al [6] defined a new type of sets called $\alpha g$ closed sets and studied their properties. In this paper we define $\alpha g$ connected spaces and derive some of its properties.

2. Preliminaries

Definition 2.1: A subset A of a topological space X is said to be
1)A preopen set if $A \subseteq \text{int cl } A$ and a preclosed set if $\text{cl int } A \subseteq A$
2)A regular open set if $A = \text{int cl } A$ and a regular closed set if $A = \text{cl int } A$.
3)A semiopen set if $A \subseteq \text{cl int } A$ and a semi closed set if $\text{int cl } A \subseteq A$
4)A $\alpha$- open set if $A \subseteq \text{int cl int } A$ and a $\alpha$- closed set if $\text{cl int cl } A \subseteq A$.

Definition 2.2: A subset A of a topological space X is called $\alpha g$ closed set if $\text{int cl int } A \subseteq U$ whenever $A \subseteq U$ and U is open in X.

The complement of $\alpha g$ closed set in X is called $\alpha g$ open set in X.

The intersection of all $\alpha g$ closed sets containing A is called $\alpha g$ closure of A and shall be denoted by $\alpha g$ cl A. In general $\alpha g$ cl A is not $\alpha g$ closed in X.

The union of all $\alpha g$ open sets that are contained in A is called $\alpha g$ interior of A and shall be denoted by $\alpha g$ int A. In general, $\alpha g$ int A is not $\alpha g$ open in X.

In what follows, let us assume arbitrary intersection of $\alpha g$ closed sets in X is $\alpha g$ closed in X.

Then $\alpha g$ cl A will be $\alpha g$ closed in X and $\alpha g$ int A is $\alpha g$ open in X.

3. $\alpha g$ Separated sets

Definition 3.1: Let X be a topological space. Two nonempty subsets A and B of X are said to be $\alpha g$ separated if and only if $A \cap \alpha g \text{ cl } B = \emptyset$ and $\alpha g \text{ cl } A \cap B = \emptyset$.

These two conditions are equivalent to a single condition $(A \cap \alpha g \text{ cl } B) \cup (\alpha g \text{ cl } A \cap B) = \emptyset$.

Remark 3.2: A and B are $\alpha g$ separated if and only if A and B are disjoint and neither of them contains the $\alpha g$ limit point of the other.

Theorem 3.3: If A and B are $\alpha g$ separated subsets of X and $C \subseteq A$ and $D \subseteq B$, then C and D are $\alpha g$ separated.

Proof: $A \cap \alpha g \text{ cl } B = \emptyset$ and $\alpha g \text{ cl } A \cap B = \emptyset$.

$C \subseteq A \Rightarrow \alpha g \text{ cl } C \subseteq \alpha g \text{ cl } A$ and $D \subseteq B \Rightarrow \alpha g \text{ cl } D \subseteq \alpha g \text{ cl } B$

$C \cap \alpha g \text{ cl } D \subseteq C \cap \alpha g \text{ cl } B = \emptyset$.
Similarly, \( \hat{a} \) closed \( \hat{g} \) separated.

**Theorem 3.4**: Two \( \hat{a} \) closed \( \hat{g} \) open subset of the topological space \( X \) are \( \hat{a} \) closed if and only if they are disjoint.

**Proof**: Since any two \( \hat{a} \) closed \( \hat{g} \) open are disjoint, we need only to show that two \( \hat{a} \) closed \( \hat{g} \) open sets are \( \hat{a} \) closed. Let \( A \) and \( B \) be disjoint and \( \hat{a} \) closed. Then \( A \cap B = \emptyset \), \( \hat{a} \) closed \( A = A \), \( \hat{a} \) closed \( B = B \). So, \( \hat{a} \) closed \( A \cap B = \emptyset \) and \( \hat{a} \) closed \( A \cap B = \emptyset \).

Hence \( A \) and \( B \) are \( \hat{a} \) closed. Let \( A \) and \( B \) be disjoint and \( \hat{a} \) open. Then \( \hat{a} \) closed \( A' = A' \) and \( \hat{a} \) closed \( B' = B' \). \( A \cap B = \emptyset \Rightarrow A \subseteq B' \) and \( B \subseteq A' \)

\[ \hat{a} \text{ closed } A \subseteq \hat{a} \text{ closed } B' \]

\[ \hat{a} \text{ closed } A \cap B = \emptyset \]

Hence \( A \) and \( B \) are \( \hat{a} \) separated.

**Theorem 3.5**: If \( A \) and \( B \) are \( \hat{a} \) closed sets of a topological space \( X \), then

i) \( A \cup B \) is \( \hat{a} \) closed \( \Rightarrow A \) and \( B \) are \( \hat{a} \) closed

ii) \( A \cup B \) is \( \hat{a} \) open \( \Rightarrow A \) and \( B \) are \( \hat{a} \) open

**Proof**: Let \( A \) and \( B \) be \( \hat{a} \) closed sets of a topological space \( X \) so that \( A \neq \emptyset \) and \( B \neq \emptyset \)

\( \hat{a} \) closed \( A \cap B = \emptyset \) and \( A \cap \hat{a} \) closed \( B = \emptyset \)

Let \( A \cup B \) be \( \hat{a} \) closed

\( \hat{a} \) closed \( (A \cup B) = A \cup B \). That is \( \hat{a} \) closed \( A \cup \hat{a} \) closed \( B = A \cup B \)

\( \hat{a} \) closed \( A = \hat{a} \) closed \( A \cap \hat{a} \) closed \( B \cap \hat{a} \) closed \( B \)

\[ \hat{a} \text{ closed } A \cap \hat{a} \text{ closed } B \]

Hence \( A \) is \( \hat{a} \) closed. Similarly, \( B \) is \( \hat{a} \) closed.

Let \( A \cup B \) be \( \hat{a} \) open.

\( \hat{a} \) open \( \hat{a} \) closed \( B \) is \( \hat{a} \) closed. Hence \( X - \hat{a} \) closed \( B \) is \( \hat{a} \) closed

\( (A \cup B) \cap (X - \hat{a} \) closed \( B) = (A \cup (X - \hat{a} \) closed \( B)) \subseteq (B \cap (X - \hat{a} \) closed \( B)) \subseteq A \cup \emptyset = A \).

Hence \( A \) is \( \hat{a} \) open. Similarly \( B \) is \( \hat{a} \) open.

**Theorem 3.6**: Two disjoint \( \hat{a} \) closed \( \hat{g} \) open sets of a topological space \( X = A \cup B \) if and only if they are both \( \hat{a} \) closed and \( \hat{a} \) open in \( X \).

**Proof**: Let disjoint \( \hat{a} \) closed \( \hat{g} \) open \( A \cap \hat{a} \) closed \( B = \emptyset \) and \( \hat{a} \) closed \( A \cap B = \emptyset \)

\( X = A \cup B \). \( \hat{a} \) closed \( A \cup \hat{a} \) closed \( B \)

\[ \hat{a} \text{ closed } A \cap \hat{a} \text{ closed } B \]

Hence \( A \) is \( \hat{a} \) closed. Similarly \( B \) is \( \hat{a} \) closed.

Since \( A \) and \( B \) are disjoint and \( A \cup B = X \), \( B \) is \( \hat{a} \) closed. Similarly \( A \) is \( \hat{a} \) closed.

Conversely, let disjoint \( \hat{a} \) closed \( \hat{g} \) open \( A \) and \( B \) be both \( \hat{a} \) closed and \( \hat{a} \) open in \( X = A \cup B \).

\( A \) is \( \hat{a} \) closed in \( X \). So, \( A = \hat{a} \) closed \( A \cap X \)

\[ \hat{a} \text{ closed } A \cap \hat{a} \text{ closed } B \]

\[ \hat{a} \text{ closed } A \cap B \]

\[ A \cup B = \emptyset \Rightarrow A \cup (\hat{a} \text{ closed } A \cap B) = \emptyset \Rightarrow \hat{a} \text{ closed } A \cap B = \emptyset \]

Similarly, \( A \cap \hat{a} \) closed \( B = \emptyset \)

So, \( A \) and \( B \) are \( \hat{a} \) closed.

**Theorem 3.7**: If \( A \) and \( B \) are proper subsets of a space \( X \) and both \( A \) and \( B \) are \( \hat{a} \) closed and \( \hat{a} \) open, then \( \hat{a} \) closed \( \hat{a} \) open \( A - B \) is \( \hat{a} \) closed and \( \hat{a} \) open.

**Proof**: \((A - B) \cap (B - A) = \emptyset \) and \( A - B \) and \( B - A \) are non empty.

\( (A - B) \cap \hat{a} \) closed \( (B - A) = (A \cap B') \subseteq (B \cap A') \subseteq (A \cap B') \cup (\hat{a} \text{ closed } B \cap \hat{a} \text{ closed } A') \)

\[ (A \cap B') \cup \hat{a} \text{ closed } B \cap \hat{a} \text{ closed } A' \]

\[ (A \cap B') \cup \hat{a} \text{ closed } B \cap \hat{a} \text{ closed } A' \]
\[(A \cap B') \cap (A \cap B' \cap A') = \phi\]

Similarly, \(\hat{\alpha} g \text{ cl} (A - B) \cap (B - A) = \phi\)

Hence \(A - B\) and \(B - A\) are \(\hat{\alpha} g\) separated.

### 4. \(\hat{\alpha} g\) Connected and \(\hat{\alpha} g\) Disconnected sets

**Definition 4.1**: Let \(X\) be a topological space. A subset \(A\) of \(X\) is said to be \(\hat{\alpha} g\) disconnected if and only if it is the union of two non-empty \(\hat{\alpha} g\) separated sets. That is, if and only if there exist non-empty sets \(C\) and \(D\) such that \(C \cap \hat{\alpha} g \text{ cl } D = \phi\), \(\hat{\alpha} g \text{ cl } C \cap D = \phi\) and \(A = C \cup D\). \(A\) is said to be \(\hat{\alpha} g\) connected if and only if it is not \(\hat{\alpha} g\) disconnected.

**Remark 4.2**: The empty set is trivially \(\hat{\alpha} g\) connected.

Also, every singleton set is \(\hat{\alpha} g\) connected.

**Definition 4.3**: Two points \(a\) and \(b\) of a topological space \(X\) is \(\hat{\alpha} g\) connected if and only if they are contained in a \(\hat{\alpha} g\) connected subset of \(X\).

**Theorem 4.4**: A topological space \(X\) is \(\hat{\alpha} g\) disconnected if and only if there exists a non-empty proper subset of \(X\) which is both \(\hat{\alpha} g\) open and \(\hat{\alpha} g\) closed in \(X\).

**Proof**: Let \(A\) be a non-empty proper subset of \(X\) which is both \(\hat{\alpha} g\) open and \(\hat{\alpha} g\) closed. Let us show that \(X\) is \(\hat{\alpha} g\) disconnected. Let \(B = A'\). As \(A\) is a proper subset, \(B\) is nonempty. Moreover, \(A \cup B = X\) and \(A \cap B = \phi\). As \(A\) is both \(\hat{\alpha} g\) open and \(\hat{\alpha} g\) closed, \(B\) is also both \(\hat{\alpha} g\) open and \(\hat{\alpha} g\) closed. Therefore, by theorem 3.6, \(A\) and \(B\) are \(\hat{\alpha} g\) separated. Hence \(X = A \cup B\), is \(\hat{\alpha} g\) disconnected.

Conversely, let \(X\) be \(\hat{\alpha} g\) disconnected. Then, there exists non-empty sets \(A\) and \(B\) such that \(A \cap \hat{\alpha} g\ cl B = \phi\), \(\hat{\alpha} g\ cl A \cap B = \phi\) and \(X = A \cup B\). Now \(A \subset \hat{\alpha} g\ cl A\). So, \(A \cap B = \phi\).

Hence \(A = B'\).

Since \(B\) is nonempty and \(B \cup X = X\), it follows \(B = A'\) is a proper subset of \(X\), \(A \cup \hat{\alpha} g\ cl B = X\) as \(A \cup B = X\). \(A \cap \hat{\alpha} g\ cl B = X\).

Also \(A \cap \hat{\alpha} g\ cl B = \phi\) \(\Rightarrow A = (\hat{\alpha} g\ cl B)'\) and similarly, \(B = (\hat{\alpha} g\ cl A)'.\) Since \(\hat{\alpha} g\ cl A\) and \(\hat{\alpha} g\ cl B\) are \(\hat{\alpha} g\) closed sets, \(B\) and \(A\) are \(\hat{\alpha} g\) open sets. Since \(A = B'\), \(A\) is also \(\hat{\alpha} g\) closed.

This completes the proof.

**Remark 4.5**: In the above theorem, we have also shown that \(B\) is also a proper subset of \(X\) which is both \(\hat{\alpha} g\) open and \(\hat{\alpha} g\) closed.

**Corollary 4.6**: A topological space \(X\) is \(\hat{\alpha} g\) connected if and only if the only nonempty subset of \(X\) which is both \(\hat{\alpha} g\) open and \(\hat{\alpha} g\) closed in \(X\) is \(X\) itself.

**Theorem 4.7**: A topological space \(X\) is \(\hat{\alpha} g\) disconnected if and only if any one of the following statement holds.

i) \(X\) is the union of two disjoint nonempty \(\hat{\alpha} g\) open sets.

ii) \(X\) is the union of two disjoint non empty \(\hat{\alpha} g\) closed sets.

**Proof**: Let \(X\) be \(\hat{\alpha} g\) disconnected. Then, there exists a non-empty proper subset \(A\) of \(X\) which is both \(\hat{\alpha} g\) open and \(\hat{\alpha} g\) closed. Then \(A'\) is also both \(\hat{\alpha} g\) open and \(\hat{\alpha} g\) closed. Also \(A \cup A' = X\).

Hence the sets \(A\) and \(A'\) satisfy the requirement of (i) and (ii).

Conversely, let \(X = A \cup B\) and \(A \cap B = \phi\) where \(A\) and \(B\) are nonempty \(\hat{\alpha} g\) open sets.

It follows that \(A = B'\), so that \(A\) is \(\hat{\alpha} g\) closed. Since \(B\) is nonempty, \(A\) is a proper subset of \(X\). Then \(A\) is a nonempty proper subset of \(X\). Then \(A\) is a nonempty proper subset of \(X\) which is both \(\hat{\alpha} g\) open and \(\hat{\alpha} g\) closed. Hence by theorem 4.4, \(X\) is \(\hat{\alpha} g\) disconnected. Similarly, assuming (ii) we can prove \(X\) is \(\hat{\alpha} g\) disconnected.

**Remark 4.8**: \(A \cup B\) is called a \(\hat{\alpha} g\) disconnection of \(X\).

**Definition 4.9**: Let \(A\) be a subset of a topological space \(X\). A point \(x\) of \(X\) is said to be \(\hat{\alpha} g\) exterior point of \(A\) if it is \(\hat{\alpha} g\) interior point of the complement \(A'\) of \(A\). The set of all \(\hat{\alpha} g\) exterior points of \(A\) is called the \(\hat{\alpha} g\) exterior of \(A\) and shall be denoted by \(\hat{\alpha} g\ ext (A)\).
Definition 4.10: A point $x$ of a topological space $X$ is said to be a $\hat{\alpha}g$ frontier point or $\hat{\alpha}g$ boundary point of a subset $A$ of $X$ if it is neither a $\hat{\alpha}g$ interior point nor $\hat{\alpha}g$ exterior point of $A$. The set of all $\hat{\alpha}g$ frontier points of $A$ is called the $\hat{\alpha}g$ frontier of $A$ and shall be denoted by $\hat{\alpha}g \text{Fr}(A)$.

Theorem 4.11: A topological space $X$ is $\hat{\alpha}g$ connected if and only if every nonempty proper subset of $X$ has a nonempty $\hat{\alpha}g$ frontier.

Proof: Let every nonempty proper subset of $X$ have a nonempty $\hat{\alpha}g$ frontier. Let us show $X$ is $\hat{\alpha}g$ connected. Let $X$ be $\hat{\alpha}g$ disconnected. Then, there exists nonempty disjoint sets $G$ and $H$ both $\hat{\alpha}g$ open and $\hat{\alpha}g$ closed in $X$ such that $X = G \cup H$. Therefore $G = \hat{\alpha}g \text{int} \ G = \hat{\alpha}g \ Cl \ G$. But $\hat{\alpha}g \text{Fr}(G) = \phi$, which is a contradiction to our hypothesis. Hence $X$ must be $\hat{\alpha}g$ connected.

Conversely, let $X$ be $\hat{\alpha}g$ connected. Let, if possible, there exists a nonempty proper subset of $A$ of $X$ such that $\hat{\alpha}g \text{Fr}(A) = \phi$ Now $\hat{\alpha}g \ cl \ A = \hat{\alpha}g \ int \ A \cup \hat{\alpha}g \ Fr(A)$ by theorem 6.2[7]

$\alpha \ = A \cup \hat{\alpha}g \ Fr(A)$ by theorem 6.4[7]

Hence $A = \hat{\alpha}g \ int \ A = \hat{\alpha}g \ cl \ A$, showing that $A$ is both $\hat{\alpha}g$ open and $\hat{\alpha}g$ closed. Therefore $X$ is $\hat{\alpha}g$ disconnected, which is a contradiction. This completes the proof.

Theorem 4.12: i) Every indiscrete topological space $X$, where $X$ is nonempty is $\hat{\alpha}g$ connected.

ii) Every discrete topological space $X$, where $X$ contains more than one point is $\hat{\alpha}g$ connected.

Proof: i) There is no proper subset of $X$, which is both $\hat{\alpha}g$ open and $\hat{\alpha}g$ closed. So, $X$ is $\hat{\alpha}g$ connected.

ii) $X$ contains more than one point. Every singleton subset of $X$ is a nonempty proper subset of $X$ which is both $\hat{\alpha}g$ open and $\hat{\alpha}g$ closed. Hence $X$ is $\hat{\alpha}g$ disconnected.

Theorem 4.13: Let $X$ be a topological space and let $E$ be $\hat{\alpha}g$ connected subset of $X$ such that $E \subseteq A \cup B$ where $A$ and $B$ are $\hat{\alpha}g$ separated sets. Then $E \subseteq A$ or $E \subseteq B$, that is $E$ cannot intersect both $A$ and $B$.

Proof: Since $A$ and $B$ are $\hat{\alpha}g$ separated, $A \cap \hat{\alpha}g \ cl \ B = \phi$ and $\hat{\alpha}g \ cl \ A \cap B = \phi$.

Now, $E \subseteq A \cup B \implies E = E \cap (A \cup B) = (E \cap A) \cup (E \cap B)$

Let us prove one of the sets $E \cap A$ or $E \cap B$ is empty. For, if possible, none of these sets is empty. That is, let $E \cap A \neq \phi$ and $E \cap B \neq \phi$.

Then $(E \cap A) \cap \hat{\alpha}g \ cl \ (E \cap B) = (E \cap A \cap \hat{\alpha}g \ cl \ B) \cap (E \cap \hat{\alpha}g \ cl \ B) = (E \cap \hat{\alpha}g \ cl \ B) \cap \phi = \phi$

Similarly, $\hat{\alpha}g \ cl \ (E \cap A) \cap (E \cap B) = \phi$

Hence $E \cap A$ and $E \cap B$ are $\hat{\alpha}g$ separated sets. So, $E$ is $\hat{\alpha}g$ disconnected, which is a contradiction.

Hence one of the sets $E \cap A$ or $E \cap B$ is empty. If $E \cap A = \phi$, then $E = E \cap B$, which implies $E \subseteq B$.

Corollary 4.14: If $E$ is a $\hat{\alpha}g$ connected subset of a space $X$ such that $E \subseteq A \cup B$, where $A$ and $B$ are disjoint $\hat{\alpha}g$ open and $\hat{\alpha}g$ closed subsets of $X$, then $A$ and $B$ are $\hat{\alpha}g$ separated.

Proof: A and $B$ are $\hat{\alpha}g$ open with $A \cap B = \phi$.

Then $A \cap B' \implies \hat{\alpha}g \ cl \ A \cap \hat{\alpha}g \ cl \ B' = B'$

$\implies \hat{\alpha}g \ cl \ A \cap B = \phi$

Similarly $A \cap \hat{\alpha}g \ cl \ B = \phi$

Hence $A$ and $B$ are $\hat{\alpha}g$ separated.

Theorem 4.15: Let $E$ be a $\hat{\alpha}g$ connected subset of a space $X$. If $F$ is a subset of $X$ such that $E \subseteq F \subseteq \hat{\alpha}g \ cl \ E$, then $F$ is $\hat{\alpha}g$ connected. In particular $\hat{\alpha}g \ cl \ E$ is $\hat{\alpha}g$ connected.

Proof: Let $F$ be $\hat{\alpha}g$ disconnected. Then there exist nonempty sets $A$ and $B$ such that $A \cap \hat{\alpha}g \ cl \ F = \phi$, $\hat{\alpha}g \ cl \ A \cap \hat{\alpha}g \ cl \ B = \phi$ and $A \cup B = F$.

Since $E \subseteq F \subseteq A \cup B$, it follows from theorem 4.13 that $E \subseteq A$ or $E \subseteq B$.

Let $E \subseteq A$. This implies $\hat{\alpha}g \ cl \ E \subseteq \hat{\alpha}g \ cl \ A$.

This implies $\hat{\alpha}g \ cl \ E \cap B \subseteq \hat{\alpha}g \ cl \ A \cap B = \phi$

That is $\hat{\alpha}g \ cl \ E \cap B = \phi$. Also $A \cup B = F \subseteq \hat{\alpha}g \ cl \ E$

$\implies \hat{\alpha}g \ cl \ E \subseteq \hat{\alpha}g \ cl \ E \cap B = B$
Hence $B = \emptyset$, a contradiction. So, $F$ must be $\hat{a}g$ connected. Again $E \subset \hat{a}g \text{cl } E \subset \hat{a}g \text{cl } E$, $\hat{a}g \text{cl } E$ is $\hat{a}g$ connected.

**Theorem 4.16**: If every two points of a subset $E$ of a topological space $X$ are contained in some $\hat{a}g$ connected subset of $E$, then $E$ is $\hat{a}g$ connected subset of $X$.

**Proof**: Let $E$ be not $\hat{a}g$ connected. Then there exist non empty subsets $A$ and $B$ of $X$ such that $A \cap \hat{a}g \text{cl } E = \emptyset$, $\hat{a}g \text{cl } A \cap B = \emptyset$ and $E = A \cup B$.

Since $A$ and $B$ are non empty, there exists a point $a \in A$ and a point $b \in B$. By hypothesis $a$ and $b$ must be contained in some $\hat{a}g$ connected subset $F$ of $E$. Since $F \subset A \cup B$, by theorem 4.13, $F \subset A$ or $F \subset B$. It follows $a$ and $b$ are both in $A$ or $B$. Let $a,b \in A$. As $b \in B$, we have $A \cap B \neq \emptyset$, a contradiction. Hence $E$ must be $\hat{a}g$ connected.

**Theorem 4.17**: A is a $\hat{a}g$ connected subset of a $\hat{a}g$ connected topological space $X$ such that $A'$ is the union of two $\hat{a}g$ separated sets $B$ and $C$. Then $A \cup B$ and $A \cup C$ are $\hat{a}g$ connected.

**Proof**: Let $A \cup B$ be $\hat{a}g$ disconnected. Then, there exist two nonempty $\hat{a}g$ separated sets $G$ and $H$, whose union is $A \cup B$. $A$ is a $\hat{a}g$ connected subset of $\hat{a}g$ disconnected set $A \cup B$ having the $\hat{a}g$ separation $A \cup = \text{G} \cup \text{H}$. Then by theorem 4.13, either $A \subset G$ or $A \subset H$.

Suppose that $A \subset G$.

Now, since $A \cup B = G \cup H$.

$A \cap G \Rightarrow A \cup B \subset G \cup B \Rightarrow G \cup H \subset G \cup B \Rightarrow H \subset B$

Since $B$ and $C$ are $\hat{a}g$ separated sets and $H \subset B$, $C \subset C$, it follows from theorem 3.3, that $H$ and $C$ are also $\hat{a}g$ separated. Thus the set $H$ is $\hat{a}g$ separated from $G$ as well as $C$ and therefore $H$ is $\hat{a}g$ separated from $G \cup C$.

But $A' = B \cup C \Rightarrow X = A \cup A' = A \cup (B \cup C)$

$A \cup (B \cup C) \Rightarrow X = (G \cup H) \cup C = (G \cup C) \cup H$

Consequently, $X$ has been expressed as the union of nonempty $\hat{a}g$ separated sets. This implies $X$ is $\hat{a}g$ disconnected, a contradiction.

Similar contradiction will arise if $A \subset H$.

Hence $A \cup$ is $\hat{a}g$ connected. Similarly $A \cup C$ is $\hat{a}g$ connected.

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**Definition 5.1**: Let $X$ and $Y$ be topological spaces. A mapping $f : X \to Y$ is said to be $\hat{a}g$ continuous if and only if the inverse image of every open set of $Y$ is $\hat{a}g$ open in $X$.

**Theorem 5.2**: If $f$ is a $\hat{a}g$ continuous mapping of a $\hat{a}g$ connected space $X$ onto an arbitrary topological space $Y$, then $Y$ is connected.

**Proof**: Let $Y$ be disconnected. Then there exists a nonempty proper subset $G$ of $Y$ which is both open and closed in $Y$. Since $f$ is $\hat{a}g$ continuous and onto $Y$, $f^{-1}(G)$ is a nonempty proper subset of $X$ which is both $\hat{a}g$ open and $\hat{a}g$ closed in $X$. Therefore $X$ is $\hat{a}g$ disconnected, a contradiction. Hence $Y$ must be connected.

**Theorem 5.3**: A topological space $X$ is $\hat{a}g$ disconnected if and only if there exists a $\hat{a}g$ continuous mapping of $X$ onto the discrete two point space $\{0,1\}$

**Proof**: Let $X$ be $\hat{a}g$ disconnected. Then, there exist two nonempty disjoint $\hat{a}g$ open subsets $G_1$ and $G_2$ of $X$ such that $X = G_1 \cup G_2$. Define a mapping $f$ of $X$ onto $\{0,1\}$ by setting $f(x) = 0$ if $x \in G_1$ and $f(x) = 1$ if $x \in G_2$. Since $\{0,1\}$ is discrete, its open sets are $\emptyset$, $\{0\}$, $\{1\}$ and $\{0,1\}$

$f^{-1}(\{0\}) = G_1$, $f^{-1}(\{1\}) = G_2$, $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(\{0,1\}) = X$

Hence $f$ is $\hat{a}g$ continuous.

Conversely, let there exists a $\hat{a}g$ continuous mapping of $X$ onto the discrete space $\{0,1\}$. If $X$ is $\hat{a}g$ connected, then $\{0,1\}$ is connected by the above theorem. But this is impossible, since every discrete space is disconnected.
Theorem 5.4 : A space $\hat{\alpha}$ connected if and only if every $\hat{\alpha}$ continuous function $f$ from $X$ into the discrete two points space $\{0,1\}$ is constant.

Proof : Let $X$ be $\hat{\alpha}$ connected and let $y \in f(x) \subset \{0,1\}$, where $f : X \rightarrow \{0,1\}$ is any $\hat{\alpha}$ continuous function. Then $\{y\}$ is both open and closed in the discrete space $\{0,1\}$. $f$ is $\hat{\alpha}$ continuous. Therefore $f^{-1}(\{y\})$ is $\hat{\alpha}$ open and $\hat{\alpha}$ closed and nonempty. Since $X$ is $\hat{\alpha}$ connected, $f^{-1}(\{y\}) = X$. Therefore $f$ is a constant function.

Conversely, let every $\hat{\alpha}$ continuous function $f : X \rightarrow \{0,1\}$ be constant. Suppose, if possible, $X$ be $\hat{\alpha}$ disconnected. Then, there exists a nonempty proper subset $A$ of $X$ which is both $\hat{\alpha}$ open and $\hat{\alpha}$ closed in $X$. Consider the characteristic function $K_A$ of $A$ defined by

$$K_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \in A' \end{cases}$$

Then $K_A^{-1}(\emptyset) = \emptyset$, $K_A^{-1}(\{1\}) = A$, $K_A^{-1}(\{0\}) = A'$, $K_A^{-1}(\{0,1\}) = X$. Therefore $K_A$ is $\hat{\alpha}$ continuous.

But $K_A$ is non constant, a contradiction. Hence $X$ must be $\hat{\alpha}$ connected.

Definition 5.5 : A map $f : X \rightarrow Y$ is called $\hat{\alpha}$ irresolute if $f^{-1}(V)$ is $\hat{\alpha}$ closed in $X$ for every $\hat{\alpha}$ closed set $V$ of $Y$. It is called irresolute if $f^{-1}(V)$ is open in $X$ for every open set $V$ of $Y$.

Theorem 5.6 : A map $f : X \rightarrow Y$ is $\hat{\alpha}$ irresolute if for every $\hat{\alpha}$ open set $A$ of $Y$, $f^{-1}(A)$ is $\hat{\alpha}$ open in $X$.

Proof : obvious

Theorem 5.7 : Let $f : X \rightarrow Y$ be a map. Then, if $X$ is $\hat{\alpha}$ connected and $f$ is $\hat{\alpha}$ irresolute surjective, then $Y$ is $\hat{\alpha}$ connected.

Proof : Let $Y$ be $\hat{\alpha}$ disconnected. Then $Y = A \cup B$, where $A$ and $B$ are disjoint nonempty $\hat{\alpha}$ open subsets of $Y$. Since $f$ is $\hat{\alpha}$ irresolute surjective, $X = f^{-1}(A) \cup f^{-1}(B)$. $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint nonempty $\hat{\alpha}$ open subset of $X$. So, $X$ is $\hat{\alpha}$ disconnected, a contradiction. This completes the proof.

References