ON $\alpha g$ CONNECTED SPACES

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Abstract. In this paper, we define $\alpha g$ connected spaces and derive some of its properties.

1. Introduction

Throughout this paper, a space means a topological space on which no separation axioms are assumed unless explicitly stated. In 1963, Levine [4] initiated semiopen sets and studied their properties. Mathematicians gave several papers containing interesting results about new types of sets. Njastad [5] defined $\alpha$ open sets. Senthilkumaran et al [6] defined a new type of sets called $\alpha g$ closed sets and studied their properties. In this paper we define $\alpha g$ connected spaces and derive some of its properties.

2. Preliminaries

Definition 2.1: A subset $A$ of a topological space $X$ is said to be
1) A preopen set if $A \subseteq \text{int cl } A$ and a preclosed set if $\text{cl int } A \subseteq A$
2) A regular open set if $A = \text{int cl } A$ and a regular closed set if $A = \text{cl int } A$.
3) A semiopen set if $A \subseteq \text{cl int } A$ and a semi closed set if $\text{int cl } A \subseteq A$
4) A $\alpha$-open set if $A \subseteq \text{int cl } \text{int } A$ and a $\alpha$-closed set if $\text{cl int cl } A \subseteq A$.

Definition 2.2: A subset $A$ of a topological space $X$ is called $\alpha g$ closed set if $\text{int cl cl int } A \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $X$.

The complement of $\alpha g$ closed set in $X$ is called $\alpha g$ open set in $X$.

The intersection of all $\alpha g$ closed sets containing $A$ is called $\alpha g$ closure of $A$ and shall be denoted by $\alpha g \text{ cl } A$. In general $\alpha g \text{ cl } A$ is not $\alpha g$ closed in $X$.

The union of all $\alpha g$ open sets that are contained in $A$ is called $\alpha g$ interior of $A$ and shall be denoted by $\alpha g \text{ int } A$. In general, $\alpha g \text{ int } A$ is not $\alpha g$ open in $X$.

In what follows, let us assume arbitrary intersection of $\alpha g$ closed sets in $X$ is $\alpha g$ closed in $X$.

Then $\alpha g \text{ cl } A$ will be $\alpha g$ closed in $X$ and $\alpha g \text{ int } A$ is $\alpha g$ open in $X$.

3. $\alpha g$ Separated sets

Definition 3.1: Let $X$ be a topological space. Two nonempty subsets $A$ and $B$ of $X$ are said to be $\alpha g$ separated if and only if $A \cap \alpha g \text{ cl } B = \phi$ and $\alpha g \text{ cl } A \cap B = \phi$.

These two conditions are equivalent to a single condition $(A \cap \alpha g \text{ cl } B) \cup (\alpha g \text{ cl } A \cap B) = \phi$.

Remark 3.2: $A$ and $B$ are $\alpha g$ separated if and only if $A$ and $B$ are disjoint and neither of them contains the $\alpha g$ limit point of the other.

Theorem 3.3: If $A$ and $B$ are $\alpha g$ separated subsets of $X$ and $C \subseteq A$ and $D \subseteq B$, then $C$ and $D$ are $\alpha g$ separated.

Proof: $A \cap \alpha g \text{ cl } B = \phi$ and $\alpha g \text{ cl } A \cap B = \phi$.

$C \subseteq A \Rightarrow \alpha g \text{ cl } C \subseteq \alpha g \text{ cl } A$ and $D \subseteq B \Rightarrow \alpha g \text{ cl } D \subseteq \alpha g \text{ cl } B$

$C \cap \alpha g \text{ cl } D \subseteq A \cap \alpha g \text{ cl } B = \phi$
Similarly, \( \hat{a} \) g cl \( A \cap B = \phi \). Hence C and D are \( \hat{a} \) g separated.

**Theorem 3.4** : Two \( \hat{a} \) g closed (\( \hat{a} \) g open) subset of the topological space \( X \) are \( \hat{a} \) g separated if and only if they are disjoint.

**Proof** : Since any two \( \hat{a} \) g separated sets are disjoint, we need only to show that two disjoint \( \hat{a} \) g closed (\( \hat{a} \) g open) sets are \( \hat{a} \) g separated. Let \( A \) and \( B \) be disjoint and \( \hat{a} \) g closed. Then \( A \cap B = \phi \), \( \hat{a} \) g cl \( A = A \), \( \hat{a} \) g cl \( B = B \). So, \( \hat{a} \) g cl \( A \cap B = \phi \) and \( A \cap A = \hat{a} \) g cl \( B = \phi \).

Hence \( A \) and \( B \) are \( \hat{a} \) g separated. Let \( A \) and \( B \) be disjoint and \( \hat{a} \) g open. Then \( \hat{a} \) g cl \( A' = A' \) and \( \hat{a} \) g cl \( B' = B' \). \( A \cap B = \phi \) \( \Rightarrow \) \( A \subset B' \) and \( B \subset A' \)
\[ \Rightarrow \hat{a} \) g cl \( A' \subset \hat{a} \) g cl \( B' \) and \( \hat{a} \) g cl \( B' \subset \hat{a} \) g cl \( A' = A' \)
\[ \Rightarrow \hat{a} \) g cl \( A \cap B = \phi \) and \( A \cap \) \( \hat{a} \) g cl \( B = \phi \)
Hence \( A \) and \( B \) are \( \hat{a} \) g separated.

**Theorem 3.5** : If \( A \) and \( B \) are \( \hat{a} \) g separated sets of a topological space \( X \), then

i) \( A \cup B \) is \( \hat{a} \) g closed \( \Rightarrow \) \( A \) and \( B \) are \( \hat{a} \) g closed
ii) \( A \cup B \) is \( \hat{a} \) g open \( \Rightarrow \) \( A \) and \( B \) are \( \hat{a} \) g open

**Proof** : Let \( A \) and \( B \) be \( \hat{a} \) g separated sets of a topological space \( X \) so that \( A \neq \phi \) and \( B \neq \phi \).
\( \hat{a} \) g cl \( A \cap B = \phi \) and \( A \cap \) \( \hat{a} \) g cl \( B = \phi \)
Let \( A \cup B \) be \( \hat{a} \) g closed
\( \hat{a} \) g cl \( (A \cup B) = A \cup B \).
That is \( \hat{a} \) g cl \( A \cup \) \( \hat{a} \) g cl \( B = A \cup B \)
\( \hat{a} \) g cl \( A = \hat{a} \) g cl \( A \cap (\hat{a} \) g cl \( A \cup \) \( \hat{a} \) g cl \( B) = \hat{a} \) g cl \( A \cap (A \cup B) \)
\[ = \hat{a} \) g cl \( A \cap (A \cup B) \)
\[ = (\hat{a} \) g cl \( A \cap A) \cup (\hat{a} \) g cl \( A \cap B) \]
\[ = A \cup \phi = A \]
Hence \( A \) is \( \hat{a} \) g closed. Similarly, \( B \) is \( \hat{a} \) g closed.

Let \( A \cup B \) be \( \hat{a} \) g open.
\( \hat{a} \) g cl \( B \) is \( \hat{a} \) g closed. Hence \( X - \hat{a} \) g cl \( B \) is \( \hat{a} \) g open
\( (A \cup B) \cap (X - \hat{a} \) g cl \( B) = (A \cap (X - \hat{a} \) g cl \( B)) \cup (B \cap (X - \hat{a} \) g cl \( B)) \)
\[ = A \cup \phi = A \]
Hence \( A \) is \( \hat{a} \) g open. Similarly \( B \) is \( \hat{a} \) g open.

**Theorem 3.6** : Two disjoint sets \( A \) and \( B \) are \( \hat{a} \) g separated in a topological space \( X = A \cup B \) if and only if they are both \( \hat{a} \) g open and \( \hat{a} \) g closed in \( X \).

**Proof** : Let disjoint sets \( A \) and \( B \) be \( \hat{a} \) g separated in \( X \), \( A \cap \) \( \hat{a} \) g cl \( B = \phi \) and \( \hat{a} \) g cl \( A \cap B = \phi \)
\( X = A \cup B \). \( \hat{a} \) g cl \( A = \hat{a} \) g cl \( A \cap X = \hat{a} \) g cl \( A \cap (A \cup B) \)
\[ = (\hat{a} \) g cl \( A \cap A) \cup (\hat{a} \) g cl \( A \cap B) \]
\[ = A \cup \phi = A \]
Hence \( A \) is \( \hat{a} \) g closed. Similarly \( B \) is \( \hat{a} \) g closed.

Since \( A \) and \( B \) are disjoint and \( A \cup B = X \), \( B \) is \( \hat{a} \) g open. Similarly \( A \) is \( \hat{a} \) g open.
Conversely, let disjoint sets \( A \) and \( B \) be both \( \hat{a} \) g open and \( \hat{a} \) g closed in \( X = A \cup B \).
\( A \) is \( \hat{a} \) g closed in \( X \). So, \( A = \hat{a} \) g cl \( A \cap X = \hat{a} \) g cl \( A \cap (A \cup B) \)
\[ = \hat{a} \) g cl \( A \cap (A \cup B) \)
\[ = A \cup (\hat{a} \) g cl \( A \cap B) \]
\[ A \cup B = \phi \Rightarrow A \cup (\hat{a} \) g cl \( A \cap B) = \phi \Rightarrow \hat{a} \) g cl \( A \cap B = \phi \)
Similarly, \( A \cap \hat{a} \) g cl \( B = \phi \)
So, \( A \) and \( B \) are \( \hat{a} \) g separated.

**Theorem 3.7** : If \( A \) and \( B \) are proper subsets of a space \( X \) and both \( A \) and \( B \) are \( \hat{a} \) g closed and \( \hat{a} \) g open, then \( A - B \) is \( \hat{a} \) g separated from \( B - A \).

**Proof** : \( (A - B) \cap (B - A) = \phi \) and \( A - B \) and \( B - A \) are non empty.
\( (A - B) \cap \) \( \hat{a} \) g cl \( (B - A) = (A \cap B') \cap \) \( \hat{a} \) cl \( (B \cap A') \subset (A \cap B') \cap (\hat{a} \) g cl \( B \cap \hat{a} \) g cl \( A') \)
\[ = (A \cap B' \cap \hat{a} \) g cl \( B) \cap (A \cap B' \cap \hat{a} \) g cl \( A') \]
=(A \cap B' \cap B) \cap (A \cap B' \cap A') = \phi 
Similarly, \( \hat{A} \hat{g} \text{ cl} (A - B) \cap (B - A) = \phi \)
Hence \( A - B \) and \( B - A \) are \( \hat{A} \hat{g} \) separated.

4. \( \hat{A} \hat{g} \) Connected and \( \hat{A} \hat{g} \) Disconnected sets

**Definition 4.1:** Let \( X \) be a topological space. A subset \( A \) of \( X \) is said to be \( \hat{A} \hat{g} \) disconnected if and only if it is the union of two non empty \( \hat{A} \hat{g} \) separated sets. That is, if and only if there exists non empty sets \( C \) and \( D \) such that \( C \cap \hat{A} \hat{g} \text{ cl} \ D = \phi \), \( \hat{A} \hat{g} \text{ cl} C \cap D = \phi \) and \( A = C \cup D \). \( A \) is said to be \( \hat{A} \hat{g} \) connected if and only if it is not \( \hat{A} \hat{g} \) disconnected.

**Remark 4.2:** The empty set is trivially \( \hat{A} \hat{g} \) connected.
Also, every singleton set is \( \hat{A} \hat{g} \) connected.

**Definition 4.3:** Two points \( a \) and \( b \) of a topological space \( X \) is \( \hat{A} \hat{g} \) connected if and only if they are contained in a \( \hat{A} \hat{g} \) connected subset of \( X \).

**Theorem 4.4:** A topological space \( X \) is \( \hat{A} \hat{g} \) disconnected if and only if there exists a non empty proper subset of \( X \) which is both \( \hat{A} \hat{g} \) open and \( \hat{A} \hat{g} \) closed in \( X \).

**Proof:** Let \( A \) be a non empty proper subset of \( X \) which is both \( \hat{A} \hat{g} \) open and \( \hat{A} \hat{g} \) closed. Let us show that \( X \) is \( \hat{A} \hat{g} \) disconnected. Let \( B = A' \). As \( A \) is a proper subset, \( B \) is nonempty. Moreover, \( A \cup B = X \) and \( A \cap B = \phi \). As \( A \) is both \( \hat{A} \hat{g} \) open and \( \hat{A} \hat{g} \) closed, \( B \) is also both \( \hat{A} \hat{g} \) open and \( \hat{A} \hat{g} \) closed. Therefore, by theorem 3.6, \( A \) and \( B \) are \( \hat{A} \hat{g} \) separated. Hence \( X = A \cup B \), is \( \hat{A} \hat{g} \) disconnected.

Conversely, let \( X \) be \( \hat{A} \hat{g} \) disconnected. Then, there exists nonempty sets \( A \) and \( B \) such that \( A \cap \hat{A} \hat{g} \text{ cl} B = \phi \), \( \hat{A} \hat{g} \text{ cl} A \cap B = \phi \) and \( X = A \cup B \). Now \( A \cap \hat{A} \hat{g} \text{ cl} A \). So, \( A \cup B = \phi \)

Hence \( A = B' \)
Since \( B \) is nonempty and \( B \cup X = X \), it follows \( B = A' \) is a proper subset of \( X \),
Now \( A \cup \hat{A} \hat{g} \text{ cl} B = X \Rightarrow A \cup \hat{A} \hat{g} \text{ cl} B = X \)
Also, \( A \cap \hat{A} \hat{g} \text{ cl} B = \phi \Rightarrow A = (\hat{A} \hat{g} \text{ cl} B)' \) and similarly, \( B = (\hat{A} \hat{g} \text{ cl} A)' \). Since \( \hat{A} \hat{g} \text{ cl} A \) and \( \hat{A} \hat{g} \text{ cl} B \) are \( \hat{A} \hat{g} \) closed sets, \( B \) and \( A \) are \( \hat{A} \hat{g} \) open sets. Since \( A = B' \), \( A \) is also \( \hat{A} \hat{g} \) closed.
This completes the proof.

**Remark 4.5:** In the above theorem, we have also shown that \( B \) is also a proper subset of \( X \) which is both \( \hat{A} \hat{g} \) open and \( \hat{A} \hat{g} \) closed.

**Corollary 4.6:** A topological space \( X \) is \( \hat{A} \hat{g} \) connected if and only if the only nonempty subset of \( X \) which is both \( \hat{A} \hat{g} \) open and \( \hat{A} \hat{g} \) closed in \( X \) is \( X \) itself.

**Theorem 4.7:** A topological space \( X \) is \( \hat{A} \hat{g} \) disconnected if and only if any one of the following statement holds.
i) \( X \) is the union of two disjoint nonempty \( \hat{A} \hat{g} \) open sets.
ii) \( X \) is the union of two disjoint non empty \( \hat{A} \hat{g} \) closed sets.

**Proof:** Let \( X \) be \( \hat{A} \hat{g} \) disconnected. Then, there exists a nonempty proper subset \( A \) of \( X \) which is both \( \hat{A} \hat{g} \) open and \( \hat{A} \hat{g} \) closed. \( A' \) is also both \( \hat{A} \hat{g} \) open and \( \hat{A} \hat{g} \) closed. Also \( A \cup A' = X \).

Hence the sets \( A \) and \( A' \) satisfy the requirement of (i) and (ii).

Conversely, let \( X = A \cup B \) and \( A \cap B = \phi \) where \( A \) and \( B \) are nonempty \( \hat{A} \hat{g} \) open sets.
It follows that \( A = B' \), so that \( A = \hat{A} \hat{g} \) closed. Since \( B \) is nonempty, \( A \) is a proper subset of \( X \). Then \( A \) is a nonempty open subset of \( X \). Then \( A \) is a nonempty proper subset of \( X \) which is both \( \hat{A} \hat{g} \) open and \( \hat{A} \hat{g} \) closed. Hence by theorem 4.4, \( X \) is \( \hat{A} \hat{g} \) disconnected. Similarly, assuming (ii) we can prove \( X \) is \( \hat{A} \hat{g} \) disconnected.

**Remark 4.8:** \( A \cup B \) is called a \( \hat{A} \hat{g} \) disconnection of \( X \).

**Definition 4.9:** Let \( A \) be a subset of a topological space \( X \). A point \( x \) of \( X \) is said to be \( \hat{A} \hat{g} \) exterior point of \( A \) if it is \( \hat{A} \hat{g} \) interior point of the complement \( A' \) of \( A \). The set of all \( \hat{A} \hat{g} \) exterior points of \( A \) is called the \( \hat{A} \hat{g} \) exterior of \( A \) and shall be denoted by \( \hat{A} \hat{g} \text{ ext} (A) \).
Definition 4.10: A point \( x \) of a topological space \( X \) is said to be a \( \hat{a} \) g frontier point or \( \hat{a} \) g boundary point of a subset \( A \) of \( X \) if it is neither a \( \hat{a} \) g interior point nor \( \hat{a} \) g exterior point of \( A \). The set of all \( \hat{a} \) g frontier points of \( A \) is called the \( \hat{a} \) g frontier of \( A \) and shall be denoted by \( \hat{a} \) g Fr\( (A) \).

Theorem 4.11: A topological space \( X \) is \( \hat{a} \) g connected if and only if every nonempty proper subset of \( X \) has a nonempty \( \hat{a} \) g frontier.

**Proof:** Let every nonempty proper subset of \( X \) have a nonempty \( \hat{a} \) g frontier. Let us show \( X \) is \( \hat{a} \) g connected. Let \( X \) be \( \hat{a} \) g disconnected. Then, there exists nonempty disjoint sets \( G \) and \( H \) both \( \hat{a} \) g open and \( \hat{a} \) g closed in \( X \) such that \( X = G \cup H \). Therefore \( G = \hat{a} \) g int \( G = \hat{a} \) g cl \( G \). But \( \hat{a} \) g Fr\( (G) \) = \( \emptyset \), which is a contradiction to our hypothesis. Hence \( X \) must be \( \hat{a} \) g connected.

Conversely, let \( X \) be \( \hat{a} \) g connected. Let, if possible, there exists a nonempty proper subset \( A \) of \( X \) such that \( \hat{a} \) g Fr\( (A) \) = \( \emptyset \). Now \( \hat{a} \) g cl \( A \) = \( \hat{a} \) g int \( A \cup \hat{a} \) g Fr\( (A) \) by theorem 6.2[7]

\[
\text{Hence } A = \hat{a} \text{ g int } A = \hat{a} \text{ g cl } A, \text{ showing that } A \text{ is both } \hat{a} \text{ g open and } \hat{a} \text{ g closed. Therefore } X \text{ is } \hat{a} \text{ g disconnected, which is a contradiction. This completes the proof.}

Theorem 4.12: i) Every indiscrete topological space \( X \), where \( X \) is nonempty is \( \hat{a} \) g connected.

ii) Every discrete topological space \( X \), where \( X \) contains more than one point is \( \hat{a} \) g connected.

**Proof** i) There is no proper subset of \( X \), which is both \( \hat{a} \) g open and \( \hat{a} \) g closed. So, \( X \) is \( \hat{a} \) g connected.

ii) \( X \) contains more than one point. Every singleton subset of \( X \) is a nonempty proper subset of \( X \) which is both \( \hat{a} \) g open and \( \hat{a} \) g closed. Hence \( X \) is \( \hat{a} \) g disconnected.

Theorem 4.13: Let \( X \) be a topological space and let \( E \) be \( \hat{a} \) g connected subset of \( X \) such that \( E \cap A \cup B \) where \( A \) and \( B \) are \( \hat{a} \) g separated sets. Then \( E \cap A \) or \( E \cap B \), that is \( E \) cannot intersect both \( A \) and \( B \).

**Proof:** Since \( A \) and \( B \) are \( \hat{a} \) g separated, \( A \cap \hat{a} \) g cl \( B \) = \( \emptyset \) and \( \hat{a} \) g cl \( A \cap B \) = \( \emptyset \)

Now, \( E \cap (A \cup B) \Rightarrow E = E \cap (A \cup B) = (E \cap A) \cup (E \cap B) \)

Let us prove one of the sets \( E \cap A \) or \( E \cap B \) is empty. For, if possible, none of these sets is empty.

That is, let \( E \cap A \neq \emptyset \) and \( E \cap B \neq \emptyset \)

Then \( (E \cap A) \cap \hat{a} \) g cl \( (E \cap B) \subset (E \cap A) \cap (\hat{a} \) g cl \( E \cap \hat{a} \) g cl \( B \) ) = \( (E \cap \hat{a} \) g cl \( E \cap \hat{a} \) g cl \( B \) ) \cap \( (A \cap (\hat{a} \) g cl \( E \cap \hat{a} \) g cl \( B \) )) \cap \( (A \cap (\hat{a} \) g cl \( E \cap \hat{a} \) g cl \( B \)) \cap \( (E \cap \hat{a} \) g cl \( B \) \cap \( \emptyset \) = \( \emptyset \)

Similarly, \( \hat{a} \) g cl \( (E \cap A) \cap (E \cap B) = \emptyset \)

Hence \( E \cap A \) and \( E \cap B \) are \( \hat{a} \) g separated sets. So, \( E \) is \( \hat{a} \) g disconnected, which is a contradiction.

Hence one of the sets \( E \cap A \) or \( E \cap B \) is empty.

If \( E \cap A = \emptyset \), then \( E = E \cap B \), which implies \( E \cap B \).

Corollary 4.14: If \( E \) is a \( \hat{a} \) g connected subset of a space \( X \) such that \( E \cap A \cup B \), where \( A \) and \( B \) are disjoint \( \hat{a} \) g open and \( \hat{a} \) g closed subsets of \( X \), then \( A \) and \( B \) are \( \hat{a} \) g separated.

**Proof:** \( A \) and \( B \) are \( \hat{a} \) g open with \( A \cap B = \emptyset \)

Then \( A \subseteq B' \Rightarrow \hat{a} \) g cl \( A \subseteq \hat{a} \) g cl \( B' = B' \)

\( \Rightarrow \hat{a} \) g cl \( A \cap B = \emptyset \)

Similarly, \( A \cap \hat{a} \) g cl \( B = \emptyset \)

Hence \( A \) and \( B \) are \( \hat{a} \) g separated.

Theorem 4.15: Let \( E \) be a \( \hat{a} \) g connected subset of a space \( X \). If \( F \) is a subset of \( X \) such that \( E \cap F \subseteq \hat{a} \) g cl \( E \), then \( F \) is \( \hat{a} \) g connected. In particular \( \hat{a} \) g cl \( E \) is \( \hat{a} \) g connected.

**Proof:** Let \( F \) be \( \hat{a} \) g disconnected. Then there exist nonempty sets \( A \) and \( B \) such that \( A \cap \hat{a} \) g cl \( B = \emptyset \), \( \hat{a} \) g cl \( A \cap B = \emptyset \) and \( A \cup B = F \),

Since \( E \cap F = A \cup B \), it follows from theorem 4.13 that \( E \cap A \) or \( E \cap B \).

Let \( E \subseteq A \). This implies \( \hat{a} \) g cl \( E \cap \hat{a} \) g cl \( A \),

This implies \( \hat{a} \) g cl \( E \cap B \subseteq \hat{a} \) g cl \( A \cap B = \emptyset \)

That is \( \hat{a} \) g cl \( E \cap B = \emptyset \). Also \( A \cup = F \subseteq \hat{a} \) g cl \( E \)

\( \Rightarrow B \subseteq F \subseteq \hat{a} \) g cl \( E \Rightarrow \hat{a} \) g cl \( E \cap B = B \)
Hence B = φ, a contradiction. So, F must be āg connected. Again E ⊂ āg cl E ⊂ āg cl E, āg cl E is āg connected.

**Theorem 4.16**: If every two points of a subset E of a topological space X are contained in some āg connected subset of E, then E is āg connected subset of X.

**Proof**: Let E be not āg connected. Then there exist non empty subsets A and B of X such that A ∩ āg cl B = φ, āg cl A ∩ B = φ and E = A ∪ B.

Since A and B are non empty, there exists a point a ∈ A and a point b ∈ B. By hypothesis a and b must be contained in some āg connected subset F of E. Since F ⊂ A ∪ B, by theorem 4.13, F ⊂ A or F ⊂ B. It follows a and b are both in A or B. Let a, b ∈ A. As b ∈ B, we have A ∩ B = φ, a contradiction. Hence E must be āg connected.

**Theorem 4.17**: A is a āg connected subset of a āg connected topological space X such that A' is the union of two āg separated sets B and C. Then A ∪ B and A ∪ C are āg connected.

**Proof**: Let A ∪ B be āg disconnected. Then, there exist two nonempty āg separated sets G and H, whose union is A ∪ B. A is a āg connected subset of āg disconnected set A ∪ B having the āg separation A ∪ = G ∪ H. Then by theorem 4.13, either A ⊂ G or A ⊂ H.

Suppose that A ⊂ G.

Now, since A ∪ B = G ∪ H

AcG ⇒ A ∪ B ⊂ G ∪ B ⇒ G ∪ H ⊂ G ∪ B ⇒ H ⊂ B

Since B and C are āg separated sets and H ⊂ B, C ⊂ C, it follows from theorem 3.3, that H and C are also āg separated. Thus the set H is āg separated from G as well as C and therefore H is āg separated from G ⊂ C

But A' = B ∪ C ⇒ X = A ∪ A' = A ∪ (B ∪ C)

⇒ X = (A ∪ B) ∪ C ⇒ X = (G ∪ H) ∪ C ⇒ X = (G ∪ C) ∪ H

Consequently, X has been expressed as the union of nonempty āg separated sets. This implies X is āg disconnected, a contradiction.

Similar contradiction will arise if A ⊂ H.

Hence A ∪ is āg connected. Similarly A ∪ C is āg connected.

**āg continuity and āg connectedness**

**Definition 5.1**: Let X and Y be topological spaces. A mapping f : X → Y is said to be āg continuous if and only if the inverse image of every open set of Y is āg open in X.

**Theorem 5.2**: If f is a āg continuous mapping of āg connected space X onto an arbitrary topological space Y, then Y is connected.

**Proof**: Let Y be disconnected. Then there exists a nonempty proper subset G of Y which is both open and closed in Y. Since f is āg continuous and onto Y, f−1(G) is a nonempty proper subset of X which is both āg open and āg closed in X. Therefore X is āg disconnected, a contradiction. Hence Y must be connected.

**Theorem 5.3**: A topological space X is āg disconnected if and only if there exists a āg continuous mapping of X onto the discrete two point space {0,1}.

**Proof**: Let X be āg disconnected. Then, there exist two nonempty disjoint āg open subsets G1 and G2 of X such that X = G1 ∪ G2. Define a mapping f of X onto {0,1} by setting f(x) = 0 if x ∈ G1 and f(x) = 1 if x ∈ G2. Since {0,1} is discrete, its open sets are φ, {0}, {1} and {0,1} 

f−1({0}) = G1, f−1({1}) = G2, f−1(φ) = φ, f−1({0,1}) = X

Hence f is āg continuous.

Conversely, let there exists a āg continuous mapping of X onto the discrete space {0,1}.

If X is āg connected, then {0,1} is connected by the above theorem. But this is impossible, since every discrete space is disconnected.
Theorem 5.4: A space $X$ is $\alpha g$ connected if and only if every $\alpha g$ continuous function $f$ from $X$ into the discrete two points space $\{0,1\}$ is constant.

Proof: Let $X$ be $\alpha g$ connected and let $y \in f(x) \subset \{0,1\}$, where $f : X \rightarrow \{0,1\}$ is any $\alpha g$ continuous function. Then $\{y\}$ is both open and closed in the discrete space $\{0,1\}$. Therefore $f^{-1}(\{y\})$ is $\alpha g$ open and $\alpha g$ closed and nonempty. Since $X$ is $\alpha g$ connected, $f^{-1}(\{y\}) = X$. Therefore $f$ is a constant function.

Conversely, let every $\alpha g$ continuous function $f : X \rightarrow \{0,1\}$ be constant.

Suppose, if possible, $X$ be $\alpha g$ disconnected. Then, there exists a nonempty proper subset $A$ of $X$ which is both $\alpha g$ open and $\alpha g$ closed in $X$. Consider the characteristic function $K_A$ of $A$ defined by

$$K_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \in A' \end{cases}$$

Then $K_A^{-1}(\emptyset) = \emptyset$, $K_A^{-1}(\{1\}) = A$, $K_A^{-1}(\{0\}) = A'$, $K_A^{-1}(\{0,1\}) = X$. Therefore $K_A$ is $\alpha g$ continuous.

But $K_A$ is non constant, a contradiction.

Hence $X$ must be $\alpha g$ connected.

Definition 5.5: A map $f : X \rightarrow Y$ is called $\alpha g$ irresolute if $f^{-1}(V)$ is $\alpha g$ closed in $X$ for every $\alpha g$ closed set $V$ of $Y$. It is called irresolute if $f^{-1}(V)$ is open in $X$ for every open set $V$ of $Y$.

Theorem 5.6: A map $f : X \rightarrow Y$ is $\alpha g$ irresolute if for every $\alpha g$ open set $A$ of $Y$, $f^{-1}(A)$ is $\alpha g$ open in $X$.

Proof: obvious

Theorem 5.7: Let $f : X \rightarrow Y$ be a map. Then, if $X$ is $\alpha g$ connected and $f$ is $\alpha g$ irresolute surjective, then $Y$ is $\alpha g$ connected.

Proof: Let $Y$ be $\alpha g$ disconnected. Then $Y = A \cup B$, where $A$ and $B$ are disjoint nonempty $\alpha g$ open subsets of $Y$. Since $f$ is $\alpha g$ irresolute surjective, $X = f^{-1}(A) \cup f^{-1}(B)$. $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint nonempty $\alpha g$ open subset of $X$. So, $X$ is $\alpha g$ disconnected, a contradiction. This completes the proof.

References