Observations on the Ternary Quadratic Equation

\[ X^2 = 24\alpha^2 + Y^2 \]

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Abstract. The ternary quadratic equation \( X^2 = 24\alpha^2 + Y^2 \) is considered. Employing its non-zero integral solutions, relations among a few special polygonal numbers are determined.

1. Introduction

In [1, 8, 10], different patterns of \(m\)-gonal numbers are presented. In [2], explicit formulas for the ranks of triangular numbers which are simultaneously equal to pentagonal, decagonal and dodecagonal in turn are presented. In [3, 5, 6], the relations among the pairs of special \(m\)-gonal numbers are generated through the solutions of the ternary quadratic equation are determined. In [4, 7, 9], the binary quadratic equations are considered and relations between special polygonal numbers are obtained through its solutions.

In this communication, we consider the ternary quadratic equation given by \( X^2 = 24\alpha^2 + Y^2 \), where \(\alpha\) is a non-zero integer and obtain the relations among the pairs of special \(m\)-gonal numbers generated through its solutions.

2. Notation

- \(t_{3,N}\) - Triangular number of rank \(N\)
- \(t_{5,P}\) - Pentagonal number of rank \(P\)
- \(t_{6,Q}\) - Hexagonal number of rank \(Q\)
- \(t_{7,H}\) - Heptagonal number of rank \(H\)
- \(t_{8,M}\) - Centered octagonal number of rank \(M\)
- \(g_4(G)\) - 4-gram number of rank \(G\)

3. Method of Analysis

Consider the Diophantine equation

\[ X^2 = 24\alpha^2 + Y^2 \]  \hspace{1cm} (1)

where \(\alpha\) is a non-zero integer. The smallest non-zero integer solution of the Pellian equation \( X^2 = 24\alpha^2 + 1 \) is given by \(\alpha_0 = 1, X_0 = 5\). Thus the general form of non-trivial integral solutions \((\alpha_n, X_n)\) are given by

\[
\alpha_n = \frac{\sqrt[4]{6}}{2^n}\left[\left(5 + 2\sqrt{6}\right)^{n+1} - \left(5 - 2\sqrt{6}\right)^{n+1}\right], \quad n = 0,1,2, \ldots
\]

\[
X_n = \frac{\sqrt[2]{6}}{2^n}\left[\left(5 + 2\sqrt{6}\right)^{n+1} + \left(5 - 2\sqrt{6}\right)^{n+1}\right]
\]  \hspace{1cm} (2)

Case 3.1

Let \(N\) and \(Q\) be the ranks of the \(n^{th}\) Triangular and Hexagonal numbers respectively.

The choice

\[ X = 2N + 1; \quad Y = 4Q - 1 \]  \hspace{1cm} (3)

in (1) leads to the relation “2 times triangular number – 2 times hexagonal number = a nasty number” From (2) and (3), the values of ranks of Triangular numbers are found to be
Numerical examples are illustrated in the table 3.1

Table 3.1

<table>
<thead>
<tr>
<th>Q</th>
<th>( t_{6Q} )</th>
<th>( N_n \quad n = 0,1,2,... )</th>
<th>( t_{3N_n} \quad n = 0,1,2,... )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>7,73,727,...</td>
<td>28,2701,264628,...</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>17,171,1697,...</td>
<td>153,14706,1440753,...</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
<td>27,269,2667,...</td>
<td>378,36315,3557778,...</td>
</tr>
</tbody>
</table>

**Case 3.2**

Let \( N \) and \( P \) be the ranks of the \( n^{th} \) Triangular and Pentagonal numbers respectively. The choice

\[
X = 2N + 1; \quad Y = 6P - 1
\]

in (1) leads to the relation “2 times triangular number – 6 times pentagonal number = a nasty number” From (2) and (4), the values of ranks of Triangular numbers are found to be

\[
N_n = \frac{6^p - 1}{4} \left[ (5 + 2\sqrt{6})^{n+1} + (5 - 2\sqrt{6})^{n+1} \right] - \frac{1}{2}, \quad n = 0,1,2,... \quad \text{and} \quad P \in \mathbb{Z}^+ \setminus \{0\}
\]

Numerical examples are illustrated in the table 3.2

Table 3.2

<table>
<thead>
<tr>
<th>P</th>
<th>( t_{5P} )</th>
<th>( N_n \quad n = 0,1,2,... )</th>
<th>( t_{3N_n} \quad n = 0,1,2,... )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>12,122,1212,...</td>
<td>78,7503,735078,...</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>27,269,2667,...</td>
<td>378,36315,3557778,...</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>42,416,4122,...</td>
<td>903,86736,8497503,...</td>
</tr>
</tbody>
</table>

**Case 3.3**

Let \( N \) and \( H \) be the ranks of the \( n^{th} \) Triangular and Heptagonal numbers respectively. The choice

\[
X = 6N + 3; \quad Y = 10H - 3
\]

in (1) leads to the relation “18 times triangular number – 10 times heptagonal number = a nasty number” From (2) and (5), the values of ranks of Triangular numbers are found to be

\[
N_n = \frac{10H - 2}{12} \left[ (5 + 2\sqrt{6})^{n+1} + (5 - 2\sqrt{6})^{n+1} \right] - \frac{1}{2}, \quad n = 0,1,2,... \quad \text{and} \quad H \equiv 0 \ (mod \ 3)
\]

Numerical examples are illustrated in the table 3.3

Table 3.3

<table>
<thead>
<tr>
<th>H</th>
<th>( t_{7H} )</th>
<th>( N_n \quad n = 0,1,2,... )</th>
<th>( t_{3N_n} \quad n = 0,1,2,... )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>18</td>
<td>22,220,2182,...</td>
<td>253,24310,2381653,...</td>
</tr>
<tr>
<td>6</td>
<td>81</td>
<td>47,465,4607,...</td>
<td>1128,108345,10614528,...</td>
</tr>
<tr>
<td>9</td>
<td>189</td>
<td>72,710,7032,...</td>
<td>2628,252405,24728028,...</td>
</tr>
</tbody>
</table>
Case 3.4  
Let $M$ and $G$ be the ranks of the $n^{th}$ Centered octagonal and 4-gram numbers respectively. The choice

$$X=8M+3; \ Y=8G-4$$  \hspace{1cm} (6)

in (1) leads to the relation “4 times centered octagonal number – 4 times 4-gram number = a nasty number” From (2) and (6), the values of ranks of centered octagonal numbers are found to be

$$N_n = \frac{8G-4}{16} \left( \left(5+2\sqrt{6} \right)^{n+1} + \left(5-2\sqrt{6} \right)^{n+1} \right) - \frac{1}{2}, \quad n=0,1,2,... \quad \text{and } G \in \mathbb{Z}^+ - \{0\}$$

Numerical examples are illustrated in the table 3.4

<table>
<thead>
<tr>
<th>$G$</th>
<th>$g_4(G)$</th>
<th>$M_n$</th>
<th>$CP_{B,M_n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2,24,242,...</td>
<td>25,2401,235225,...</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
<td>7,73,727,...</td>
<td>225,21609,2117025,...</td>
</tr>
<tr>
<td>3</td>
<td>25</td>
<td>12,122,1212,...</td>
<td>625,60025,5880625,...</td>
</tr>
</tbody>
</table>

CONCLUSION

To conclude, one may determine the relations between the special polygonal numbers employing the integer solutions of various ternary quadratic equations.

REFERENCES