OBSERVATIONS ON THE TERNARY QUADRATIC EQUATION

$X^2 = 24\alpha^2 + Y^2$

M.A. Gopalan¹, V. Sangeetha², Manju Somanath³

¹Professor, Department of Mathematics, Srimathi Indira Gandhi College, Trichy-2, India.
², ³Assistant Professor, Department of Mathematics, National College, Trichy-1, India.

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Abstract. The ternary quadratic equation $X^2 = 24\alpha^2 + Y^2$ is considered. Employing its non-zero integral solutions, relations among a few special polygonal numbers are determined.

1. INTRODUCTION

In [1, 8, 10], different patterns of m-gonal numbers are presented. In [2], explicit formulas for the ranks of triangular numbers which are simultaneously equal to pentagonal, decagonal and dodecagonal in turn are presented. In [3, 5, 6], the relations among the pairs of special m-gonal numbers are generated through the solutions of the ternary quadratic equation are determined. In [4, 7, 9], the binary quadratic equations are considered and relations between special polygonal numbers are obtained through its solutions.

In this communication, we consider the ternary quadratic equation given by $X^2 = 24\alpha^2 + Y^2$, where $\alpha$ is a non-zero integer and obtain the relations among the pairs of special m-gonal numbers generated through its solutions.

2. NOTATION

- $t_{3,N}$ · Triangular number of rank N
- $t_{5,P}$ · Pentagonal number of rank P
- $t_{6,Q}$ · Hexagonal number of rank Q
- $t_{7,H}$ · Heptagonal number of rank H
- $t_{8,M}$ · Centered octagonal number of rank M
- $g_4(G)$ · 4-gran number of rank G

3. METHOD OF ANALYSIS

Consider the Diophantine equation

$X^2 = 24\alpha^2 + Y^2$  \hspace{1cm} (1)

where $\alpha$ is a non-zero integer. The smallest non-zero integer solution of the Pellian equation $X^2 = 24\alpha^2 + 1$ is given by $\alpha_0 = 1, X_0 = 5$. Thus the general form of non-trivial integral solutions $(\alpha_n, X_n)$ are given by

$$\alpha_n = \frac{\sqrt{5}}{4\sqrt{6}} \left[ (5 + 2\sqrt{6})^{n+1} - (5 - 2\sqrt{6})^{n+1} \right], \hspace{0.5cm} n = 0, 1, 2, \ldots$$

$$X_n = \frac{\sqrt{5}}{2} \left[ (5 + 2\sqrt{6})^{n+1} + (5 - 2\sqrt{6})^{n+1} \right]$$  \hspace{1cm} (2)

Case 3.1

Let N and Q be the ranks of the $n^{th}$ Triangular and Hexagonal numbers respectively.

The choice

$X = 2N + 1; Y = 4Q - 1$  \hspace{1cm} (3)

in (1) leads to the relation “2 times triangular number – 2 times hexagonal number = a nasty number” From (2) and (3), the values of ranks of Triangular numbers are found to be

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\[
N_n = \frac{4Q-1}{4} \left[ (5 + 2\sqrt{6})^{n+1} + (5 - 2\sqrt{6})^{n+1} \right] - \frac{1}{2}, \quad n = 0, 1, 2, \ldots \quad \text{and} \quad Q \in \mathbb{Z}^+ - \{0\}
\]

Numerical examples are illustrated in the table 3.1

<table>
<thead>
<tr>
<th>Q</th>
<th>(t_{5,Q})</th>
<th>(\frac{N_n}{n=0,1,2,\ldots})</th>
<th>(t_{3,N_n})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>7,73,727,\ldots</td>
<td>28,2701,264628,\ldots</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>17,171,1697,\ldots</td>
<td>153,14706,1440753,\ldots</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
<td>27,269,2667,\ldots</td>
<td>378,36315,3557778,\ldots</td>
</tr>
</tbody>
</table>

Case 3.2

Let \(N\) and \(P\) be the ranks of the \(n^{th}\) Triangular and Pentagonal numbers respectively. The choice \(X=2N+1; \ Y=6P-1\) in (1) leads to the relation “2 times triangular number – 6 times pentagonal number = a nasty number” From (2) and (4), the values of ranks of Triangular numbers are found to be

\[
N_n = \frac{6P-1}{4} \left[ (5 + 2\sqrt{6})^{n+1} + (5 - 2\sqrt{6})^{n+1} \right] - \frac{1}{2}, \quad n = 0, 1, 2, \ldots \quad \text{and} \quad P \in \mathbb{Z}^+ - \{0\}
\]

Numerical examples are illustrated in the table 3.2

<table>
<thead>
<tr>
<th>P</th>
<th>(t_{5,P})</th>
<th>(\frac{N_n}{n=0,1,2,\ldots})</th>
<th>(t_{3,N_n})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>12,122,1212,\ldots</td>
<td>78,7503,735078,\ldots</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>27,269,2667,\ldots</td>
<td>378,36315,3557778,\ldots</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>42,416,4122,\ldots</td>
<td>903,86736,8497503,\ldots</td>
</tr>
</tbody>
</table>

Case 3.3

Let \(N\) and \(H\) be the ranks of the \(n^{th}\) Triangular and Heptagonal numbers respectively. The choice \(X=6N+3; \ Y=10H-3\) in (1) leads to the relation “18 times triangular number – 10 times heptagonal number = a nasty number” From (2) and (5), the values of ranks of Triangular numbers are found to be

\[
N_n = \frac{10H-3}{12} \left[ (5 + 2\sqrt{6})^{n+1} + (5 - 2\sqrt{6})^{n+1} \right] - \frac{1}{2}, \quad n = 0, 1, 2, \ldots \quad \text{and} \quad H \equiv 0 \pmod{3}
\]

Numerical examples are illustrated in the table 3.3

<table>
<thead>
<tr>
<th>H</th>
<th>(t_{7,H})</th>
<th>(\frac{N_n}{n=0,1,2,\ldots})</th>
<th>(t_{3,N_n})</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>18</td>
<td>22,220,2182,\ldots</td>
<td>253,24310,2381653,\ldots</td>
</tr>
<tr>
<td>6</td>
<td>81</td>
<td>47,465,4607,\ldots</td>
<td>1128,108345,10614528,\ldots</td>
</tr>
<tr>
<td>9</td>
<td>189</td>
<td>72,710,7032,\ldots</td>
<td>2628,252405,24728028,\ldots</td>
</tr>
</tbody>
</table>
Case 3.4
Let M and G be the ranks of the \( n^{th} \) centered octagonal and 4-gram numbers respectively. The choice

\[
X = 8M + 3; \quad Y = 8G - 4
\]  

in (1) leads to the relation “4 times centered octagonal number – 4 times 4-gram number = a nasty number”. From (2) and (6), the values of ranks of centered octagonal numbers are found to be

\[
N_n = \frac{8G - 4}{16} \left[ \left(5 + 2\sqrt{6}\right)^n + \left(5 - 2\sqrt{6}\right)^n \right] - \frac{1}{2}, \quad n = 0, 1, 2, \ldots \quad \text{and} \quad G \in \mathbb{Z}^+ - \{0\}
\]

Numerical examples are illustrated in the table 3.4

<table>
<thead>
<tr>
<th>G</th>
<th>( g_4(G) )</th>
<th>( M_n )</th>
<th>( c_{P_6,M_n} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2,24,242,...</td>
<td>25,2401,235225,...</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
<td>7,73,727,...</td>
<td>225,21609,2117025,...</td>
</tr>
<tr>
<td>3</td>
<td>25</td>
<td>12,122,1212,...</td>
<td>625,60025,5880625,...</td>
</tr>
</tbody>
</table>

**CONCLUSION**

To conclude, one may determine the relations between the special polygonal numbers employing the integer solutions of various ternary quadratic equations.

**REFERENCES**


