OBSERVATIONS ON THE TERNARY QUADRATIC EQUATION 

\[ X^2 = 24\alpha^2 + Y^2 \]

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Keywords. Ternary quadratic, polygonal numbers, integral solutions.

Abstract. The ternary quadratic equation \( X^2 = 24\alpha^2 + Y^2 \) is considered. Employing its non-zero integral solutions, relations among a few special polygonal numbers are determined.

1. INTRODUCTION

In [1, 8, 10], different patterns of m-gonal numbers are presented. In [2], explicit formulas for the ranks of triangular numbers which are simultaneously equal to pentagonal, decagonal and dodecagonal in turn are presented. In [3, 5, 6], the relations among the pairs of special m-gonal numbers are generated through the solutions of the ternary quadratic equation are determined. In [4, 7, 9], the binary quadratic equations are considered and relations between special polygonal numbers are obtained through its solutions.

In this communication, we consider the ternary quadratic equation given by \( X^2 = 24\alpha^2 + Y^2 \), where \( \alpha \) is a non-zero integer and obtain the relations among the pairs of special m-gonal numbers generated through its solutions.

2. NOTATION

- \( t_{3,N} \) · Triangular number of rank N
- \( t_{5,P} \) · Pentagonal number of rank P
- \( t_{6,Q} \) · Hexagonal number of rank Q
- \( t_{7,H} \) · Heptagonal number of rank H
- \( C_{8,M} \) · Centered octagonal number of rank M
- \( g_4(G) \) · 4-gram number of rank G

3. METHOD OF ANALYSIS

Consider the Diophantine equation

\[ X^2 = 24\alpha^2 + Y^2 \]

where \( \alpha \) is a non-zero integer. The smallest non-zero integer solution of the Pellian equation \( X^2 = 24\alpha^2 + 1 \) is given by \( \alpha_0 = 1, X_0 = 5 \). Thus the general form of non-trivial integral solutions \((\alpha_n, X_n)\) are given by

\[ \alpha_n = \frac{\sqrt{2}}{4\sqrt{6}} \left[ (5 + 2\sqrt{6})^{n+1} - (5 - 2\sqrt{6})^{n+1} \right], \quad n = 0, 1, 2, \ldots \]
\[ X_n = \frac{\sqrt{2}}{2} \left[ (5 + 2\sqrt{6})^{n+1} + (5 - 2\sqrt{6})^{n+1} \right] \]

Case 3.1

Let \( N \) and \( Q \) be the ranks of the \( n^{th} \) Triangular and Hexagonal numbers respectively.

The choice

\[ X = 2N + 1; \quad Y = 4Q - 1 \]

in (1) leads to the relation “ 2 times triangular number – 2 times hexagonal number = a nasty number” From (2) and (3), the values of ranks of Triangular numbers are found to be...
Let $N$ and $P$ be the ranks of the $n^{th}$ Triangular and Pentagonal numbers respectively. The choice
\[ X = 2N + 1; \quad Y = 6P - 1 \]
leads to the relation “2 times triangular number – 6 times pentagonal number = a nasty number.” From (2) and (4), the values of ranks of Triangular numbers are found to be
\[ N_n = \frac{6^p - 1}{4} \left[ \left(5 + 2\sqrt{6}\right)^{n+1} + \left(5 - 2\sqrt{6}\right)^{n+1} \right] - \frac{1}{2}, \quad n = 0, 1, 2, \ldots \quad \text{and} \quad P \in \mathbb{Z}^+ - \{0\} \]

Numerical examples are illustrated in the table 3.2

<table>
<thead>
<tr>
<th>P</th>
<th>$t_{5,p}$</th>
<th>$N_n$</th>
<th>$t_{3,N_n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>12,122,1212,...</td>
<td>78,7503,735078,...</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>27,269,2667,...</td>
<td>378,36315,3557778,...</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>42,416,4122,...</td>
<td>903,86736,8497503,...</td>
</tr>
</tbody>
</table>

Let $N$ and $H$ be the ranks of the $n^{th}$ Triangular and Heptagonal numbers respectively. The choice
\[ X = 6N + 3; \quad Y = 10H - 3 \]
leads to the relation “18 times triangular number – 10 times heptagonal number = a nasty number.” From (2) and (5), the values of ranks of Triangular numbers are found to be
\[ N_n = \frac{10H - 3}{12} \left[ \left(5 + 2\sqrt{6}\right)^{n+1} + \left(5 - 2\sqrt{6}\right)^{n+1} \right] - \frac{1}{2}, \quad n = 0, 1, 2, \ldots \quad \text{and} \quad H \equiv 0 \pmod{3} \]

Numerical examples are illustrated in the table 3.3

<table>
<thead>
<tr>
<th>H</th>
<th>$t_{7,H}$</th>
<th>$N_n$</th>
<th>$t_{3,N_n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>18</td>
<td>22,220,2182,...</td>
<td>253,24310,2381653,...</td>
</tr>
<tr>
<td>6</td>
<td>81</td>
<td>47,465,4607,...</td>
<td>1128,108345,10614528,...</td>
</tr>
<tr>
<td>9</td>
<td>189</td>
<td>72,710,7032,...</td>
<td>2628,252405,24728028,...</td>
</tr>
</tbody>
</table>
Case 3.4  
Let M and G be the ranks of the $n^{th}$ Centered octagonal and 4-gram numbers respectively. The choice
\[ X=8M+3; \quad Y=8G-4 \]  
in (1) leads to the relation “4 times centered octagonal number – 4 times 4-gram number = a nasty number” From (2) and (6), the values of ranks of centered octagonal numbers are found to be
\[ N_n = \frac{8G-4}{16} \left[ (5+2\sqrt{6})^n + (5-2\sqrt{6})^n \right] - \frac{1}{2}, \quad n = 0, 1, 2, ... \quad \text{and} \quad G \in \mathbb{Z}^+ - \{0\} \]
Numerical examples are illustrated in the table 3.4

<table>
<thead>
<tr>
<th>G</th>
<th>$g_4(G)$</th>
<th>$M_n$</th>
<th>$C_{P_{8,M_n}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>$2, 24, 242, ...$</td>
<td>$25, 2401, 235225, ...$</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
<td>$7, 73, 727, ...$</td>
<td>$225, 21609, 2117025, ...$</td>
</tr>
<tr>
<td>3</td>
<td>25</td>
<td>$12, 122, 1212, ...$</td>
<td>$625, 60025, 5880625, ...$</td>
</tr>
</tbody>
</table>

CONCLUSION  
To conclude, one may determine the relations between the special polygonal numbers employing the integer solutions of various ternary quadratic equations.

REFERENCES