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OBSERVATIONS ON THE TERNARY QUADRATIC EQUATION $X^2=24\alpha^2+Y^2$

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Abstract. The ternary quadratic equation $X^2=24\alpha^2+Y^2$ is considered.Employing its non-zero integral solutions, relations among a few special polygonal numbers are determined.

1. INTRODUCTION

In [1,8,10],different patterns of m-gonal numbers are presented. In [2], explicit formulas for the ranks of triangular numbers which are simultaneously equal to pentagonal, decagonal and dodecagonal in turn are presented. In [3,5,6], the relations among the pairs of special m-gonal numbers are generated through the solutions of the ternary quadratic equation are determined. In [4,7,9], the binary quadratic equations are considered and relations between special polygonal numbers are obtained through its solutions.

In this communication, we consider the ternary quadratic equation given by $X^2=24\alpha^2+Y^2$, where α is a non-zero integer and obtain the relations among the pairs of special m-gonal numbers generated through its solutions.

2. NOTATION

- $t_{3,N}$ · Triangular number of rank N
- $t_{5,P}$ · Pentagonal number of rank P
- $t_{6,Q}$ · Hexagonal number of rank Q
- $t_{7,H}$ · Heptagonal number of rank H
- $cp_{8,M}$ · Centered octagonal number of rank M
- $g_4(G)$ · 4-gram number of rank G

3. METHOD OF ANALYSIS

Consider the Diophantine equation

$$X^2=24\alpha^2+Y^2 \tag{1}$$

where α is a non-zero integer. The smallest non-zero integer solution of the Pellian equation $X^2=24\alpha^2+1$ is given by $\alpha_0=1$, $X_0=5$. Thus the general form of non-trivial integral solutions (α_n,X_n) are given by

$$\alpha_n = \frac{Y}{4\sqrt{6}}\left[\left(5+2\sqrt{6}\right)^{n+1}-\left(5-2\sqrt{6}\right)^{n+1}\right], \quad n=0,1,2,\ldots..$$
$$X_n = \frac{Y}{2}\left[\left(5+2\sqrt{6}\right)^{n+1}+\left(5-2\sqrt{6}\right)^{n+1}\right] \tag{2}$$

Case 3.1

Let N and Q be the ranks of the n^{th} Triangular and Hexagonal numbers respectively.

The choice

$$X=2N+1;\ Y=4Q-1 \tag{3}$$

in (1) leads to the relation " 2 times triangular number – 2 times hexagonal number = a nasty number" From (2) and (3), the values of ranks of Triangular numbers are found to be

$$N_n = \frac{4Q-1}{4}\left[\left(5+2\sqrt{6}\right)^{n+1} + \left(5-2\sqrt{6}\right)^{n+1}\right] - \frac{1}{2}, \quad n = 0,1,2,\dots \text{ and } Q \in Z^+ - \{0\}$$

Numerical examples are illustrated in the table 3.1

Table 3.1

Q	$t_{6,Q}$	N_n $n = 0,1,2,\dots$	t_{3,N_n} $n = 0,1,2,\dots$
1	1	7,73,727,….	28,2701,264628,….
2	6	17,171,1697,…	153,14706,1440753,….
3	15	27,269,2667,…	378,36315,3557778,…

Case 3.2

Let N and P be the ranks of the n^{th} Triangular and Pentagonal numbers respectively.

The choice

$$X=2N+1;\ Y=6P-1 \tag{4}$$

in (1) leads to the relation " 2 times triangular number – 6 times pentagonal number = a nasty number" From (2) and (4), the values of ranks of Triangular numbers are found to be

$$N_n = \frac{6P-1}{4}\left[\left(5+2\sqrt{6}\right)^{n+1} + \left(5-2\sqrt{6}\right)^{n+1}\right] - \frac{1}{2}, \quad n = 0,1,2,\dots \text{ and } P \in Z^+ - \{0\}$$

Numerical examples are illustrated in the table 3.2

Table 3.2

P	$t_{5,P}$	N_n $n = 0,1,2,\dots$	t_{3,N_n} $n = 0,1,2,\dots$
1	1	12,122,1212,….	78,7503,735078,….
2	5	27,269,2667,…	378,36315,3557778,…
3	12	42,416,4122,…	903,86736,8497503,….

Case 3.3

Let N and H be the ranks of the n^{th} Triangular and Heptagonal numbers respectively.

The choice

$$X=6N+3;\ Y=10H-3 \tag{5}$$

in (1) leads to the relation " 18 times triangular number – 10 times heptagonal number = a nasty number" From (2) and (5), the values of ranks of Triangular numbers are found to be

$$N_n = \frac{10H-3}{12}\left[\left(5+2\sqrt{6}\right)^{n+1} + \left(5-2\sqrt{6}\right)^{n+1}\right] - \frac{1}{2}, \quad n = 0,1,2,\dots \text{ and } H \equiv 0 \ (mod\ 3)$$

Numerical examples are illustrated in the table 3.3

Table 3.3

H	$t_{7,H}$	N_n $n = 0,1,2,\dots$	t_{3,N_n} $n = 0,1,2,\dots$
3	18	22,220,2182,…	253,24310,2381653,….
6	81	47,465,4607,…	1128,108345,10614528,….
9	189	72,710,7032,…	2628,252405,24728028,…

Case 3.4

Let M and G be the ranks of the n^{th} Centered octagonal and 4-gram numbers respectively. The choice

$$X=8M+3;\ Y=8G-4 \tag{6}$$

in (1) leads to the relation " 4 times centered octagonal number – 4 times 4-gram number = a nasty number" From (2) and (6), the values of ranks of centered octagonal numbers are found to be

$$N_n=\frac{8G-4}{16}\left[\left(5+2\sqrt{6}\right)^{n+1}+\left(5-2\sqrt{6}\right)^{n+1}\right]-\frac{1}{2},\quad n=0,1,2,\ldots \text{ and } G\in Z^+-\{0\}$$

Numerical examples are illustrated in the table 3.4

Table 3.4

G	$g_4(G)$	M_n $n=0,1,2,\ldots$	cp_{8,M_n} $n=0,1,2,\ldots$
1	1	2,24,242,….	25,2401,235225,….
2	9	7,73,727,…	225,21609,2117025,…
3	25	12,122,1212,…	625,60025,5880625,….

CONCLUSION

To conclude, one may determine the relations between the special polygonal numbers employing the integer solutions of various ternary quadratic equations.

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