OBSERVATIONS ON THE TERNARY QUADRATIC EQUATION

\[ X^2 = 24\alpha^2 + Y^2 \]

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Abstract. The ternary quadratic equation \( X^2 = 24\alpha^2 + Y^2 \) is considered. Employing its non-zero integral solutions, relations among a few special polygonal numbers are determined.

1. INTRODUCTION

In [1, 8, 10], different patterns of \( m \)-gonal numbers are presented. In [2], explicit formulas for the ranks of triangular numbers which are simultaneously equal to pentagonal, decagonal and dodecagonal in turn are presented. In [3, 5, 6], the relations among the pairs of special \( m \)-gonal numbers are generated through the solutions of the ternary quadratic equation are determined. In [4, 7, 9], the binary quadratic equations are considered and relations between special polygonal numbers are obtained through its solutions.

In this communication, we consider the ternary quadratic equation given by \( X^2 = 24\alpha^2 + Y^2 \), where \( \alpha \) is a non-zero integer and obtain the relations among the pairs of special \( m \)-gonal numbers generated through its solutions.

2. NOTATION

- \( t_{3,N} \) - Triangular number of rank \( N \)
- \( t_{5,P} \) - Pentagonal number of rank \( P \)
- \( t_{6,Q} \) - Hexagonal number of rank \( Q \)
- \( t_{7,H} \) - Heptagonal number of rank \( H \)
- \( cP_{8,M} \) - Centered octagonal number of rank \( M \)
- \( g_4(G) \) - 4-gram number of rank \( G \)

3. METHOD OF ANALYSIS

Consider the Diophantine equation

\[ X^2 = 24\alpha^2 + Y^2 \]  \hspace{1cm} (1)

where \( \alpha \) is a non-zero integer. The smallest non-zero integer solution of the Pellian equation \( X^2 = 24\alpha^2 + 1 \) is given by \( \alpha_0 = 1, X_0 = 5 \). Thus the general form of non-trivial integral solutions \( (\alpha_n, X_n) \) are given by

\[
\alpha_n = \frac{\sqrt{24} \left[ (5 + 2\sqrt{6})^n + (5 - 2\sqrt{6})^n \right]}{4\sqrt{3}}, \quad n = 0, 1, 2, \ldots \]

\[
X_n = \frac{\sqrt{2} \left[ (5 + 2\sqrt{6})^n + (5 - 2\sqrt{6})^n \right]}{2}, \quad n = 0, 1, 2, \ldots \]  \hspace{1cm} (2)

Case 3.1

Let \( N \) and \( Q \) be the ranks of the \( n^{th} \) Triangular and Hexagonal numbers respectively.

The choice

\[ X = 2N + 1; \quad Y = 4Q - 1 \]  \hspace{1cm} (3)

in (1) leads to the relation “2 times triangular number – 2 times hexagonal number = a nasty number” From (2) and (3), the values of ranks of Triangular numbers are found to be...
Numerical examples are illustrated in the table 3.1

Table 3.1

<table>
<thead>
<tr>
<th>Q</th>
<th>$r_{6,Q}$</th>
<th>$N_n$</th>
<th>$r_{3,N_n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>7,73,727,...</td>
<td>28,2701,264628,...</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>17,171,1697,...</td>
<td>153,14706,1440753,...</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
<td>27,269,2667,...</td>
<td>378,36315,3557778,...</td>
</tr>
</tbody>
</table>

Case 3.2

Let N and P be the ranks of the $n^{th}$ Triangular and Pentagonal numbers respectively. The choice

$$X=2N+1; \quad Y=6P-1$$

in (1) leads to the relation “2 times triangular number – 6 times pentagonal number = a nasty number” From (2) and (4), the values of ranks of Triangular numbers are found to be

$$N_n = \frac{6^p-1}{4} \left[ (5 + 2\sqrt{6})^{n+1} + (5 - 2\sqrt{6})^{n+1} \right] - \frac{1}{2}, \quad n = 0,1,2,...$$

and $P \in \mathbb{Z}^+ \setminus \{0\}$

Numerical examples are illustrated in the table 3.2

Table 3.2

<table>
<thead>
<tr>
<th>P</th>
<th>$r_{5,P}$</th>
<th>$N_n$</th>
<th>$r_{3,N_n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>12,122,1212,...</td>
<td>78,7503,735078,...</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>27,269,2667,...</td>
<td>378,36315,3557778,...</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>42,416,4122,...</td>
<td>903,86736,8497503,...</td>
</tr>
</tbody>
</table>

Case 3.3

Let N and H be the ranks of the $n^{th}$ Triangular and Heptagonal numbers respectively. The choice

$$X=6N+3; \quad Y=10H-3$$

in (1) leads to the relation “18 times triangular number – 10 times heptagonal number = a nasty number” From (2) and (5), the values of ranks of Triangular numbers are found to be

$$N_n = \frac{10H-3}{12} \left[ (5 + 2\sqrt{6})^{n+1} + (5 - 2\sqrt{6})^{n+1} \right] - \frac{1}{2}, \quad n = 0,1,2,...$$

and $H = 0 \pmod{3}$

Numerical examples are illustrated in the table 3.3

Table 3.3

<table>
<thead>
<tr>
<th>H</th>
<th>$r_{7,H}$</th>
<th>$N_n$</th>
<th>$r_{3,N_n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>18</td>
<td>22,220,2182,...</td>
<td>253,24310,2381653,...</td>
</tr>
<tr>
<td>6</td>
<td>81</td>
<td>47,465,4607,...</td>
<td>1128,108345,10614528,...</td>
</tr>
<tr>
<td>9</td>
<td>189</td>
<td>72,710,7032,...</td>
<td>2628,252405,24728028,...</td>
</tr>
</tbody>
</table>
Case 3.4  
Let M and G be the ranks of the \( n \)th Centered octagonal and 4-gram numbers respectively. The choice

\[
X=8M+3; \quad Y=8G-4
\]

in (1) leads to the relation “ 4 times centered octagonal number – 4 times 4-gram number = a nasty number” From (2) and (6), the values of ranks of centered octagonal numbers are found to be

\[
N_n = \frac{8G-4}{16} \left[ \left(5 + 2\sqrt{6}\right)^{n+1} + \left(5 - 2\sqrt{6}\right)^{n+1} \right] - \frac{1}{2}, \quad n = 0,1,2,\ldots \quad \text{and} \quad G \in \mathbb{Z} - \{0\}
\]

Numerical examples are illustrated in the table 3.4

<table>
<thead>
<tr>
<th>G</th>
<th>( G_4(G) )</th>
<th>( M_n ) ( n = 0,1,2,\ldots )</th>
<th>( CP_{G,M_n} ) ( n = 0,1,2,\ldots )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2,24,242,\ldots</td>
<td>25,240,1,235225,\ldots</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
<td>7,73,727,\ldots</td>
<td>225,21609,2117025,\ldots</td>
</tr>
<tr>
<td>3</td>
<td>25</td>
<td>12,122,1212,\ldots</td>
<td>625,60025,5880625,\ldots</td>
</tr>
</tbody>
</table>

CONCLUSION
To conclude, one may determine the relations between the special polygonal numbers employing the integer solutions of various ternary quadratic equations.

REFERENCES