OBSERVATIONS ON THE TERNARY QUADRATIC EQUATION

\[ X^2 = 24\alpha^2 + Y^2 \]

M.A. Gopalan\(^1\), V. Sangeetha\(^2\), Manju Somanath\(^3\)

\(^1\)Professor, Department of Mathematics, Srimathi Indira Gandhi College, Trichy-2, India.
\(^2,3\)Assistant Professor, Department of Mathematics, National College, Trichy-1, India.

Keywords. Ternary quadratic, polygonal numbers, integral solutions.

Abstract. The ternary quadratic equation \[ X^2 = 24\alpha^2 + Y^2 \] is considered. Employing its non-zero integral solutions, relations among a few special polygonal numbers are determined.

1. INTRODUCTION

In [1,8,10], different patterns of m-gonal numbers are presented. In [2], explicit formulas for the ranks of triangular numbers which are simultaneously equal to pentagonal, decagonal and dodecagonal in turn are presented. In [3,5,6], the relations among the pairs of special m-gonal numbers are generated through the solutions of the ternary quadratic equation are determined. In [4,7,9], the binary quadratic equations are considered and relations between special polygonal numbers are obtained through its solutions.

In this communication, we consider the ternary quadratic equation given by \[ X^2 = 24\alpha^2 + Y^2 \], where \( \alpha \) is a non-zero integer and obtain the relations among the pairs of special m-gonal numbers generated through its solutions.

2. NOTATION

- \( t_{3,N} \): Triangular number of rank \( N \)
- \( t_{5,P} \): Pentagonal number of rank \( P \)
- \( t_{6,Q} \): Hexagonal number of rank \( Q \)
- \( t_{7,H} \): Heptagonal number of rank \( H \)
- \( t_{8,M} \): Centered octagonal number of rank \( M \)
- \( g_4(G) \): 4-gram number of rank \( G \)

3. METHOD OF ANALYSIS

Consider the Diophantine equation

\[ X^2 = 24\alpha^2 + Y^2 \]  \( \text{(1)} \)

where \( \alpha \) is a non-zero integer. The smallest non-zero integer solution of the Pellian equation \[ X^2 = 24\alpha^2 + 1 \] is given by \( \alpha_0 = 1, X_0 = 5 \). Thus the general form of non-trivial integral solutions \((\alpha_n, X_n)\) are given by

\[ \alpha_n = \frac{\sqrt{24}}{4\sqrt{6}} \left[ (5 + 2\sqrt{6})^{n+1} - (5 - 2\sqrt{6})^{n+1} \right], \quad n = 0, 1, 2, \ldots \]

\[ X_n = \frac{\sqrt{2}}{2} \left[ (5 + 2\sqrt{6})^{n+1} + (5 - 2\sqrt{6})^{n+1} \right] \]  \( \text{(2)} \)

Case 3.1

Let \( N \) and \( Q \) be the ranks of the \( n^{th} \) Triangular and Hexagonal numbers respectively. The choice

\[ X = 2N + 1; \quad Y = 4Q - 1 \]

in \( \text{(1)} \) leads to the relation “2 times triangular number – 2 times hexagonal number = a nasty number” From \( \text{(2)} \) and \( \text{(3)} \), the values of ranks of Triangular numbers are found to be
Numerical examples are illustrated in the table 3.1

Table 3.1

<table>
<thead>
<tr>
<th>Q</th>
<th>( r_{6,Q} )</th>
<th>( N_n ) ( n = 0,1,2,... )</th>
<th>( t_{3,N_n} ) ( n = 0,1,2,... )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>7,737,727,.....</td>
<td>28,2701,264628,.....</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>17,171,1697,.....</td>
<td>153,14706,1440753,.....</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
<td>27,269,2667,.....</td>
<td>378,36315,3557778,.....</td>
</tr>
</tbody>
</table>

Case 3.2

Let \( N \) and \( P \) be the ranks of the \( n^{th} \) Triangular and Pentagonal numbers respectively.

The choice

\[
X = 2N + 1; \quad Y = 6P - 1
\]

(4)

in (1) leads to the relation “ 2 times triangular number – 6 times pentagonal number = a nasty number” From (2) and (4), the values of ranks of Triangular numbers are found to be

\[
N_n = \frac{6P-1}{4} \left[(5+2\sqrt{6})^{n+1} + (5-2\sqrt{6})^{n+1}\right] - \frac{1}{2}, \quad n = 0,1,2,... \quad \text{and} \quad P \in \mathbb{Z}^+ - \{0\}
\]

Numerical examples are illustrated in the table 3.2

Table 3.2

<table>
<thead>
<tr>
<th>P</th>
<th>( r_{5,P} )</th>
<th>( N_n ) ( n = 0,1,2,... )</th>
<th>( t_{3,N_n} ) ( n = 0,1,2,... )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>12,122,1212,.....</td>
<td>78,7503,735078,.....</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>27,269,2667,.....</td>
<td>378,36315,3557778,.....</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
<td>42,416,4122,.....</td>
<td>903,86736,8497503,.....</td>
</tr>
</tbody>
</table>

Case 3.3

Let \( N \) and \( H \) be the ranks of the \( n^{th} \) Triangular and Heptagonal numbers respectively.

The choice

\[
X = 6N + 3; \quad Y = 10H - 3
\]

(5)

in (1) leads to the relation “ 18 times triangular number – 10 times heptagonal number = a nasty number” From (2) and (5), the values of ranks of Triangular numbers are found to be

\[
N_n = \frac{10H-3}{2} \left[(5+2\sqrt{6})^{n+1} + (5-2\sqrt{6})^{n+1}\right] - \frac{1}{2}, \quad n = 0,1,2,... \quad \text{and} \quad H \equiv 0 \pmod{3}
\]

Numerical examples are illustrated in the table 3.3

Table 3.3

<table>
<thead>
<tr>
<th>H</th>
<th>( r_{7,H} )</th>
<th>( N_n ) ( n = 0,1,2,... )</th>
<th>( t_{3,N_n} ) ( n = 0,1,2,... )</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>18</td>
<td>22,220,2182,.....</td>
<td>253,24310,2381653,.....</td>
</tr>
<tr>
<td>6</td>
<td>81</td>
<td>47,465,4607,.....</td>
<td>1128,108345,10614528,.....</td>
</tr>
<tr>
<td>9</td>
<td>189</td>
<td>72,710,7032,.....</td>
<td>2628,252405,24728028,.....</td>
</tr>
</tbody>
</table>
Case 3.4  
Let M and G be the ranks of the $n^{th}$ Centered octagonal and 4-gram numbers respectively. 

The choice 

$$X=8M+3; \ Y=8G-4$$  \hspace{1cm} (6) 

in (1) leads to the relation “4 times centered octagonal number – 4 times 4-gram number = a nasty number” From (2) and (6), the values of ranks of centered octagonal numbers are found to be 

$$N_n = \frac{8G-4}{16} \left[ (5+2\sqrt{6})^{n+1} + (5-2\sqrt{6})^{n+1} \right] - \frac{1}{2}, \ n = 0,1,2,... \text{ and } G \in \mathbb{Z}^+ - \{0\}.$$ 

Numerical examples are illustrated in the table 3.4

<table>
<thead>
<tr>
<th>G</th>
<th>$g_4(G)$</th>
<th>$M_n$</th>
<th>$CP_{8,M_n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2,24,242,...</td>
<td>25,2401,235225,...</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
<td>7,73,727,...</td>
<td>225,21609,2117025,...</td>
</tr>
<tr>
<td>3</td>
<td>25</td>
<td>12,122,1212,...</td>
<td>625,60025,5880625,...</td>
</tr>
</tbody>
</table>

CONCLUSION

To conclude, one may determine the relations between the special polygonal numbers employing the integer solutions of various ternary quadratic equations.

REFERENCES