On Fuzzy Orders and Metrics

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Abstract: In this paper we present results regarding fixed point theory using notion of fuzzy orders.We propose an approach based on quasi metrics which in a way unifies the fixed point theory for ordered sets and metric spaces.

1. Introduction:

Much mathematical effort is expanded on solving fixed point equations i.e.equations of the form $f(x) = x$ where $f : X \rightarrow Y$ is a map. A solution of such fixed point equations,when one exists,often has to be obtained by process of successive approximations.Order theory plays a role when X carries an order and when solution can be realized as joint of elements which approximates it. At some instances we find that fixed point theorems for ordered sets appear to be insufficient . In such cases , it is helpful to apply fixed point techniques in metric spaces.

In this paper give some results in this direction. By using the notion of quasi metric spaces we prove a theorem simultaneously generalizing fixed point theorems for ordered structures and metric spaces.All results are interpreted in terms of fuzzy set theory.

2. Preliminaries:

We recall some basic notions, $I = [0,1]$ *is a complete completely distributive lattice.*

Definition 2.1 *Let X be a nonempty set,a fuzzy subset μ of X is a function μ:X→I.*

Given two fuzzy sets μ_1, μ_2, we set $\mu_1 \subseteq \mu_2$ provided that $\mu_1(x) \leq \mu_2(x)$ for every $x \in X$.

Given a fuzzy set $\mu, \alpha \in I, \mu_\alpha = \{x \in X \mid \mu(x) \geq \alpha\}$ is called α-cut of μ.

The notion of fuzzy order and similarity was introduced by L.Zadeh in 1971. It was further developed by P.Venugopalan, Kundu, Beg and Islam, Negoita and Rolescu etc..

Definition 2.2 *A fuzzy relation ρ on X is a map ρ:X⨯X→I.*

ρ is called

1. reflexive, if $\rho(x, x) = 1$, for all $x \in X$.
2. symmetric, if $\rho(x, y) = \rho(y, x)$, for all $x, y \in X$
3. antisymmetric, if $\rho(x, y) \wedge \rho(y, x) = 0$ whenever $x \neq y$, for all $x, y \in X$
4. transitive, if $\rho(x, z) \wedge \rho(z, y) \leq \rho(x, y)$, for all $x, y, z \in X$.

A reflexive and transitive fuzzy relation is called a fuzzy preordering. Moreover a preordering which is antisymmetric is called a fuzzy order relation.A fuzzy symmetric fuzzy preorder is called a fuzzy equivalence.

A set X equipped with fuzzy order relation ρ, denoted as (X, ρ),is called a fuzzy ordered set(foset).

The domain of ρ is the fuzzy set on X, denoted by Dom ρ, whose membership is defined as

$$\text{Dom } \rho(x) = \bigvee_{x \neq y} \{\rho(x, y) \mid y \in X\}$$

Similarly, the range of ρ, denoted as Ran ρ is defined as

$$\text{Ran} \rho(y) = \bigvee_{x \neq y} \{\rho(x, y) \mid x \in X\}$$

The height of ρ, denoted by $h(R)$ is defined as

$$h(\rho) = \bigvee_{\{(x,y)\mid x \neq y\}} \{\rho(x, y)\}.$$

Definition 2.3 *If Y is a subset of a foset (X, ρ) then the fuzzy order ρ is a fuzzy order on Y,called the induced fuzzy order.*

Definition 2.4 *A fuzzy order ρ is linear(or total)on X if for every we have $x, y \in X$ we have $\rho(x, y) > 0$ or $\rho(y, x) > 0$. fuzzy ordered set (X, ρ) in which ρ is total is called a ρ-fuzzy chain. Conversely, if for $x, y \in X$, $\rho(x, y) > 0$ iff x=y, then (X, ρ) is called a ρ-fuzzy antichain.*

Given a fuzzy preorder ρ on X, we define $\rho_{op} : X \times X \rightarrow L$ by $\rho_{op}(x, y) = \rho(y, x)$. Then, ρ_{op} is also a preorder on X, called the opposite of ρ.

ρ is a fuzzy partial order (fuzzy equivalence) iff ρ_{op} is a fuzzy partial order(fuzzy equivalence).

Suppose that $(\rho_i)_{i \in I}$ is a collection of fuzzy preorders on X. Then ,the pointwise intersection $\rho(x, y) = \bigwedge_{i \in I} \rho_i(x, y)$ is also a fuzzy preorder on X.

If ρ is a fuzzy order on X then $\rho \wedge \rho_{op}$ is a fuzzy equivalence on X.

Proposition 2.5 *A fuzzy relation ρ on a set X is a fuzzy order iff every cut ρ_α is a fuzzy order.*
Proof: Suppose ρ is a fuzzy order relation on X. Let $x \in X$. Since ρ is reflexive we have, for any $\alpha \in I$, $\rho(x, x) = 1 \geq \alpha$, and therefore $x \rho_\alpha x$. Thus ρ_α is reflexive.
For $x, y \in X$ Let $x \rho_\alpha y$. Then, $\rho(y, x) = \rho(x, y) \geq \alpha$. So, $y \rho_\alpha x$. That is, ρ_α is symmetric.
For $x, y, z \in X$. suppose $x \rho_\alpha y$ and $y \rho_\alpha z$ then $\rho(x, y) \geq \alpha$ and $\rho(y, z) \geq \alpha$. By transitivity of ρ it follows that $\rho(x, z) \geq \rho(x, y) \wedge \rho(y, z) \geq \alpha$. So, $x \rho_\alpha z$. Thus. ρ_α is transitive.This proves that ρ_α is a fuzzy order.

Conversely, suppose ρ_α a fuzzy order for each $\alpha \in I$. For $x \in X$. we have $x \rho_\alpha x$, so $\rho(x, x) \geq \alpha$,for all $\alpha \in L$. Hence, $\rho(x, x) = 1$. Thus ρ is reflexive. For $x, y \in X$ suppose $x \rho y$. Then $\rho(x, y) \geq \alpha$ for some $\alpha \in L$. But, $\rho(y, x) = \rho(x, y) \geq \alpha$ as ρ_α is symmetric. Hence, $y \rho x$. Thus ρ is symmetric. Lastly,suppose for $x, y, z \in X$. we have $x \rho y$ and $y \rho x$ that is, $\rho(x, y) \geq \alpha$ and $\rho(y, z) \geq \alpha$ for some $\alpha \in L$. So $x \rho_\alpha y$ and $y \rho_\alpha z$. It follows that $x \rho_\alpha z$ as ρ_α is transitive. Thus. $\rho(y, z) \geq \alpha$. In particular setting $\alpha = \rho(x, y) \wedge \rho(y, z)$ we get $\rho(x, z) \geq \rho(x, y) \wedge \rho(y, z)$. This proves that ρ is transitive. Hence, ρ is a fuzzy order on X.

If ρ is a fuzzy preorder and $\alpha \neq 1$, then in general the cut ρ_α is not a preorder .But, ρ_1 is always a preorder ,called the preorder associated with ρ.

3. Relationship between Ordered structures and Topological Structures

It is well known that crisp topological structures and classical ordered structures are closely related.We observe that if $(X, \leq)$ is a preordered set and 2^X is the corresponding power set with inclusion ordering ,we can define a map $f : X \rightarrow 2^X$ as $f(x) = \{y \in X \mid y \leq x\}$ which is a homomorphism. If $\leq$ is an order then $f : X \rightarrow f(X)$ is an isomorphism.We extend this idea in the fuzzy setting.

Definition 3.1: *A fuzzy set $\mu:X \rightarrow I$ on a preordered set (X, ρ) is called an upper set if $\mu(x) \wedge \rho(x, y) \leq \mu(y)$ for any $x, y \in X$.*
Dually, μ is called a lower set if $\mu(y) \wedge \rho(x, y) \leq \mu(x)$ for any $x, y \in X$.
A fuzzy set μ is an upper set in (X, ρ) iff it is a lower set in (X, ρ_{op})
In particular, if ρ is a fuzzy equivalence relation then a fuzzy set μ is an upper set in (X, ρ) iff it is an lower set in (X, ρ).

Definition 3.2: *Let (X, ρ) be a fuzzy preordered set and $z \in X$ then the fuzzy set $u(z)(x) = \rho(z, x)$ is an upper set, called the principal upper set generated by z.*
Similarly, the fuzzy set $l(z)(x) = \rho(x, z)$ is a down set, called the principal down set generated by z.

Definition 3.3: *Let* (X, ρ) *and* (Y, σ) *be preordered fuzzy sets.We say that h:X→Y.*
    1. order preserving, if $\rho(x, y) \leq \sigma(h(x), h(y))$ for all $x, y \in X$
    2. order homomorphism, if $\rho(x, y) \leq \sigma(h(x), h(y))$, for all $x, y \in X$
    3. order isomorphism, if h is an injective order homomorphism.

Proposition 3.4: *If* (X, ρ) *is a foset ,then any homomorphism defined on* (X, ρ) *is injective.*
Proof: Let (X, ρ) be a foset and $h:(X, \rho) \to (Y, \sigma)$ a homomorphism. Suppose $h(x)=h(y)$ then $\rho(x, y) = \sigma(h(x), h(y)) = 1$ and $\rho(y, x) = \sigma(h(y), h(x)) = 1$ imply x=y. Hence, h is injective.

$$\rho : I \times I \to I \text{ be a fuzzy order.Define } \varepsilon : I^X \times I^X \to I \text{ as}$$

Proposition 3.5 *Let* $\quad\quad\quad \varepsilon(\mu, \nu) = \bigwedge (\rho(\mu(x), \nu(x)) \mid x \in X)$

Then, ε is a fuzzy order on I^X.
Proof: Clearly, $\varepsilon(\mu, \mu) = \bigwedge (\rho(\mu(x), \mu(x)) \mid x \in X) = 1$. So, ε is reflexive.
Using definition $\varepsilon(\mu, \nu) = \varepsilon(\nu, \mu)$ Thus, ε is symmetric.

$$\varepsilon(\mu, \omega) \wedge \varepsilon(\omega, \nu) = (\bigwedge \{\rho(\mu(x), \omega(x)) \mid x \in X\}) \wedge (\bigwedge \{\rho(\omega(y), \nu(y)) \mid y \in X\})$$
$$\leq (\bigwedge_{x \in X} \{\rho(\mu(x), \omega(x))\}) \wedge (\bigwedge \{\rho(\omega(y), \nu(y)) \mid y \in X\})$$
$$\leq \bigwedge_{x \in X} \bigwedge_{y \in X} \{\rho(\mu(x), \omega(x)) \wedge \rho(\omega(y), \nu(y))\}$$
$$\leq \bigwedge_{x \in X} \{\rho(\mu(x), \omega(x)) \wedge \rho(\omega(x), \nu(x))\}$$
$$\leq \bigwedge \{\rho(\mu(x), \nu(x)) \mid x \in X\}$$
$$= \varepsilon(\mu, \nu).$$

Thus, ε is a fuzzy order on I^X.

Theorem 3.6: *Let* (X, ρ) *be a fuzzy preordered set and z ∈X .We define l:X→I.*
as $l(z)(x) = \rho(x, z)$. *Then l is a homomorphism from* (X, ρ) *to* (I^X,ε). *Furher if ρ is a fuzzy order then l gives an isomorphism between* (X, ρ) *and (l(X),ε).*
Proof: Let z,w ∈X. By transitivity of ρ we have $\rho(x, z) \wedge \rho(z, w) \leq \rho(x, w)$.
So, $\rho(z, w) \leq \rho(x, z) \wedge \rho(x, w)$, therefore
$$\varepsilon(l(z), l(w)) = \bigwedge (l(z)(x), l(w)(x)) = \bigwedge (\rho(x, z) \wedge \rho(x, w) \mid x \in X) \geq \rho(z, w)$$
Also, $\varepsilon(l(z), l(w)) = \bigwedge (\rho(x, z) \wedge \rho(x, w) \mid x \in X) \leq (\rho(z, z) \wedge \rho(z, w)) = 1 \wedge \rho(z, w) = \rho(z, w)$.
Hence, $\varepsilon(l(z), l(w)) = \rho(z, w)$ for any $z, w \in X$ So, *l* s a homomorphism.
Now, suppose *l* is a fuzzy order and z,w ∈X such that *l(z)=l(w)*
Then $l(z)(z) = l(w)(z)$ and $l(z)(w) = l(w)(w)$ This implies $\rho(z, z) = \rho(w, z) = 1$ and $\rho(z, w) = \rho(w, w) = 1$. Thus, $z = w$. Hence, *l:X→l(X)* is an isomorphism.

4. Orders and Metrics

In this section we show that fuzzy preorders on a set X are closely related to pseudo quasi metric on X.

Definition 4.1: *Let X be a nonempty set and* $\delta : X \times X \to [0,1]$ *is a mapping.Consider the following axioms for any* $x, y, z \in X$.
    i) $\delta(x, x) = 0$.
    ii) $\delta(x, y) = \delta(y, x) = 0 \Rightarrow x = y$.
    iii) $\delta(x, y) + \delta(y, z) \geq \delta(x, z)$
    iv) $\delta(x, y) = \delta(y, x)$
    v) $\delta(x, y) \vee \delta(y, z) \geq \delta(x, z)$

Then (X, δ) is called
- a pseudo metric space if it satisfies i),iii) and iv).
- a quasi metric spqce if it satisfies i),ii) and iii).
- a pseudo quasi metric space if it satisfies i) and iii).
- a pseudo quasi ultrametric space if it satisfies i) and v).
- a quasi ultrametric space if it satisfies i),ii) and v)

Remark 4.2: *Associated with every metric* $d : X \times X \to [0, \infty)$ *there is an equivalent bounded metric* $\delta : X \times X \to [0,1]$. *So,we are using* [0,1] *as range of* δ *in the definition.*

Theorem 4.3: *Let X be a nonempty set.* $d : X \times X \to [0, \infty)$ *a map and set* $\rho(x.y) = 1 - d(x, y)$ *for any* $x, y \in X$.*Then*
    1) ρ is a fuzzy preorder iff δ is a pseudo quasi ultra metic.
    2) ρ is a fuzzy order iff δ is a quasi ultra metic
Proof*:*
1) Suppose ρ is a fuzzy preorder then $\delta(x, x) = 1 - \rho(x, x) = 1 - 1 = 0$.
Also,
$$\delta(x, y) \vee \delta(y, z) = (1 - \rho(x, y)) \vee (1 - \rho(y, z)) = 1 - (\rho(x, y) \wedge \rho(y, z)) \geq 1 - (\rho(x, z)) = \delta(x, z)$$
Hence, δ is a pseudo quasi ultra metic. Converse can be proved similarly.
2) Suppose ρ is a fuzzy order then by i) δ is a pseudo quasi ultra metrics.
Further, if $\delta(x, y) = \delta(y, x) = 0$ then $1 - \rho(x, y) = 1 - \rho(y, x) = 0$
    implies $\rho(x, y) = \rho(y, x) = 1$ so $x = y$.
Hence, ρ is a fuzzy order. Converse can be proved similarly.

5. Fixed Point Theorem for Fuzzy Ordered Structures

Here we introduce order preserving sequences for fuzzy orders

Definition 5.1: *Let* ρ *be a fuzzy preorder relation on a set X. We say that a sequence in* $(x_n)_{n \in N}$ *is almost* ρ_α *preserving if for* $\alpha \in I, \alpha \neq 1$ *there exists* $k \in N$ *such that* $\rho(x_n, x_{n+1}) \geq \alpha$ *whenever* $n \geq k$
Further a sequence in $(x_n)_{n \in N}$ is almost ρ-preserving if for each $\alpha \in I, \alpha \neq 1$ the sequence $(x_n)_{n \in N}$ is almost ρ_α-preserving.
Definition 5.2 : *Almost* ρ-*preserving sequence* $(x_n)_{n \in N}$ *is said to converge to* $l \in X$ *and we write* $l = lim(x_n)$ *if* $lim(\rho(x_n, x)) = \rho(l, x)$ *for all* $x \in X$.
Definition 5.3: *The fposet* (X, ρ) *is said to be order complete if every almost* ρ-*preserving sequence is convergent.*
Proposition 5.4: *Let* (X, ρ) *be a fuzzy ordered set then limit of an almost* ρ-*preserving sequence* $(x_n)_{n \in N}$,*if it exists, is unique.*
Proof: Suppose that $lim(x_n) = l$ and $lim(x_n) = m$.
By definition, $\rho(l, x) = lim(\rho(x_n, x))$ and $\rho(m, x) = lim(\rho(x_n, x))$ for all $x \in X$.
In particular by setting $x = l$ we have $lim(\rho(x_n, x)) = \rho(l, l) = 1 = \rho(m, l)$
Similarly by setting $x = m$ we get $\rho(l, m) = 1$. Thus $\rho(l, m) = \rho(m, l) = 1$.
So, $l=m$.
Definition 5.5: *Let* (X, ρ) *be a fuzzy partial ordered set,a function* $f : X \to X$ *is order preserving if* $\rho(f(x), f(y)) \geq \rho(x, y)$ *for all* $x, y \in X$.
A order preserving map is said to be fuzzy order continuous if for every almost ρ-preserving sequence $(x_n)_{n \in N}$ with $lim(x_n) = l$ imply $lim(f(x_n)) = f(l)$.

Definition 5.6: *Let* (X, ρ) *be a fuzzy ordered set and* $f : X \to X$. *We say that* $x \in X$ *is a fixed point of f if* $f(x) = x$,*that is* , $\rho(x, f(x)) = \rho(f(x), x) = 1.$

Theorem 5.7: *Let* (X, ρ) *be a complete ordered set and* $f : X \to X$ *is a continuous map such that* $\rho(x, f(x)) = 1$ *for a suitable* $x \in X$. *Then,f has a fixed point.*

Proof: Consider the sequence $(x, f(x), f^2(x), \cdots)$. Since f is order preserving,this sequence is almost ρ preserving.

So, we have $1 = \rho(x, f(x)) \leq \rho(f(x), f^2(x)) \leq \cdots \leq (f^n(x), f^{n+1}(x))$

Thus for each $m \geq n$ we have

$\rho(f^n(x), f^m(x)) \geq \rho(f^n(x)^{n+1}(x)) \wedge \cdots \wedge \rho(f^{m-1}(x), f^m(x)) \geq 1 \wedge \cdots \wedge 1 = 1.$

Hence, sequence $(f^n(x))$ is almost ρ-preserving. It follows from the completeness of ρ that there is a limit *l* of the sequence $(f^n(x))$.

By continuity of f , *f(l)* is the limit of sequence $f^{n+1}(x)$ and hence of $(f^n(x))$.

So, by uniqueness of limit of almost ρ-preserving sequences we have *f(l)=l*.

Thus *l* is a fixed point of f.

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