

A Connection between Hardy-Ramanujan Number and Special Pythagorean Triangles

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Abstract: Special Pythagorean Triangles are obtained in relation with the Hardy-Ramanujan Number 1729. Some special cases are also discussed. A few interesting results are obtained.

Introduction:

The famous story about the Ramanujan number 1729, sometimes also called Hardy-Ramanujan Number leaves the reader in stupor of awesome admiration for Srinivas Ramanujan. So also is the case with Pythagoras, who gave the world scintillating Pythagorean Theorem, which continues to illuminate the minds of mathematicians all over the world till today. The contribution of this theorem is that it generated, very often, quite elementary-seeming problems, which led to the development of techniques to solve a number of Diophantine Equations including Fermat's Last Theorem that has directed to the advancement of much of modern algebra and number theory. So, combining the two, to find out the Special Pythagorean Triangles, in connection with the Ramanujan number, is the main objective of this paper.

2. Method of Analysis:

2.1 Definition: The number 1729 is called Hardy-Ramanujan Number. It is the smallest natural number which can be expressed as the sum of two cubes in two different ways:

$$\begin{aligned} 1729 &= m^3 + n^3 \\ &= 10^3 + 9^3 \\ &= 12^3 + 1^3, m, n \in \mathbb{N} \text{ and } m > n. \end{aligned}$$

Table 2.1: m, n of Hardy-Ramanujan Number

S.N.	m	n
1.	10	9
2.	12	1

2.2 Definition: If X, Y and Z are natural numbers and satisfy Pythagorean equation,

$$X^2 + Y^2 = Z^2,$$

then (X, Y, Z) is called a *Pythagorean Triangle*. X and Y are called its two legs and Z is called its hypotenuse.

2.3 Definition: A solution of Pythagorean Equation is called *Primitive* if X, Y and Z are natural numbers and have no common divisor greater than one, i.e., $(X, Y, Z) = 1$. The primitive solutions of the Pythagorean Equation,

$$X^2 + Y^2 = Z^2 \tag{2.1}$$

is given by [6]

$$X = m^2 - n^2, Y = 2mn, Z = m^2 + n^2 \tag{2.2}$$

2.4 Special Pythagorean Triangles

2.4.1 Case 1: When m, n are of Ramanujan Number (see Table 2.1), we get two Pythagorean Triangles. One of them has one leg and hypotenuse consecutive (Table 2.2)

Table 2.2: Pythagorean Triangles with m, n of Ramanujan Number

S.N.	m	n	X	Y	Z	X^2	Y^2	$X^2 + Y^2 = Z^2$
1	10	9	19	180	181	361	32400	32761
2	12	1	143	24	145	20449	576	21025

2.4.2 Case 2: When $X = 1729$, then

$$X = m^2 - n^2 = 1729 \tag{2.3}$$

which is a Quadratic Diophantine Equation. Solving equation (2.3) for m, n we get four solutions as shown in Table 2.3 below:

Table 2.3: (X, Y, Z) when $X = 1729$

S.N.	m	n	X	Y	Z	X^2	Y^2	$X^2 + Y^2 = Z^2$
1	55	36	1729	3960	4321	2989441	15681600	18671041
2	73	60	1729	8760	8929	2989441	76737600	79727041
3	127	120	1729	30480	30529	2989441	929030400	932019841
4	865	864	1729	1494720	1494721	2989441	2234187878400	2234190867841

2.4.3 Case 3: When $Y = 1729$ then

$Y = 2mn = 1729$, which is impossible as Y is even.

2.4.4 Case 4: When $Z = 1729$, then

$Z = m^2 + n^2 = 1729$, we again get no solution.

4.2.5 Case 5: Hypotenuse and one leg are consecutive and one leg equals to 1729:

When hypotenuse and one leg are consecutive, then in such case either $Z = X + 1$ or $Z = Y + 1$.

Now, $Z \neq X + 1$, for if $Z = X + 1$, then by equation (2.2) we get

$$m^2 + n^2 = m^2 - n^2 + 1$$

$$\Rightarrow 2n^2 = 1$$

which gives n as irrational number, which is a contradiction. Assuming $Z = Y + 1$, then equation (2.2) gives

$$m^2 + n^2 = 2mn + 1$$

$$\Rightarrow (m - n)^2 = 1$$

$$\Rightarrow m = n + 1$$

$$\therefore X = 2n + 1, Y = 2n^2 + 2n, Z = 2n^2 + 2n + 1 \tag{2.4}$$

Taking $X = 1729$, we get just one solution, which is listed in Table 2.4 below:

Table 2.4: (X, Y, Z) with $X = 1729$ and $Z = Y + 1$

S.N.	M	n	X	Y	Z
1	865	864	1729	1494720	1494721

3. Observations and Conclusion:

The following observations are made:

1. $1729 = 10^3 + 9^3 = 12^3 + 1^3$
2. $1729 = 55^2 - 36^2 = 73^2 - 60^2 = 127^2 - 120^2 = 865^2 - 864^2$
3. $X + Y + Z = 0 \pmod{2}$.
4. $Y + Z - X = 0 \pmod{2}$.
5. $(X + Y + Z)(X + Y - Z) = 0 \pmod{8}$.
6. $(Y + Z - X)^2 = 2(Y + Z)(Z - X)$.
7. $(X + 2Y + Z)^2 = (Z - X)^2 + 4(X + Y)(Y + Z)$.

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