

A Connection between Hardy-Ramanujan Number and Special Pythagorean Triangles

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Abstract: Special Pythagorean Triangles are obtained in relation with the Hardy-Ramanujan Number 1729. Some special cases are also discussed. A few interesting results are obtained.

Introduction:

The famous story about the Ramanujan number 1729, sometimes also called Hardy-Ramanujan Number leaves the reader in stupor of awesome admiration for Srinivas Ramanujan. So also is the case with Pythagoras, who gave the world scintillating Pythagorean Theorem, which continues to illuminate the minds of mathematicians all over the world till today. The contribution of this theorem is that it generated, very often, quite elementary-seeming problems, which led to the development of techniques to solve a number of Diophantine Equations including Fermat's Last Theorem that has directed to the advancement of much of modern algebra and number theory. So, combining the two, to find out the Special Pythagorean Triangles, in connection with the Ramanujan number, is the main objective of this paper.

2. Method of Analysis:

2.1 Definition: The number 1729 is called Hardy-Ramanujan Number. It is the smallest natural number which can be expressed as the sum of two cubes in two different ways:

$$\begin{aligned} 1729 &= m^3 + n^3 \\ &= 10^3 + 9^3 \\ &= 12^3 + 1^3, m, n \in \mathbb{N} \text{ and } m > n. \end{aligned}$$

Table 2.1: m, n of Hardy-Ramanujan Number

| S.N. | m | n |
|------|----|---|
| 1. | 10 | 9 |
| 2. | 12 | 1 |

2.2 Definition: If X, Y and Z are natural numbers and satisfy Pythagorean equation,

$$X^2 + Y^2 = Z^2,$$

then (X, Y, Z) is called a *Pythagorean Triangle*. X and Y are called its two legs and Z is called its hypotenuse.

2.3 Definition: A solution of Pythagorean Equation is called *Primitive* if X, Y and Z are natural numbers and have no common divisor greater than one, i.e., $(X, Y, Z) = 1$. The primitive solutions of the Pythagorean Equation,

$$X^2 + Y^2 = Z^2 \tag{2.1}$$

is given by [6]

$$X = m^2 - n^2, Y = 2mn, Z = m^2 + n^2 \tag{2.2}$$

2.4 Special Pythagorean Triangles

2.4.1 Case 1: When m, n are of Ramanujan Number (see Table 2.1), we get two Pythagorean Triangles. One of them has one leg and hypotenuse consecutive (Table 2.2)

Table 2.2: Pythagorean Triangles with m, n of Ramanujan Number

| S.N. | m | n | X | Y | Z | X^2 | Y^2 | $X^2 + Y^2 = Z^2$ |
|------|-----|-----|-----|-----|-----|-------|-------|-------------------|
| 1 | 10 | 9 | 19 | 180 | 181 | 361 | 32400 | 32761 |
| 2 | 12 | 1 | 143 | 24 | 145 | 20449 | 576 | 21025 |

2.4.2 Case 2: When $X = 1729$, then

$$X = m^2 - n^2 = 1729 \tag{2.3}$$

which is a Quadratic Diophantine Equation. Solving equation (2.3) for m, n we get four solutions as shown in Table 2.3 below:

Table 2.3: (X, Y, Z) when $X = 1729$

| S.N. | m | n | X | Y | Z | X^2 | Y^2 | $X^2 + Y^2 = Z^2$ |
|------|-----|-----|------|---------|---------|---------|---------------|-------------------|
| 1 | 55 | 36 | 1729 | 3960 | 4321 | 2989441 | 15681600 | 18671041 |
| 2 | 73 | 60 | 1729 | 8760 | 8929 | 2989441 | 76737600 | 79727041 |
| 3 | 127 | 120 | 1729 | 30480 | 30529 | 2989441 | 929030400 | 932019841 |
| 4 | 865 | 864 | 1729 | 1494720 | 1494721 | 2989441 | 2234187878400 | 2234190867841 |

2.4.3 Case 3: When $Y = 1729$ then

$Y = 2mn = 1729$, which is impossible as Y is even.

2.4.4 Case 4: When $Z = 1729$, then

$Z = m^2 + n^2 = 1729$, we again get no solution.

4.2.5 Case 5: Hypotenuse and one leg are consecutive and one leg equals to 1729:

When hypotenuse and one leg are consecutive, then in such case either $Z = X + 1$ or $Z = Y + 1$.

Now, $Z \neq X + 1$, for if $Z = X + 1$, then by equation (2.2) we get

$$m^2 + n^2 = m^2 - n^2 + 1$$

$$\Rightarrow 2n^2 = 1$$

which gives n as irrational number, which is a contradiction. Assuming $Z = Y + 1$, then equation (2.2) gives

$$m^2 + n^2 = 2mn + 1$$

$$\Rightarrow (m - n)^2 = 1$$

$$\Rightarrow m = n + 1$$

$$\therefore X = 2n + 1, Y = 2n^2 + 2n, Z = 2n^2 + 2n + 1 \tag{2.4}$$

Taking $X = 1729$, we get just one solution, which is listed in Table 2.4 below:

Table 2.4: (X, Y, Z) with $X = 1729$ and $Z = Y + 1$

| S.N. | M | n | X | Y | Z |
|------|-----|-----|------|---------|---------|
| 1 | 865 | 864 | 1729 | 1494720 | 1494721 |

3. Observations and Conclusion:

The following observations are made:

1. $1729 = 10^3 + 9^3 = 12^3 + 1^3$
2. $1729 = 55^2 - 36^2 = 73^2 - 60^2 = 127^2 - 120^2 = 865^2 - 864^2$
3. $X + Y + Z = 0 \pmod{2}$.
4. $Y + Z - X = 0 \pmod{2}$.
5. $(X + Y + Z)(X + Y - Z) = 0 \pmod{8}$.
6. $(Y + Z - X)^2 = 2(Y + Z)(Z - X)$.
7. $(X + 2Y + Z)^2 = (Z - X)^2 + 4(X + Y)(Y + Z)$.

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