

Special Dio 3 – tuples

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Abstract. We search for three distinct polynomials with integer coefficient such that the product of any two members of the set minus their sum and increased by a non- zero integer (or polynomial with integer coefficient) is a perfect square.

Introduction:

The problem of constructing the set with property that the product of any two its distinct elements is one less than a square has a very long history and such sets were studied by Diophantus. A set of m positive integers $\{a_1, a_2, \dots, a_m\}$ is called a Diophantine m -tuple

$$a_i * a_j + 1 \text{ is a perfect square,} \quad (1)$$

a perfect square for all $1 \leq i < j \leq m$. Many generalizations of this problem (1) were considered since antiquity, for example by adding a fixed integer n instead of 1, looking k th powers instead of squares or considering the powers over domains other than Z or Q . Many mathematicians consider the problem of the existence of Diophantine quadruples with the property $D(n)$ for any arbitrary integer n and also for any linear polynomials in n . In this context one may refer [1-16]. The above results motivated us the following definition:

A set of three distinct polynomials with integer coefficient (a_1, a_2, a_3) is said to be a special dio 3- tuple with property $D(n)$ if $a_i * a_j - (a_i + a_j) + n$ is a perfect square for all $1 \leq i < j \leq 3$.

In the above definition n may be a non – zero integer or polynomial with integer coefficients. In this communication we consider a few special dio 3 tuples of polygonal numbers from $t_{6,n}$ to $t_{10,n}$ and centered polygonal numbers from $ct_{6,n}$ to $ct_{10,n}$ with their corresponding properties.

Notations:

$$t_{m,n} = n \left(1 + \frac{(n-1)(m-2)}{2} \right) = \text{Polygonal number of rank } n \text{ with sides } m$$

$$ct_{m,n} = \frac{mn(n+1)}{2} + 1 = \text{Centered Polygonal number of rank } n \text{ with sides } m$$

Construction of Dio 3 - tuples for Hexagonal numbers

Let $a = t_{6,n}$, $b = t_{6,n-2}$ be Hexagonal number of rank n and $n-2$ respectively such that $ab - (a+b) + 4n^2 - 10n + 11$ is a perfect square say γ^2
 Let c be any non zero integer such that

$$ac - (a + c) + 4n^2 - 10n + 11 = \alpha^2 \tag{2}$$

$$bc - (b + c) + 4n^2 - 10n + 11 = \beta^2 \tag{3}$$

On solving equations (2) and (3), we get

$$(b - 1)\alpha^2 - (a - 1)\beta^2 = (a - b) + (4n^2 - 10n + 11)(b - a) \tag{4}$$

Assuming $\alpha = x + (a - 1)T$ and $\beta = x + (b - 1)T$, in (4) it reduces to

$$x^2 = (b - 1)(a - 1)T^2 + (4n^2 - 10n + 11) \tag{5}$$

The initial solution of equation (5) is given by

$$T_0 = 1 \text{ and } x_0 = n^2 - 5n + 1 \tag{6}$$

Therefore, $\alpha = 4n^2 - 6n$ (7)

On substituting the values of α and a in equation (2), we get

$$\begin{aligned} c &= 8n^2 - 20n + 11 \\ &= t_{6,2n-2} - 2n + 1 \end{aligned}$$

Therefore, triple $(t_{6,n}, t_{6,n-2}, t_{6,2n-2} - 2n + 1)$ is Dio 3- tuple with property $D(4n^2 - 10n + 11)$

For simplicity, we present in the table below a few Dio 3- tuples for polygonal numbers from $t_{6,n}$ to $t_{10,n}$ with suitable properties.

a	b	c	D(n)
$t_{6,n}$	$t_{6,n-2}$	$t_{18,n} - 13n + 9$	$D(10)$
	$t_{6,n-1}$	$t_{6,2n-2} + 6n - 4$	$D(10n^2 - 15n + 7)$
$2t_{7,n}$	$2t_{7,n-2}$	$2t_{7,2n-2} - 6n + 5$	$D(10n^2 - 26n + 35)$
		$t_{42,n} - 33n + 29$	$D(30)$
	$2t_{7,n-1}$	$2t_{7,2n-2} + 14n + 13$	$D(25n^2 - 40n + 17)$
$t_{8,n}$	$t_{8,n-2}$	$t_{8,2n-2} - 4n + 7$	$D(18n^2 - 48n + 32)$
		$t_{26,n} - 21n + 7$	$D(17)$
	$t_{8,n-1}$	$t_{8,2n-2} + 8n - 6$	$D(18n^2 - 30n + 14)$
$2t_{9,n}$	$2t_{9,n-2}$	$2t_{9,2n-2} - 10n + 11$	$D(28n^2 - 76n + 74)$
		$t_{58,n} - 49n + 45$	$D(54)$
	$2t_{9,n-1}$	$2t_{9,2n-2} + 18n - 19$	$D(35n^2 - 60n + 28)$
$t_{10,n}$	$t_{10,n-2}$	$t_{10,2n-2} + 6n - 1$	$D(24n^2 - 66n + 47)$
		$t_{34,n} - 29n + 25$	$D(26)$
	$t_{10,n-1}$	$t_{10,2n-2} + 10n - 10$	$D(20n^2 - 35n + 16)$

Construction of Dio 3 - tuples for Centered Hexagonal numbers

Let $a = ct_{6,n}$, $b = ct_{6,n-2}$ be Centered Hexagonal number of rank n and n-2 respectively such that $ab - (a + b) + 24n^2 - 30n + 2$ is a perfect square say γ^2

Let c be any non zero integer such that

$$ac - (a + c) + 24n^2 - 30n + 2 = \alpha^2 \tag{8}$$

$$bc - (b + c) + 24n^2 - 30n + 4 = \beta^2 \tag{9}$$

On solving equations (8) and (9), we get

$$(b - 1)\alpha^2 - (a - 1)\beta^2 = (a - b) + (24n^2 - 30n + 2)(b - a) \tag{10}$$

Assuming $\alpha = x + (a - 1)T$ and $\beta = x + (b - 1)T$ in (10), it reduces to

$$x^2 = (b - 1)(a - 1)T^2 + (24n^2 - 30n + 1) \tag{11}$$

The initial solution of equation (11) is given by

$$T_0 = 1 \text{ and } x_0 = 3n^2 - 3n + 1 \tag{12}$$

Therefore, $\alpha = 6n^2 + 1$ (13)

On substituting the values of α and a in equation (8), we get

$$c = 12n^2 - 12n + 9 \\ = ct_{6,2n-2} + 6n + 8$$

Therefore, triple $(ct_{6,n}, ct_{6,n-2}, ct_{6,2n-2} + 6n + 8)$ is Dio 3- tuple with property $D(24n^2 - 30n + 2)$

For simplicity, we present in the table below a few Dio 3- tuples for Centered polygonal numbers from $ct_{6,n}$ to $ct_{10,n}$ with suitable properties.

a	b	C	D(n)
$ct_{6,n}$	$ct_{6,n-2}$	$ct_{24,n} - 24n$	$D(10)$
	$ct_{6,n-1}$	$ct_{6,2n-2} + 14n - 4$	$D(-12n^3 + 19n^2 - 4n + 2)$
$2ct_{7,n}$	$2ct_{7,n-2}$	$2ct_{7,2n-2} + 14n + 9$	$D(140n^2 - 140n + 2)$
		$ct_{56,n} - 56n + 4$	$D(22)$
	$2ct_{7,n-1}$	$2ct_{7,2n-2} + 32n - 9$	$D(-70n^3 + 88n^2 - 20n + 4)$
$ct_{8,n}$	$ct_{8,n-2}$	$ct_{8,2n-2} + 8n - 8$	$D(40n^2 - 40n + 2)$
		$ct_{32,n} + 32n$	$D(17)$
	$ct_{8,n-1}$	$ct_{8,2n-2} + 20n - 4$	$D(-16n^3 + 28n^2 - 4n + 2)$
$2ct_{9,n}$	$2ct_{9,n-2}$	$2ct_{9,2n-2} + 18n + 11$	$D(234n^2 - 234n + 7)$
		$ct_{72,n} - 72n + 4$	$D(46)$
	$2ct_{9,n-1}$	$2ct_{9,2n-2} + 24n - 13$	$D(-126n^3 + 148n^2 - 28n + 4)$
$ct_{10,n}$	$ct_{10,n-2}$	$ct_{10,2n-2} + 10n + 2$	$D(60n^2 - 60n + 2)$
		$ct_{40,n} - 40n$	$D(26)$
	$ct_{10,n-1}$	$ct_{10,2n-2} + 26n - 8$	$D(-20n^3 + 39n^2 - 4n + 2)$

Conclusion:

In this paper we have presented a few examples of constructing a special Dio 3 tuples for polygonal numbers and centered polygonal numbers with suitable properties. To conclude one may search for Dio 3 – tuples for higher order polygonal numbers and centered polygonal numbers with their corresponding suitable properties.

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