Special Dio 3 – tuples
M.A.Gopalan1*, K.Geetha2, Manju Somanath3

1. Professor, Department of Mathematics, Shrimathi Indira Gandhi College, Trichy,
2. Assistant Professor, Department of Mathematics, Cauvery College for Women, Trichy,
3. Assistant Professor, Department of Mathematics, National College, Trichy,

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Abstract. We search for three distinct polynomials with integer coefficient such that the product of any two members of the set minus their sum and increased by a non-zero integer (or polynomial with integer coefficient) is a perfect square.

Introduction:
The problem of constructing the set with property that the product of any two its distinct elements is one less than a square has a very long history and such sets were studied by Diophantus. A set of m positive integers \( \{a_1, a_2, \ldots, a_m\} \) is called a Diophantine m-tuple

\[ a_i * a_j + 1 \text{ is a perfect square,} \]  

(1)
a perfect square for all \( 1 \leq i < j \leq m \). Many generalizations of this problem (1) were considered since antiquity, for example by adding a fixed integer \( n \) instead of 1, looking \( k \)th powers instead of squares or considering the powers over domains other than \( Z \) or \( Q \). Many mathematicians consider the problem of the existence of Diophantine quadruples with the property \( D(n) \) for any arbitrary integer \( n \) and also for any linear polynomials in \( n \). In this context one may refer [1-16]. The above results motivated us the following definition:

A set of three distinct polynomials with integer coefficient \( (a_1, a_2, a_3) \) is said to be a special dio 3-tuple with property \( D(n) \) if

\[ a_i * a_j - (a_i + a_j) + n \text{ is a perfect square for all } 1 \leq i < j \leq 3. \]

In the above definition \( n \) may be a non-zero integer or polynomial with integer coefficients. In this communication we consider a few special dio 3 tuples of polygonal numbers from \( t_{6,n} \) to \( t_{10,n} \) and centered polygonal numbers from \( C_{t_{6,n}} \) to \( C_{t_{10,n}} \) with their corresponding properties.

Notations:
\[ t_{m,n} = n \left( 1 + \frac{(n-1)(m-2)}{2} \right) \text{ Polynomial number of rank } n \text{ with sides } m \]
\[ c_{t_{m,n}} = \frac{nm(n+1)}{2} + 1 \text{ Centered Polygonal number of rank } n \text{ with sides } m \]

Construction of Dio 3 - tuples for Hexagonal numbers

Let \( a = t_{6,n}, \quad b = t_{6,n-2} \) be Hexagonal number of rank \( n \) and \( n-2 \) respectively such that

\[ ab - (a+b) + 4n^2 - 10n + 11 \text{ is a perfect square say } \gamma^2 \]

Let \( c \) be any non zero integer such that
On solving equations (2) and (3), we get

\[ (b-1)\alpha^2 - (a-1)\beta^2 = (a-b) + \left(4n^2 - 10n + 11\right)(b-a) \]  

Assuming \( \alpha = x + (a-1)T \) and \( \beta = x + (b-1)T \), in (4) it reduces to

\[ x^2 = (b-1)(a-1)T^2 + \left(4n^2 - 10n + 10\right) \]  

The initial solution of equation (5) is given by

\[ T_0 = 1 \quad \text{and} \quad x_0 = n^2 - 5n + 1 \]  

Therefore, \( \alpha = 4n^2 - 6n \)  

On substuting the values of \( \alpha \) and \( a \) in equation (2), we get

\[ c = 8n^2 - 20n + 11 = t_{6,2n-2} - 2n + 1 \]  

Therefore, triple \( \left(t_{6,n}, t_{6,n-2}, t_{6,2n-2} - 2n + 1\right) \) is Dio 3- tuple with property \( D\left(4n^2 - 10n + 11\right) \)

For simplicity, we present in the table below a few Dio 3- tuples for polygonal numbers from \( t_{6,n} \) to \( t_{10,n} \) with suitable properties.

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>c</th>
<th>D(n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_{6,n} )</td>
<td>( t_{6,n-2} )</td>
<td>( t_{18,n} - 13n + 9 )</td>
<td>( D(10) )</td>
</tr>
<tr>
<td>( t_{6,n-1} )</td>
<td>( t_{6,2n-2} + 6n - 4 )</td>
<td>( D(10n^2 - 15n + 7) )</td>
<td></td>
</tr>
<tr>
<td>( 2t_{7,n} )</td>
<td>( 2t_{7,2n-2} - 6n + 5 )</td>
<td>( t_{42,n} - 33n + 29 )</td>
<td>( D(10n^2 - 26n + 35) )</td>
</tr>
<tr>
<td></td>
<td>( 2t_{7,n-1} )</td>
<td>( 2t_{7,2n-2} + 14n + 13 )</td>
<td>( D(25n^2 - 40n + 17) )</td>
</tr>
<tr>
<td>( t_{8,n} )</td>
<td>( t_{8,n-2} )</td>
<td>( t_{8,2n-2} - 4n + 7 )</td>
<td>( D(18n^2 - 48n + 32) )</td>
</tr>
<tr>
<td></td>
<td>( t_{26,n} - 21n + 7 )</td>
<td>( D(17) )</td>
<td></td>
</tr>
<tr>
<td>( t_{8,n-1} )</td>
<td>( t_{8,2n-2} + 8n - 6 )</td>
<td>( D(18n^2 - 30n + 14) )</td>
<td></td>
</tr>
<tr>
<td>( 2t_{9,n} )</td>
<td>( 2t_{9,n-2} )</td>
<td>( 2t_{9,2n-2} - 10n + 11 )</td>
<td>( D(28n^2 - 76n + 74) )</td>
</tr>
<tr>
<td></td>
<td>( t_{58,n} - 49n + 45 )</td>
<td>( D(54) )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( 2t_{9,n-1} )</td>
<td>( 2t_{9,2n-2} + 18n - 19 )</td>
<td>( D(35n^2 - 60n + 28) )</td>
</tr>
<tr>
<td>( t_{10,n} )</td>
<td>( t_{10,n-2} )</td>
<td>( t_{10,2n-2} + 6n - 1 )</td>
<td>( D(24n^2 - 66n + 47) )</td>
</tr>
<tr>
<td></td>
<td>( t_{34,n} - 29n + 25 )</td>
<td>( D(26) )</td>
<td></td>
</tr>
<tr>
<td></td>
<td>( t_{10,n-1} )</td>
<td>( t_{10,2n-2} + 10n - 10 )</td>
<td>( D(20n^2 - 35n + 16) )</td>
</tr>
</tbody>
</table>
Construction of Dio 3 - tuples for Centered Hexagonal numbers

Let \( a = ct_{6,n} \), \( b = ct_{6,n-2} \) be Centered Hexagonal number of rank \( n \) and \( n-2 \) respectively such that \( ab - (a + b) + 24n^2 - 30n + 2 \) is a perfect square say \( \gamma^2 \).

Let \( c \) be any non zero integer such that

\[
ac - (a + c) + 24n^2 - 30n + 2 = \alpha^2
\]

\[
b(c - (b + c) + 24n^2 - 30n + 4 = \beta^2
\]

(8) \hspace{1cm} (9)

On solving equations (8) and (9), we get

\[
(b - 1)\alpha^2 - (a - 1)\beta^2 = (a - b) + \left(24n^2 - 30n + 2\right)(b - a)
\]

(10)

Assuming \( \alpha = x + \left(a - 1\right)T \) and \( \beta = x + \left(b - 1\right)T \) in (10), it reduces to

\[
x^2 = (b - 1)\left(a - 1\right)T^2 + \left(24n^2 - 30n + 1\right)
\]

(11)

The initial solution of equation (11) is given by

\[
I_0 = 1 \text{ and } x_0 = 3n^2 - 3n + 1
\]

(12)

Therefore,

\[
\alpha = 6n^2 + 1
\]

(13)

On substituting the values of \( \alpha \) and \( a \) in equation (8), we get

\[
c = 12n^2 - 12n + 9
\]

\[
= ct_{6,2n-2} + 6n + 8
\]

Therefore, triple \( \left(ct_{6,n} \cdot ct_{6,n-2} \cdot ct_{6,2n-2} + 6n + 8\right) \) is Dio 3- tuple with property \( D\left(24n^2 - 30n + 2\right) \).

For simplicity, we present in the table below a few Dio 3- tuples for Centered polygonal numbers from \( ct_{6,n} \) to \( ct_{10,n} \) with suitable properties.

<table>
<thead>
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<th>a</th>
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<tbody>
<tr>
<td>( ct_{6,n} )</td>
<td>( ct_{6,n-2} )</td>
<td>( ct_{24,n} - 24n )</td>
<td>( D\left(10\right) )</td>
</tr>
<tr>
<td>( ct_{6,n-1} )</td>
<td>( ct_{6,2n-2} + 14n - 4 )</td>
<td>( D\left(-12n^3 + 19n^2 - 4n + 2\right) )</td>
<td></td>
</tr>
<tr>
<td>( 2ct_{7,n} )</td>
<td>( 2ct_{7,2n-2} + 14n + 9 )</td>
<td>( D\left(140n^2 - 140n + 2\right) )</td>
<td></td>
</tr>
<tr>
<td>( ct_{5,6n} - 56n + 4 )</td>
<td>( D\left(22\right) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 2ct_{7,n-1} )</td>
<td>( 2ct_{7,2n-2} + 32n - 9 )</td>
<td>( D\left(-70n^3 + 88n^2 - 20n + 4\right) )</td>
<td></td>
</tr>
<tr>
<td>( ct_{8,n} )</td>
<td>( ct_{8,2n-2} + 8n - 8 )</td>
<td>( D\left(40n^2 - 40n + 2\right) )</td>
<td></td>
</tr>
<tr>
<td>( ct_{3,2n} + 32n )</td>
<td>( D\left(17\right) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( ct_{8,n-1} )</td>
<td>( ct_{8,2n-2} + 20n - 4 )</td>
<td>( D\left(-16n^3 + 28n^2 - 4n + 2\right) )</td>
<td></td>
</tr>
<tr>
<td>( 2ct_{9,n} )</td>
<td>( 2ct_{9,2n-2} + 18n + 11 )</td>
<td>( D\left(234n^2 - 234n + 7\right) )</td>
<td></td>
</tr>
<tr>
<td>( ct_{7,2n} - 72n + 4 )</td>
<td>( D\left(46\right) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 2ct_{9,n-1} )</td>
<td>( 2ct_{9,2n-2} + 24n - 13 )</td>
<td>( D\left(-126n^3 + 148n^2 - 28n + 4\right) )</td>
<td></td>
</tr>
<tr>
<td>( ct_{10,n} )</td>
<td>( ct_{10,2n-2} + 10n + 2 )</td>
<td>( D\left(60n^2 - 60n + 2\right) )</td>
<td></td>
</tr>
<tr>
<td>( ct_{4,0n} - 40n )</td>
<td>( D\left(26\right) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( ct_{10,n-1} )</td>
<td>( ct_{10,2n-2} + 26n - 8 )</td>
<td>( D\left(-20n^3 + 39n^2 - 4n + 2\right) )</td>
<td></td>
</tr>
</tbody>
</table>
Conclusion:
In this paper we have presented a few examples of constructing a special Dio 3 tuples for polygonal numbers and centered polygonal numbers with suitable properties. To conclude one may search for Dio 3 – tuples for higher order polygonal numbers and centered polygonal numbers with their corresponding suitable properties.

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