Special Dio 3 – tuples

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Abstract. We search for three distinct polynomials with integer coefficient such that the product of any two members of the set minus their sum and increased by a non-zero integer (or polynomial with integer coefficient) is a perfect square.

Introduction:

The problem of constructing the set with property that the product of any two its distinct elements is one less than a square has a very long history and such sets were studied by Diophantus. A set of m positive integers \( \{a_1, a_2, \ldots, a_m\} \) is called a Diophantine m-tuple

\[
a_i \cdot a_j + 1 \text{ is a perfect square,}
\]

(1)

a perfect square for all \( 1 \leq i < j \leq m \). Many generalizations of this problem (1) were considered since antiquity, for example by adding a fixed integer \( n \) instead of 1, looking \( k \)th powers instead of squares or considering the powers over domains other than \( \mathbb{Z} \) or \( \mathbb{Q} \). Many mathematicians consider the problem of the existence of Diophantine quadruples with the property \( D(n) \) for any arbitrary integer \( n \) and also for any linear polynomials in \( n \). In this context one may refer [1-16]. The above results motivated us the following definition:

A set of three distinct polynomials with integer coefficient \( \{a_1, a_2, a_3\} \) is said to be a special dio 3- tuple with property \( D(n) \) if \( a_i \cdot a_j - (a_i + a_j) + n \) is a perfect square for all \( 1 \leq i < j \leq 3 \).

In the above definition \( n \) may be a non-zero integer or polynomial with integer coefficients. In this communication we consider a few special dio 3 tuples of polygonal numbers from \( t_{6,n} \) to \( t_{10,n} \) and centered polygonal numbers from \( C_{6,n} \) to \( C_{10,n} \) with their corresponding properties.

Notations:

\[
t_{m,n} = n \left( 1 + \frac{(n-1)(m-2)}{2} \right) \quad \text{Polygonal number of rank n with sides m}
\]
\[
c_{m,n} = \frac{mm(n+1)}{2} + 1 \quad \text{Centered Polygonal number of rank n with sides m}
\]

Construction of Dio 3 - tuples for Hexagonal numbers

Let \( a = t_{6,n}, b = t_{6,n-2} \) be Hexagonal number of rank \( n \) and \( n-2 \) respectively such that

\[
a \cdot b - (a + b) + 4n^2 - 10n + 11 \text{ is a perfect square say } \gamma^2.
\]

Let \( c \) be any non zero integer such that
On solving equations (2) and (3), we get

\[ a^2 - (a + c) + 4n^2 - 10n + 11 = \alpha^2 \]
\[ b^2 - (b + c) + 4n^2 - 10n + 11 = \beta^2 \]

(2) \hspace{1cm} (3)

On solving equations (2) and (3), we get

\[ (b - 1)\alpha^2 - (a - 1)\beta^2 = (a - b) + (4n^2 - 10n + 11)(b - a) \]

(4)

Assuming \( \alpha = x + (a - 1)T \) and \( \beta = x + (b - 1)T \), in (4) it reduces to

\[ x^2 = (b - 1)(a - 1)T^2 + (4n^2 - 10n + 10) \]

(5)

The initial solution of equation (5) is given by

\[ T_0 = 1 \text{ and } x_0 = n^2 - 5n + 1 \]

(6)

Therefore,

\[ \alpha = 4n^2 - 6n \]

(7)

On substuting the values of \( \alpha \) and \( a \) in equation (2), we get

\[ c = 8n^2 - 20n + 11 \]
\[ = t_{6,2n-2} - 2n + 1 \]

Therefore, triple \( (t_{6,n}, t_{6,n-2}, t_{6,2n-2} - 2n + 1) \) is Dio 3- tuple with property \( D(4n^2 - 10n + 11) \)

For simplicity, we present in the table below a few Dio 3- tuples for polygonal numbers from \( t_{6,n} \) to \( t_{10,n} \) with suitable properties.

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>c</th>
<th>D(n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_{6,n} )</td>
<td>( t_{6,n-2} )</td>
<td>( t_{18,n} - 13n + 9 )</td>
<td>( D(10) )</td>
</tr>
<tr>
<td>( t_{6,n} )</td>
<td>( t_{6,n-1} )</td>
<td>( t_{6,2n-2} + 6n - 4 )</td>
<td>( D(10n^2 - 15n + 7) )</td>
</tr>
<tr>
<td>2( t_{7,n} )</td>
<td>2( t_{7,n-2} )</td>
<td>( 2t_{7,2n-2} - 6n + 5 )</td>
<td>( D(10n^2 - 26n + 35) )</td>
</tr>
<tr>
<td>2( t_{7,n} )</td>
<td>2( t_{7,n-1} )</td>
<td>( 2t_{7,2n-2} + 14n + 13 )</td>
<td>( D(25n^2 - 40n + 17) )</td>
</tr>
<tr>
<td>( t_{8,n} )</td>
<td>( t_{8,n-2} )</td>
<td>( t_{8,2n-2} - 4n + 7 )</td>
<td>( D(18n^2 - 48n + 32) )</td>
</tr>
<tr>
<td>( t_{8,n} )</td>
<td>( t_{8,n-1} )</td>
<td>( t_{8,2n-2} + 8n - 6 )</td>
<td>( D(18n^2 - 30n + 14) )</td>
</tr>
<tr>
<td>2( t_{9,n} )</td>
<td>2( t_{9,n-2} )</td>
<td>( 2t_{9,2n-2} - 10n + 11 )</td>
<td>( D(28n^2 - 76n + 74) )</td>
</tr>
<tr>
<td>2( t_{9,n} )</td>
<td>( t_{58,n} - 49n + 45 )</td>
<td>( D(54) )</td>
<td></td>
</tr>
<tr>
<td>2( t_{9,n} )</td>
<td>( t_{9,n-1} )</td>
<td>( 2t_{9,2n-2} + 18n - 19 )</td>
<td>( D(35n^2 - 60n + 28) )</td>
</tr>
<tr>
<td>( t_{10,n} )</td>
<td>( t_{10,n-2} )</td>
<td>( t_{10,2n-2} + 6n - 1 )</td>
<td>( D(24n^2 - 66n + 47) )</td>
</tr>
<tr>
<td>( t_{10,n} )</td>
<td>( t_{34,n} - 29n + 25 )</td>
<td>( D(26) )</td>
<td></td>
</tr>
<tr>
<td>( t_{10,n} )</td>
<td>( t_{10,n-1} )</td>
<td>( t_{10,2n-2} + 10n - 10 )</td>
<td>( D(20n^2 - 35n + 16) )</td>
</tr>
</tbody>
</table>
Construction of Dio 3 - tuples for Centered Hexagonal numbers

Let \( a = ct_{6,n} \), \( b = ct_{6,n-2} \) be Centered Hexagonal number of rank \( n \) and \( n-2 \) respectively such that \( ab - (a+b) + 24n^2 - 30n + 2 \) is a perfect square say \( \varphi^2 \).

Let \( c \) be any non zero integer such that

\[
ac - (a+c) + 24n^2 - 30n + 2 = \alpha^2 
\]

(8)

\[
b_{c}-(b+c) + 24n^2 - 30n + 4 = \beta^2 
\]

(9)

On solving equations (8) and (9), we get

\[
(b-1)\alpha^2 - (a-1)\beta^2 = (a-b) + (24n^2 - 30n + 2)(b-a) \tag{10}
\]

Assuming \( \alpha = x + (a-1)T \) and \( \beta = x + (b-1)T \) in (10), it reduces to

\[
x^2 = (b-1)(a-1)T^2 + (24n^2 - 30n + 1) \tag{11}
\]

The initial solution of equation (11) is given by

\[
T_0 = 1 \text{ and } x_0 = 3n^2 - 3n + 1 \tag{12}
\]

Therefore,

\[
\alpha = 6n^2 + 1 \tag{13}
\]

On substuting the values of \( \alpha \) and \( a \) in equation (8), we get

\[
c = 12n^2 - 12n + 9 = ct_{6,2n-2} + 6n + 8
\]

Therefore, triple \( \left( ct_{6,n}, ct_{6,n-2} \cdot ct_{6,2n-2} + 6n + 8 \right) \) is Dio 3- tuple with property \( D \left( 24n^2 - 30n + 2 \right) \).

For simplicity, we present in the table below a few Dio 3- tuples for Centered polygonal numbers from \( ct_{6,n} \) to \( ct_{10,n} \) with suitable properties.

<table>
<thead>
<tr>
<th>( a )</th>
<th>( b )</th>
<th>( c )</th>
<th>( D(n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( ct_{6,n} )</td>
<td>( ct_{6,n-2} )</td>
<td>( ct_{24,n} - 24n )</td>
<td>( D(10) )</td>
</tr>
<tr>
<td>( ct_{6,n-1} )</td>
<td>( ct_{6,2n-2} + 14n - 4 )</td>
<td>( D(-12n^3 + 19n^2 - 4n + 2) )</td>
<td></td>
</tr>
<tr>
<td>( 2ct_{7,n} )</td>
<td>( 2ct_{7,2n-2} + 14n + 9 )</td>
<td>( D(140n^2 - 140n + 2) )</td>
<td></td>
</tr>
<tr>
<td>( ct_{56,n} - 56n + 4 )</td>
<td>( D(22) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 2ct_{7,n-1} )</td>
<td>( 2ct_{7,2n-2} + 32n - 9 )</td>
<td>( D(-70n^3 + 88n^2 - 20n + 4) )</td>
<td></td>
</tr>
<tr>
<td>( ct_{8,n} )</td>
<td>( ct_{8,2n-2} + 8n - 8 )</td>
<td>( D(40n^2 - 40n + 2) )</td>
<td></td>
</tr>
<tr>
<td>( ct_{32,n} + 32n )</td>
<td>( D(17) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( ct_{8,n-1} )</td>
<td>( ct_{8,2n-2} + 20n - 4 )</td>
<td>( D(-16n^3 + 28n^2 - 4n + 2) )</td>
<td></td>
</tr>
<tr>
<td>( 2ct_{9,n} )</td>
<td>( 2ct_{9,2n-2} + 18n + 11 )</td>
<td>( D(234n^2 - 234n + 7) )</td>
<td></td>
</tr>
<tr>
<td>( ct_{72,n} - 72n + 4 )</td>
<td>( D(46) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 2ct_{9,n-1} )</td>
<td>( 2ct_{9,2n-2} + 24n - 13 )</td>
<td>( D(-126n^3 + 148n^2 - 28n + 4) )</td>
<td></td>
</tr>
<tr>
<td>( ct_{10,n} )</td>
<td>( ct_{10,2n-2} + 10n + 2 )</td>
<td>( D(60n^2 - 60n + 2) )</td>
<td></td>
</tr>
<tr>
<td>( ct_{40,n} - 40n )</td>
<td>( D(26) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( ct_{10,n-1} )</td>
<td>( ct_{10,2n-2} + 26n - 8 )</td>
<td>( D(-20n^3 + 39n^2 - 4n + 2) )</td>
<td></td>
</tr>
</tbody>
</table>
Conclusion:
In this paper we have presented a few examples of constructing a special Dio 3 tuples for polygonal numbers and centered polygonal numbers with suitable properties. To conclude one may search for Dio 3 – tuples for higher order polygonal numbers and centered polygonal numbers with their corresponding suitable properties.

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References:
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