DIO 3 – TUPLES FOR SPECIAL NUMBERS – I

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Abstract. We search for three distinct polynomials with integer coefficients such that the product of any two members of the set added with their sum and increased by a non- zero integer (or polynomial with integer coefficients) is a perfect square.

Introduction:

The problem of constructing the set with property that the product of any two its distinct elements is one less than a square has a very long history and such sets were studied by Diophantus. A set of m positive integers $\{a_1, a_2, \ldots a_m\}$ is called a Diophantine m-tuple if

$$a_i * a_j + 1 \quad \text{is a perfect square} \tag{1}$$

a perfect square for all $1 \le i \prec j \le m$. Many generalizations of this problem (1) were considered since antiquity, for example by adding a fixed integer n instead of 1, looking kth powers instead of squares or considering the powers over domains other than Z or Q. Many mathematicians consider the problem of the existence of Diophantine quadruples with the property D(n) for any arbitrary integer n and also for any linear polynomials in n. In this context one may refer [1-16]. The above results motivated us the following definition:

A set of three distinct polynomials with integer coefficient (a_1, a_2, a_3) is said to be a special dio 3- tuple with property D(n) if $a_i * a_j + (a_i + a_j) + n$ is a perfect square for all $1 \le i \prec j \le 3$.

In the above definition n may be a non – zero integer or polynomial with integer coefficients. In this communication we consider a few special dio 3 tuples of polygonal numbers from $t_{11,n}$ to $t_{15,n}$, centered polygonal numbers from $ct_{11,n}$ to $ct_{15,n}$, linear polynomials and jacobasthal - lucas number with their corresponding properties.

Notations:

$$t_{m,n} = n\left(1 + \frac{(n-1)(m-2)}{2}\right) =$$
Polygonal number of rank n with sides m

$$ct_{m,n} = \frac{mn(n+1)}{2} + 1 =$$
Centered Polygonal number of rank n with sides m

$$j_n = 2^n + 1 =$$ Jacobasthal-Lucas number of rank n

Construction of Dio 3-tuples for Hendecagonal number:

Let $a = 2t_{11,n}$, $b = 2t_{11,n-2}$ be Hendecagonal number of rank n and n-2 respectively such that $ab + (a+b) + (-18n^2 + 50n - 1)$ is a perfect square say γ^2

Let c be any non zero integer such that

$$ac + (a+c) + (-18n^2 + 50n - 1) = \alpha^2 \qquad (2)$$

$$bc + (b+c) + (-18n^2 + 50n - 1) = \beta^2 \qquad (3)$$

On solving equations (2) and (3), we get

$$(b+1)\alpha^2 - (a+1)\beta^2 = (a-b) + \left(-18n^2 + 50n - 1\right)(b-a) \qquad (4)$$

Assume $\alpha = x + (a+1)T$ and $\beta = x + (b+1)T$ and it reduces to

$$x^2 = (b+1)(a+1)T^2 + \left(-18n^2 + 50n - 2\right)^` \qquad (5)$$

The initial solution of equation (5) is given by

$$T_o = 1 \text{ and } x_0 = 9n^2 - 25n + 7 \qquad (6)$$

Therefore, $\alpha = 18n^2 - 32n + 8$ (7)

On substuting the values of α and a in equation (2), we get

$$c = 36n^2 - 100n + 65$$

$$= t_{11,n-2} + 72n + 135$$

Therefore triple $\left(2t_{11,n}, 2t_{11,n-2}, t_{11,n-2} + 72n + 135\right)$ is Dio 3- tuple with property $D(-18n^2 + 50n - 1)$

For simplicity, we present below the Dio 3-tuple for polygonal numbers from $t_{11,n}$ to $t_{15,n}$ with suitable properties.

a	b	c	D(n)
$2t_{11,n}$	$2t_{11,n-2}$	$8t_{11,n} - 72n + 67$	$D(14)$
	$2t_{11,n-1}$	$8t_{11,n-1} + 36n - 39$	$D(-9n^2 + 16n)$
$t_{12,n}$	$t_{12,n-2}$	$t_{12,n} + t_{12,n-1} + 2t_{12,n-2} + 12n - 39$	$D(-20n^2 + 56n - 19)$
		$t_{12,n} + t_{12,n-2} - 28n + 11$	$D(-3)$
	$t_{12,n-1}$	$2t_{12,n-1} - 6n - 4$	$D(10n^3 - 27n^2 + 22n - 5)$
$2t_{13,n}$	$2t_{13,n-2}$	$8t_{13,n-2} + 88n - 169$	$D(-44n^2 + 124n + 2)$
		$2t_{13,2n-1} + 62n - 63$	$D(38)$
	$t_{8,n-1}$	$2t_{13,2n-2} - 26n + 33$	$D(-33n^2 + 60n - 4)$
$t_{14,n}$	$t_{14,n-2}$	$t_{50,n} - 45(n-1)$	$D(28n^2 - 76n + 74)$
		$t_{14,2(n-1)} - 10n + 13$	$D(2)$
	$t_{14,n-1}$	$2t_{14,n} + 2t_{14,n-1} - 4$	$D(-6n^2 + 11n - 2)$
$2t_{15,n}$	$2t_{15,n-2}$	$6t_{15,n} + 2t_{15,n-1} - 78n + 67$	$D(-104n^2 + 296n - 10)$
		$8t_{15,n-2} + 104n - 197$	$D(70)$
	$2t_{15,n-1}$	$8t_{14,n} + 4t_{4,n} - 56n - 37$	$D(-13n^2 + 24n + 12)$

Construction of Dio 3-tuples for Centered Hendecagonal number:

Let $a = 2ct_{11,n}$, $b = 2ct_{11,n-2}$ be Centered Hendecagonal number of rank n and n-2 respectively such that $ab + (a+b) + 352n^2 - 352n - 10$ is a perfect square say γ^2

Let c be any non zero integer such that

$$ac + (a+c) + 352n^2 - 352n - 10 = \alpha^2 \tag{8}$$

$$bc + (b+c) + 352n^2 - 352n - 10 = \beta^2 \tag{9}$$

On solving equations (8) and (9), we get

$$(b+1)\alpha^2 - (a+1)\beta^2 = (a-b) + \left(352n^2 - 352n - 10\right)(b-a) \tag{10}$$

Assume $\alpha = x + (a+1)T$ and $\beta = x + (b+1)T$, in (10) and it reduces to

$$x^2 = (b+1)(a+1)T^2 + \left(352n^2 - 352n - 11\right) \tag{11}$$

The initial solution of equation (11) is given by

$$T_o = 1 \text{ and } x_0 = 11n^2 - 11n + 8 \tag{12}$$

Therefore, $\quad \alpha = 22n^2 + 11 \tag{13}$

On substuting the values of α and a in equation (8), we get

$$c = 44n^2 - 44n + 43$$

$$= t_{90,n} - n + 43$$

Therefore triple $\left(2ct_{11,n}, 2ct_{11,n-2}, t_{90,n} - n + 43\right)$ is Dio 3- tuple with property $D(352n^2 - 352n - 10)$

For simplicity, we present below the Diophantine triples for polygonal numbers from $ct_{11,n}$ to $ct_{15,n}$ with suitable properties.

a	b	c	D(n)
$2ct_{11,n}$	$2ct_{11,n-2}$	$ct_{22,2n-1} - 22n + 10$	$D(-10)$
	$2ct_{11,n-1}$	$ct_{88,n} - 44n$	$D(11n^2 - 4)$
$ct_{12,n}$	$ct_{12,n-2}$	$2ct_{12,n-2} + ct_{24,n} - 4$	$D(96n^2 - 96n - 11)$
		$6ct_{8,n} - 48n + 1$	$D(-11)$
	$ct_{12,n-1}$	$4ct_{12,n} - 24n - 1$	$D(12n^2 - 3)$
$2ct_{13,n}$	$2ct_{13,n-2}$	$8ct_{13,n-1} + 43$	$D(520n^2 - 520n + 2)$
		$8ct_{13,n} - 104n + 3$	$D(14)$
	$2ct_{13,n-1}$	$2ct_{13,n-2} + 39n - 29$	$D(13n^2 + 1)$
$ct_{14,n}$	$ct_{14,n-2}$	$ct_{14,2n-1} - 14n + 26$	$D(140n^2 - 140n - 6)$
		$7ct_{8,n-1}$	$D(6)$

	$ct_{14,n-1}$	$ct_{14,2n} - 14n$	$D(7n^2 - 2)$
$2ct_{15,n}$	$2ct_{15,n-2}$	$ct_{15,2(n-2)} + 270n - 393$	$D(720n^2 - 720n + 46)$
		$8ct_{15,n-1} + 3$	$D(46)$
	$2ct_{15,n-1}$	$3(ct_{40,n-1} + 20n - 2)$	$D(15n^2 + 8)$

Construction of Dio 3 - tuples for Linear Polynomials

Case.1:

Let $a = r-1, b = r-2$ be two linear polynomials such that $ab + a + b + (k^2 + 1 + (2k+1)r)$ is a perfect square say γ^2

Let c be any non zero integer such that

$$ac + a + c + (k^2 + 1 + (2k+1)r) = \alpha^2 \tag{14}$$

$$bc + b + c + (k^2 + 1 + (2k+1)r) = \beta^2 \tag{15}$$

On solving equation, (14) and (15), we get

$$(b+1)\alpha^2 - (a+1)\beta^2 = (a-b) + \left(k^2 + 1 + (2k+1)r\right)(b-a) \tag{16}$$

Assuming $\alpha = x + (a+1)T$ and $\beta = x + (b+1)T$ in (16), it reduces to

$$x^2 = (b+1)(a+1)T^2 + \left(k^2 + 1 + (2k+1)r\right) - 1 \tag{17}$$

The initial solution of equation (17) is given by

$$T_o = 1 \text{ and } x_0 = r + k \tag{18}$$

Therefore, $\quad \alpha = 2r + k \tag{19}$

On substuting the values of α and a in equation (14), we get

$$c = 4r + 2k - 2$$

Therefore, triple $(r-1, r-2, 4r+2k-2)$ is Dio 3-tuple with property $D\left(k^2 + 1 + (2k+1)r\right)$

Case.2:

Let $a = r+1, b = r+2$ be two linear polynomials such that $ab + a + b + r + 4$ is a perfect square say γ^2

Let c be any non zero integer such that

$$ac + a + c + r + 4 = \alpha^2 \tag{20}$$

$$bc + b + c + r + 4 = \beta^2 \tag{21}$$

On solving equation, (20) and (21), we get

$$(b+1)\alpha^2 - (a+1)\beta^2 = (a-b) + (r+4)(b-a) \tag{22}$$

Assuming $\alpha = x + (a+1)T$ and $\beta = x + (b+1)T$ in (22), it reduces to

$$x^2 = (b+1)(a+1)T^2 + (r+4) - 1 \tag{23}$$

The initial solution of equation (17) is given by

$$T_o = 1 \text{ and } x_0 = r + 3 \tag{24}$$

Therefore, $\qquad \alpha = 2r + 5$ $\hfill (25)$

On substuting the values of α and a in equation (20), we get

$$c = 4r + 10$$

Therefore, triple $(r+1, r+2, 4r+10)$ is Dio 3-tuple with property $D(r+4)$

Construction of Dio 3 - tuples for Jacobasthal -Lucas number:

Let $a = j_{2n}, b = j_{2n+2}$ be Jacobasthal -Lucas numbers of rank 2n and 2n+2 respectively, such that $ab + a + b + 2.2^{2n} + 6$ is a perfect square say γ^2

Let c be any non zero integer such that

$$ac + a + c + 2.2^{2n} + 6 = \alpha^2 \hfill (26)$$

$$bc + b + c + 2.2^{2n} + 6 = \beta^2 \hfill (27)$$

On solving equation, (26) and (27), we get

$$(b+1)\alpha^2 - (a+1)\beta^2 = (a-b) + 2.2^{2n} + 6(b-a) \hfill (28)$$

Assuming $\alpha = x + (a+1)T$ and $\beta = x + (b+1)T$ in (28), it reduces to

$$x^2 = (b+1)(a+1)T^2 + 2.2^{2n} + 5 \hfill (29)$$

The initial solution of equation (29) is given by

$$T_o = 1 \text{ and } x_0 = 2.2^{2n} + 3 \hfill (30)$$

Therefore, $\qquad \alpha = 3.2^{2n} + 5$ $\hfill (31)$

On substuting the values of α and a in equation (26), we get

$$c = 9.2^{2n} + 9$$

Therefore, triple $(j_{2n}, j_{2n+2}, 9j_{2n})$ is Dio 3- tuple with property $D(2.2^{2n} + 6)$

In general, it is noted that the triple $(j_{2n}, j_{2n+2}, 92^{2n} + 2k + 1)$ is a Dio 3-tuple with property $D\left[(4k-14)2^{2n} + k^2 - 2k - 2\right]$

Conclusion:

In this paper we have presented a few examples of constructing a special Dio 3 tuples for Polygonal numbers, Centered polygonal numbers, linear polynomials and Jacobasthal-lucas numbers with suitable properties. To conclude one may search for Dio 3 – tuples for higher order polygonal numbers and centered polygonal numbers with their corresponding suitable properties.

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