DIO 3 – TUPLES FOR SPECIAL NUMBERS – I

M.A.Gopalan¹, V.Geetha², S.Vidhyalakshmi³*

¹,³. Professors, Department of Mathematics, Shrimathi Indira Gandhi College, Trichy,
². Assistant Professor, Department of Mathematics, Cauvery College for Women, Trichy.

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Abstract. We search for three distinct polynomials with integer coefficients such that the product of any two members of the set added with their sum and increased by a non-zero integer (or polynomial with integer coefficients) is a perfect square.

Introduction:

The problem of constructing the set with property that the product of any two its distinct elements is one less than a square has a very long history and such sets were studied by Diophantus. A set of m positive integers \{a₁, a₂, ..., aₘ\} is called a Diophantine m-tuple if

\[ a_i \cdot a_j + 1 \text{ is a perfect square} \quad (1) \]

a perfect square for all \(1 \leq i < j \leq m\). Many generalizations of this problem (1) were considered since antiquity, for example by adding a fixed integer n instead of 1, looking kth powers instead of squares or considering the powers over domains other than Z or Q. Many mathematicians consider the problem of the existence of Diophantine quadruples with the property D(n) for any arbitrary integer n and also for any linear polynomials in n. In this context one may refer [1-16]. The above results motivated us the following definition:

A set of three distinct polynomials with integer coefficient \((a₁, a₂, a₃)\) is said to be a special dio 3-tuple with property D(n) if \(a_i \cdot a_j + (a_i + a_j) + n\) is a perfect square for all \(1 \leq i < j \leq 3\).

In the above definition n may be a non-zero integer or polynomial with integer coefficients. In this communication we consider a few special dio 3 tuples of polygonal numbers from \(t_{11,n}\) to \(t_{15,n}\), centered polygonal numbers from \(c_{11,n}\) to \(c_{15,n}\), linear polynomials and jacobasthal-lucas number with their corresponding properties.

Notations:

\[ t_{m,n} = n \left( 1 + \frac{(n-1)(m-2)}{2} \right) \text{ \quad PolyGonal number of rank n with sides m} \]
\[ c_{t_{m,n}} = \frac{mn(n+1)}{2} + 1 \text{ \quad Centered PolyGonal number of rank n with sides m} \]
\[ j_n = 2^n + 1 \text{ \quad Jacobasthal-Lucas number of rank n} \]
Construction of Dio 3-tuples for Hendecagonal number:

Let \( a = 2t_{11,n} \), \( b = 2t_{11,n-2} \) be Hendecagonal number of rank \( n \) and \( n-2 \) respectively such that \( ab + (a + b) + (-18n^2 + 50n - 1) \) is a perfect square say \( \gamma^2 \)

Let \( c \) be any non zero integer such that

\[
ac + (a + c) + (-18n^2 + 50n - 1) = \alpha^2
\]

\[
b + (b + c) + (-18n^2 + 50n - 1) = \beta^2
\]

On solving equations (2) and (3), we get

\[
(b+1)\alpha^2 - (a+1)\beta^2 = (a-b) + (-18n^2 + 50n - 1)(b-a)
\]

Assume \( \alpha = x + (a+1)T \) and \( \beta = x + (b+1)T \) and it reduces to

\[
x^2 = (b+1)(a+1)T^2 + (-18n^2 + 50n - 2)
\]

The initial solution of equation (5) is given by

\[
T_0 = 1 \text{ and } x_0 = 9n^2 - 25n + 7
\]

Therefore, \( \alpha = 18n^2 - 32n + 8 \)

On substituting the values of \( \alpha \) and \( \beta \) in equation (2), we get

\[
c = 36n^2 - 100n + 65
\]

Therefore triple \( (2t_{11,n}, 2t_{11,n-2}, 2t_{11,n-2} + 72n + 135) \) is Dio 3-tuple with property \( D(-18n^2 + 50n - 1) \)

For simplicity, we present below the Dio 3-tuple for polygonal numbers from \( t_{11,n} \) to \( t_{15,n} \) with suitable properties.

<table>
<thead>
<tr>
<th>( a )</th>
<th>( b )</th>
<th>( c )</th>
<th>( D(n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2t_{11,n} )</td>
<td>( 2t_{11,n-2} )</td>
<td>( 8t_{11,n} - 72n + 67 )</td>
<td>( D(14) )</td>
</tr>
<tr>
<td>( 2t_{11,n-1} )</td>
<td>( 8t_{11,n-1} + 36n - 39 )</td>
<td>( D(-9n^2 + 16n) )</td>
<td></td>
</tr>
<tr>
<td>( t_{12,n-2} )</td>
<td>( t_{12,n-1} + 2t_{12,n-2} + 12n - 39 )</td>
<td>( t_{12,n} + t_{12,n-2} - 28n + 11 )</td>
<td>( D(-20n^2 + 56n - 19) )</td>
</tr>
<tr>
<td>( t_{12,n-1} )</td>
<td>( 2t_{12,n-1} - 6n - 4 )</td>
<td>( D(10n^3 - 27n^2 + 22n - 5) )</td>
<td></td>
</tr>
<tr>
<td>( 2t_{13,n} )</td>
<td>( 8t_{13,n} + 88n - 169 )</td>
<td>( 2t_{13,n-1} + 62n - 63 )</td>
<td>( D(-44n^2 + 124n + 2) )</td>
</tr>
<tr>
<td>( t_{13,n-1} )</td>
<td>( 2t_{13,n-2} - 26n + 33 )</td>
<td>( D(-33n^2 + 60n - 4) )</td>
<td></td>
</tr>
<tr>
<td>( t_{14,n} )</td>
<td>( t_{50,n} - 45(n-1) )</td>
<td>( t_{14,n-1} - 10n + 13 )</td>
<td>( D(28n^2 - 76n + 74) )</td>
</tr>
<tr>
<td>( t_{14,n-1} )</td>
<td>( 2t_{14,n} + 2t_{14,n-1} - 4 )</td>
<td>( D(-6n^2 + 11n - 2) )</td>
<td></td>
</tr>
<tr>
<td>( 2t_{15,n} )</td>
<td>( 2t_{15,n-2} + 78n + 67 )</td>
<td>( 2t_{15,n-2} + 104n - 197 )</td>
<td>( D(-104n^2 + 296n - 10) )</td>
</tr>
<tr>
<td>( 2t_{15,n-1} )</td>
<td>( 8t_{14,n} + 4t_{14,n} - 56n - 37 )</td>
<td>( D(-13n^2 + 24n + 12) )</td>
<td></td>
</tr>
</tbody>
</table>
Construction of Dio 3-tuples for Centered Hendecagonal number:

Let \( a = 2ct_{11,n} \) and \( b = 2ct_{11,n-2} \) be Centered Hendecagonal number of rank \( n \) and \( n-2 \) respectively such that \( ab + (a+b) + 352n^2 - 352n - 10 \) is a perfect square say \( \gamma^2 \)

Let \( c \) be any non zero integer such that

\[
\begin{align*}
ac + (a+c) + 352n^2 - 352n - 10 &= \alpha^2 \\
bc + (b+c) + 352n^2 - 352n - 10 &= \beta^2
\end{align*}
\]

On solving equations (8) and (9), we get

\[
(b+1)\alpha^2 - (a+1)\beta^2 = (a-b) + (352n^2 - 352n - 10)(b-\alpha)
\]

Assume \( \alpha = x + (a+1)T \) and \( \beta = x + (b+1)T \), in (10) and it reduces to

\[
x^2 = (b+1)(a+1)T^2 + (352n^2 - 352n - 11)
\]

The initial solution of equation (11) is given by

\[
T_0 = 1 \text{ and } x_0 = 11n^2 - 11n + 8
\]

Therefore,

\[
\alpha = 22n^2 + 11
\]

On substituting the values of \( \alpha \) and \( a \) in equation (8), we get

\[
c = 44n^2 - 44n + 43
\]

Therefore triple \( (2ct_{11,n}, 2ct_{11,n-2}, t_{90,n} - n + 43) \) is Dio 3-tuple with property \( D(352n^2 - 352n - 10) \)

For simplicity, we present below the Diophantine triples for polygonal numbers from \( c_{11,n} \) to \( c_{15,n} \) with suitable properties.

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>c</th>
<th>D(n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2ct_{11,n} )</td>
<td>( 2ct_{11,n-2} )</td>
<td>( ct_{22,n-1} - 22n + 10 )</td>
<td>( D(-10) )</td>
</tr>
<tr>
<td>( 2ct_{11,n-1} )</td>
<td></td>
<td>( ct_{88,n} - 44n )</td>
<td>( D(11n^2 - 4) )</td>
</tr>
<tr>
<td>( ct_{12,n} )</td>
<td>( ct_{12,n-2} )</td>
<td>( 2ct_{24,n} - 4 )</td>
<td>( D(96n^2 - 96n - 11) )</td>
</tr>
<tr>
<td></td>
<td>( 6ct_{8,n} - 48n + 1 )</td>
<td></td>
<td>( D(-11) )</td>
</tr>
<tr>
<td></td>
<td>( ct_{12,n-1} )</td>
<td>( 4ct_{12,n} - 24n - 1 )</td>
<td>( D(12n^2 - 3) )</td>
</tr>
<tr>
<td>( 2ct_{13,n} )</td>
<td>( 2ct_{13,n-2} )</td>
<td>( 8ct_{13,n-1} + 43 )</td>
<td>( D(520n^2 - 520n + 2) )</td>
</tr>
<tr>
<td></td>
<td>( 8ct_{13,n} - 104n + 3 )</td>
<td></td>
<td>( D(14) )</td>
</tr>
<tr>
<td></td>
<td>( 2ct_{13,n-1} )</td>
<td>( 2ct_{13,n-2} + 39n - 29 )</td>
<td>( D(13n^2 + 1) )</td>
</tr>
<tr>
<td>( ct_{14,n} )</td>
<td>( ct_{14,n-2} )</td>
<td>( ct_{14,2n-1} - 14n + 26 )</td>
<td>( D(140n^2 - 140n - 6) )</td>
</tr>
<tr>
<td></td>
<td>( 7ct_{8,n} )</td>
<td></td>
<td>( D(6) )</td>
</tr>
</tbody>
</table>
### Construction of Dio 3 - tuples for Linear Polynomials

#### Case.1:
Let \(a = r - 1, b = r - 2\) be two linear polynomials such that \(ab + a + b + (k^2 + 1 + (2k + 1)r)\) is a perfect square say \(\gamma^2\).

Let \(c\) be any non zero integer such that
\[
\begin{align*}
ac + a + c + (k^2 + 1 + (2k + 1)r) &= \alpha^2 \\
bc + b + c + (k^2 + 1 + (2k + 1)r) &= \beta^2
\end{align*}
\]

On solving equation, (14) and (15), we get
\[
(b + 1)\alpha^2 - (a + 1)\beta^2 = (a - b) + (k^2 + 1 + (2k + 1)r)(b - a)
\]

Assuming \(\alpha = x + (a + 1)T\) and \(\beta = x + (b + 1)T\) in (16), it reduces to
\[
x^2 = (b + 1)(a + 1)T^2 + (k^2 + 1 + (2k + 1)r) - 1
\]

The initial solution of equation (17) is given by
\[
T_o = 1 \quad \text{and} \quad x_0 = r + k
\]

Therefore, \(\alpha = 2r + k\)

On substuting the values of \(\alpha\) and \(a\) in equation (14), we get
\[
c = 4r + 2k - 2
\]

Therefore, triple \((r - 1, r - 2, 4r + 2k - 2)\) is Dio 3-tuple with property
\[
D\left(k^2 + 1 + (2k + 1)r\right)
\]

#### Case.2:
Let \(a = r + 1, b = r + 2\) be two linear polynomials such that \(ab + a + b + r + 4\) is a perfect square say \(\gamma^2\).

Let \(c\) be any non zero integer such that
\[
\begin{align*}
ac + a + c + r + 4 &= \alpha^2 \\
bc + b + c + r + 4 &= \beta^2
\end{align*}
\]

On solving equation, (20) and (21), we get
\[
(b + 1)\alpha^2 - (a + 1)\beta^2 = (a - b) + (r + 4)(b - a)
\]

Assuming \(\alpha = x + (a + 1)T\) and \(\beta = x + (b + 1)T\) in (22), it reduces to
\[
x^2 = (b + 1)(a + 1)T^2 + (r + 4) - 1
\]

The initial solution of equation (17) is given by
\[
T_o = 1 \quad \text{and} \quad x_0 = r + 3
\]
Therefore, \( \alpha = 2r + 5 \)  

On substituting the values of \( \alpha \) and \( a \) in equation (20), we get  
\[ c = 4r + 10 \]  

Therefore, triple \( (r + 1, r + 2, 4r + 10) \) is \( D(1) \)-tuple with property \( D(r + 4) \)

**Construction of \( D(3) \)-tuples for Jacobsthal-Lucas number:**

Let \( a = j_{2n}, b = j_{2n+2} \) be Jacobsthal-Lucas numbers of rank \( 2n \) and \( 2n+2 \) respectively, such that \( ab + a + b + 2.2^{2n} + 6 \) is a perfect square say \( \gamma^2 \).

Let \( c \) be any non zero integer such that  
\[ ac + a + c + 2.2^{2n} + 6 = \alpha^2 \]  
\[ bc + b + c + 2.2^{2n} + 6 = \beta^2 \]  

On solving equation, (26) and (27), we get  
\[ (b+1)\alpha^2 - (a+1)\beta^2 = (a-b) + 2.2^{2n} + 6(b-a) \]  

Assuming \( \alpha = x + (a+1) T \) and \( \beta = x + (b+1) T \) in (28), it reduces to  
\[ x^2 = (b+1)(a+1) T^2 + 2.2^{2n} + 5 \]  

The initial solution of equation (29) is given by  
\[ T_0 = 1 \text{ and } x_0 = 2.2^{2n} + 3 \]  

Therefore,  
\[ \alpha = 3.2^{2n} + 5 \]  

On substituting the values of \( \alpha \) and \( a \) in equation (26), we get  
\[ c = 9.2^{2n} + 9 \]  

Therefore, triple \( (j_{2n}, j_{2n+2}, 9j_{2n}) \) is \( D(3) \)-tuple with property \( D(2.2^{2n} + 6) \)

In general, it is noted that the triple \( (j_{2n}, j_{2n+2}, 92^{2n} + 2k + 1) \) is a \( D(3) \)-tuple with property  
\[ D\left( 4k - 14 \right) 2^{2n} + k^2 - 2k - 2 \]

**Conclusion:**

In this paper we have presented a few examples of constructing a special \( D(3) \) tuples for Polygonal numbers, Centered polygonal numbers, linear polynomials and Jacobsthal-lucas numbers with suitable properties. To conclude one may search for \( D(3) \)-tuples for higher order polygonal numbers and centered polygonal numbers with their corresponding suitable properties.

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