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DIO 3 – TUPLES FOR SPECIAL NUMBERS – I

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Abstract. We search for three distinct polynomials with integer coefficients such that the product of any two members of the set added with their sum and increased by a non- zero integer (or polynomial with integer coefficients) is a perfect square.

Introduction:

The problem of constructing the set with property that the product of any two its distinct elements is one less than a square has a very long history and such sets were studied by Diophantus. A set of m positive integers $\{a_1, a_2, \ldots a_m\}$ is called a Diophantine m-tuple if

$$a_i * a_j + 1 \text{ is a perfect square} \tag{1}$$

a perfect square for all $1 \le i \prec j \le m$. Many generalizations of this problem (1) were considered since antiquity, for example by adding a fixed integer n instead of 1, looking kth powers instead of squares or considering the powers over domains other than Z or Q. Many mathematicians consider the problem of the existence of Diophantine quadruples with the property D(n) for any arbitrary integer n and also for any linear polynomials in n. In this context one may refer [1-16]. The above results motivated us the following definition:

A set of three distinct polynomials with integer coefficient (a_1, a_2, a_3) is said to be a special dio 3- tuple with property D(n) if $a_i * a_j + (a_i + a_j) + n$ is a perfect square for all $1 \le i \prec j \le 3$.

In the above definition n may be a non – zero integer or polynomial with integer coefficients. In this communication we consider a few special dio 3 tuples of polygonal numbers from $t_{11,n}$ to $t_{15,n}$, centered polygonal numbers from $ct_{11,n}$ to $ct_{15,n}$, linear polynomials and jacobasthal - lucas number with their corresponding properties.

Notations:

$t_{m,n} = n\left(1 + \frac{(n-1)(m-2)}{2}\right) =$ Polygonal number of rank n with sides m

$ct_{m,n} = \frac{mn(n+1)}{2} + 1 =$ Centered Polygonal number of rank n with sides m

$j_n = 2^n + 1 =$ Jacobasthal-Lucas number of rank n

Construction of Dio 3-tuples for Hendecagonal number:

Let $a = 2t_{11,n}$, $b = 2t_{11,n-2}$ be Hendecagonal number of rank n and n-2 respectively such that $ab+(a+b)+(-18n^2+50n-1)$ is a perfect square say γ^2

Let c be any non zero integer such that

$$ac+(a+c)+(-18n^2+50n-1)=\alpha^2 \tag{2}$$

$$bc+(b+c)+(-18n^2+50n-1)=\beta^2 \tag{3}$$

On solving equations (2) and (3), we get

$$(b+1)\alpha^2-(a+1)\beta^2=(a-b)+\left(-18n^2+50n-1\right)(b-a) \tag{4}$$

Assume $\alpha = x+(a+1)T$ and $\beta = x+(b+1)T$ and it reduces to

$$x^2=(b+1)(a+1)T^2+\left(-18n^2+50n-2\right) \tag{5}$$

The initial solution of equation (5) is given by

$$T_o=1 \text{ and } x_0=9n^2-25n+7 \tag{6}$$

Therefore, $\alpha = 18n^2-32n+8$ (7)

On substuting the values of α and *a* in equation (2), we get

$$c = 36n^2-100n+65$$

$$= t_{11,n-2}+72n+135$$

Therefore triple $\left(2t_{11,n}, 2t_{11,n-2}, t_{11,n-2}+72n+135\right)$ is Dio 3- tuple with property $D(-18n^2+50n-1)$

For simplicity, we present below the Dio 3-tuple for polygonal numbers from $t_{11,n}$ to $t_{15,n}$ with suitable properties.

a	b	c	D(n)
$2t_{11,n}$	$2t_{11,n-2}$	$8t_{11,n}-72n+67$	$D(14)$
	$2t_{11,n-1}$	$8t_{11,n-1}+36n-39$	$D(-9n^2+16n)$
$t_{12,n}$	$t_{12,n-2}$	$t_{12,n}+t_{12,n-1}+2t_{12,n-2}+12n-39$	$D(-20n^2+56n-19)$
		$t_{12,n}+t_{12,n-2}-28n+11$	$D(-3)$
	$t_{12,n-1}$	$2t_{12,n-1}-6n-4$	$D(10n^3-27n^2+22n-5)$
$2t_{13,n}$	$2t_{13,n-2}$	$8t_{13,n-2}+88n-169$	$D(-44n^2+124n+2)$
		$2t_{13,2n-1}+62n-63$	$D(38)$
	$t_{8,n-1}$	$2t_{13,2n-2}-26n+33$	$D(-33n^2+60n-4)$
$t_{14,n}$	$t_{14,n-2}$	$t_{50,n}-45(n-1)$	$D(28n^2-76n+74)$
		$t_{14,2(n-1)}-10n+13$	$D(2)$
	$t_{14,n-1}$	$2t_{14,n}+2t_{14,n-1}-4$	$D(-6n^2+11n-2)$
$2t_{15,n}$	$2t_{15,n-2}$	$6t_{15,n}+2t_{15,n-1}-78n+67$	$D(-104n^2+296n-10)$
		$8t_{15,n-2}+104n-197$	$D(70)$
	$2t_{15,n-1}$	$8t_{14,n}+4t_{4,n}-56n-37$	$D(-13n^2+24n+12)$

Construction of Dio 3-tuples for Centered Hendecagonal number:

Let $a = 2ct_{11,n}$, $b = 2ct_{11,n-2}$ be Centered Hendecagonal number of rank n and n-2 respectively such that $ab+(a+b)+352n^2-352n-10$ is a perfect square say γ^2

Let c be any non zero integer such that

$$ac+(a+c)+352n^2-352n-10=\alpha^2 \quad (8)$$

$$bc+(b+c)+352n^2-352n-10=\beta^2 \quad (9)$$

On solving equations (8) and (9), we get

$$(b+1)\alpha^2-(a+1)\beta^2=(a-b)+\left(352n^2-352n-10\right)(b-a) \quad (10)$$

Assume $\alpha = x+(a+1)T$ and $\beta = x+(b+1)T$, in (10) and it reduces to

$$x^2=(b+1)(a+1)T^2+\left(352n^2-352n-11\right) \quad (11)$$

The initial solution of equation (11) is given by

$$T_o=1 \text{ and } x_0=11n^2-11n+8 \quad (12)$$

Therefore, $\alpha = 22n^2+11$ (13)

On substuting the values of α and *a* in equation (8), we get

$$c = 44n^2-44n+43$$

$$= t_{90,n}-n+43$$

Therefore triple $\left(2ct_{11,n}, 2ct_{11,n-2}, t_{90,n}-n+43\right)$ is Dio 3- tuple with property $D(352n^2-352n-10)$

For simplicity, we present below the Diophantine triples for polygonal numbers from $ct_{11,n}$ to $ct_{15,n}$ with suitable properties.

a	b	c	D(n)
$2ct_{11,n}$	$2ct_{11,n-2}$	$ct_{22,2n-1}-22n+10$	$D(-10)$
	$2ct_{11,n-1}$	$ct_{88,n}-44n$	$D(11n^2-4)$
$ct_{12,n}$	$ct_{12,n-2}$	$2ct_{12,n-2}+ct_{24,n}-4$	$D(96n^2-96n-11)$
		$6ct_{8,n}-48n+1$	$D(-11)$
	$ct_{12,n-1}$	$4ct_{12,n}-24n-1$	$D(12n^2-3)$
$2ct_{13,n}$	$2ct_{13,n-2}$	$8ct_{13,n-1}+43$	$D(520n^2-520n+2)$
		$8ct_{13,n}-104n+3$	$D(14)$
	$2ct_{13,n-1}$	$2ct_{13,n-2}+39n-29$	$D(13n^2+1)$
$ct_{14,n}$	$ct_{14,n-2}$	$ct_{14,2n-1}-14n+26$	$D(140n^2-140n-6)$
		$7ct_{8,n-1}$	$D(6)$

	$ct_{14,n-1}$	$ct_{14,2n}-14n$	$D(7n^2-2)$
$2ct_{15,n}$	$2ct_{15,n-2}$	$ct_{15,2(n-2)}+270n-393$	$D(720n^2-720n+46)$
		$8ct_{15,n-1}+3$	$D(46)$
	$2ct_{15,n-1}$	$3(ct_{40,n-1}+20n-2)$	$D(15n^2+8)$

Construction of Dio 3 - tuples for Linear Polynomials

Case.1:

Let $a=r-1, b=r-2$ be two linear polynomials such that $ab+a+b+(k^2+1+(2k+1)r)$ is a perfect square say γ^2

Let c be any non zero integer such that

$$ac+a+c+(k^2+1+(2k+1)r)=\alpha^2 \tag{14}$$

$$bc+b+c+(k^2+1+(2k+1)r)=\beta^2 \tag{15}$$

On solving equation, (14) and (15), we get

$$(b+1)\alpha^2-(a+1)\beta^2=(a-b)+\left(k^2+1+(2k+1)r\right)(b-a) \tag{16}$$

Assuming $\alpha=x+(a+1)T$ and $\beta=x+(b+1)T$ in (16), it reduces to

$$x^2=(b+1)(a+1)T^2+\left(k^2+1+(2k+1)r\right)-1 \tag{17}$$

The initial solution of equation (17) is given by

$$T_o=1 \text{ and } x_0=r+k \tag{18}$$

Therefore, $\alpha=2r+k$ (19)

On substuting the values of α and *a* in equation (14), we get

$$c=4r+2k-2$$

Therefore, triple $(r-1, r-2, 4r+2k-2)$ is Dio 3-tuple with property $D\left(k^2+1+(2k+1)r\right)$

Case.2:

Let $a=r+1, b=r+2$ be two linear polynomials such that $ab+a+b+r+4$ is a perfect square say γ^2

Let c be any non zero integer such that

$$ac+a+c+r+4=\alpha^2 \tag{20}$$

$$bc+b+c+r+4=\beta^2 \tag{21}$$

On solving equation, (20) and (21), we get

$$(b+1)\alpha^2-(a+1)\beta^2=(a-b)+(r+4)(b-a) \tag{22}$$

Assuming $\alpha=x+(a+1)T$ and $\beta=x+(b+1)T$ in (22), it reduces to

$$x^2=(b+1)(a+1)T^2+(r+4)-1 \tag{23}$$

The initial solution of equation (17) is given by

$$T_o=1 \text{ and } x_0=r+3 \tag{24}$$

Therefore, $\alpha = 2r+5$ (25)

On substuting the values of α and *a* in equation (20), we get

$$c = 4r+10$$

Therefore, triple $(r+1, r+2, 4r+10)$ is Dio 3-tuple with property $D(r+4)$

Construction of Dio 3 - tuples for Jacobasthal -Lucas number:

Let $a = j_{2n}, b = j_{2n+2}$ be Jacobasthal -Lucas numbers of rank 2n and 2n+2 respectively, such that $ab+a+b+2.2^{2n}+6$ is a perfect square say γ^2

Let c be any non zero integer such that

$$ac+a+c+2.2^{2n}+6 = \alpha^2 \quad (26)$$

$$bc+b+c+2.2^{2n}+6 = \beta^2 \quad (27)$$

On solving equation, (26) and (27), we get

$$(b+1)\alpha^2 - (a+1)\beta^2 = (a-b)+2.2^{2n}+6(b-a) \quad (28)$$

Assuming $\alpha = x+(a+1)T$ and $\beta = x+(b+1)T$ in (28), it reduces to

$$x^2 = (b+1)(a+1)T^2 + 2.2^{2n} + 5 \quad (29)$$

The initial solution of equation (29) is given by

$$T_o = 1 \text{ and } x_0 = 2.2^{2n} + 3 \quad (30)$$

Therefore, $\alpha = 3.2^{2n}+5$ (31)

On substuting the values of α and *a* in equation (26), we get

$$c = 9.2^{2n}+9$$

Therefore, triple $(j_{2n}, j_{2n+2}, 9j_{2n})$ is Dio 3- tuple with property $D(2.2^{2n}+6)$

In general, it is noted that the triple $\left(j_{2n}, j_{2n+2}, 92^{2n}+2k+1\right)$ is a Dio 3-tuple with property $D\left[(4k-14)2^{2n}+k^2-2k-2\right]$

Conclusion:

In this paper we have presented a few examples of constructing a special Dio 3 tuples for Polygonal numbers, Centered polygonal numbers, linear polynomials and Jacobasthal-lucas numbers with suitable properties. To conclude one may search for Dio 3 – tuples for higher order polygonal numbers and centered polygonal numbers with their corresponding suitable properties.

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