Keywords: Dio 3 - tuples, Pell equation, Polygonal numbers and Centered polygonal numbers, Linear Polynomials and Jacobasthal-Lucas number.

Abstract. We search for three distinct polynomials with integer coefficients such that the product of any two members of the set added with their sum and increased by a non-zero integer (or polynomial with integer coefficients) is a perfect square.

Introduction:

The problem of constructing the set with property that the product of any two its distinct elements is one less than a square has a very long history and such sets were studied by Diophantus. A set of m positive integers \(\{a_1, a_2, \ldots, a_m\}\) is called a Diophantine m-tuple if

\[
a_i \cdot a_j + 1 \text{ is a perfect square}
\]

for all \(1 \leq i < j \leq m\). Many generalizations of this problem (1) were considered since antiquity, for example by adding a fixed integer \(n\) instead of 1, looking \(k\)th powers instead of squares or considering the powers over domains other than \(\mathbb{Z}\) or \(\mathbb{Q}\). Many mathematicians consider the problem of the existence of Diophantine quadruples with the property \(D(n)\) for any arbitrary integer \(n\) and also for any linear polynomials in \(n\). In this context one may refer [1-16]. The above results motivated us the following definition:

A set of three distinct polynomials with integer coefficient \(\{a_1, a_2, a_3\}\) is said to be a special dio 3- tuple with property \(D(n)\) if

\[
a_i \cdot a_j + (a_i + a_j) + n \text{ is a perfect square for all } 1 \leq i < j \leq 3.
\]

In the above definition \(n\) may be a non-zero integer or polynomial with integer coefficients. In this communication we consider a few special dio 3 tuples of polygonal numbers from \(t_{11,n}\) to \(t_{15,n}\), centered polygonal numbers from \(c_{11,n}\) to \(c_{15,n}\), linear polynomials and Jacobasthal-Lucas number with their corresponding properties.

Notations:

\[
t_{m,n} = n \left(1 + \frac{(n - 1)(m - 2)}{2}\right) = \text{Polygonal number of rank } n \text{ with sides } m
\]

\[
c_{m,n} = \frac{mn(n+1)}{2} + 1 = \text{Centered Polygonal number of rank } n \text{ with sides } m
\]

\[
j_n = 2^n + 1 = \text{Jacobasthal-Lucas number of rank } n
\]
Construction of Dio 3-tuples for Hendecagonal number:

Let \( a = \frac{2t_{11,n}}{} \) and \( b = \frac{2t_{11,n-2}}{} \) be Hendecagonal number of rank \( n \) and \( n-2 \) respectively such that \( ab + (a + b) + (-18n^2 + 50n - 1) \) is a perfect square say \( r^2 \).

Let \( c \) be any non-zero integer such that

\[
ac + (a + c) + (-18n^2 + 50n - 1) = \alpha^2 \\
bc + (b + c) + (-18n^2 + 50n - 1) = \beta^2
\]

On solving equations (2) and (3), we get

\[
\frac{(b + 1)\alpha^2 - (a + 1)\beta^2}{(b^2 - \alpha^2)} = \frac{(a - b) + (-18n^2 + 50n - 1)(b - a)}{(b^2 - \alpha^2)}
\]

Assume \( \alpha = x + (a + 1)T \) and \( \beta = x + (b + 1)T \) and it reduces to

\[
x^2 = (b + 1)(a + 1)T^2 + (-18n^2 + 50n - 2).
\]

The initial solution of equation (5) is given by

\[
T_0 = 1 \text{ and } x_0 = 9n^2 - 25n + 7
\]

Therefore, \( \alpha = 18n^2 - 32n + 8 \)

On substituting the values of \( \alpha \) and \( \alpha \) in equation (2), we get

\[
c = 36n^2 - 100n + 65
\]

\[
= t_{11,n-2} + 72n + 135
\]

Therefore triple \( \left( 2t_{11,n}, 2t_{11,n-2}, t_{11,n-2} + 72n + 135 \right) \) is Dio 3-tuple with property \( D(-18n^2 + 50n - 1) \)

For simplicity, we present below the Dio 3-tuple for polygonal numbers from \( t_{11,n} \) to \( t_{15,n} \) with suitable properties.

<table>
<thead>
<tr>
<th>( a )</th>
<th>( b )</th>
<th>( c )</th>
<th>( D(n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2t_{11,n} )</td>
<td>( 2t_{11,n-2} )</td>
<td>( 8t_{11,n} - 72n + 67 )</td>
<td>( D(14) )</td>
</tr>
<tr>
<td>( 2t_{11,n-1} )</td>
<td>( 8t_{11,n-1} + 36n - 39 )</td>
<td>( D(-9n^2 + 16n) )</td>
<td></td>
</tr>
<tr>
<td>( t_{12,n} )</td>
<td>( t_{12,n-2} )</td>
<td>( t_{12,n} + t_{12,n-1} + 2t_{12,n-2} + 12n - 39 )</td>
<td>( D(-20n^2 + 56n - 19) )</td>
</tr>
<tr>
<td>( t_{12,n-1} )</td>
<td>( 2t_{12,n-1} - 6n - 4 )</td>
<td>( D(10n^2 - 27n^2 + 22n - 5) )</td>
<td></td>
</tr>
<tr>
<td>( 2t_{13,n} )</td>
<td>( 8t_{13,n-2} + 88n - 169 )</td>
<td>( D(-44n^2 + 124n + 2) )</td>
<td></td>
</tr>
<tr>
<td>( 2t_{13,n-1} )</td>
<td>( 2t_{13,n-1} + 62n - 63 )</td>
<td>( D(38) )</td>
<td></td>
</tr>
<tr>
<td>( t_{8,n-1} )</td>
<td>( 2t_{13,n-2} - 26n + 33 )</td>
<td>( D(-33n^2 + 60n - 4) )</td>
<td></td>
</tr>
<tr>
<td>( t_{14,n} )</td>
<td>( t_{50,n} - 45(n - 1) )</td>
<td>( D(28n^2 - 76n + 74) )</td>
<td></td>
</tr>
<tr>
<td>( t_{14,n-1} )</td>
<td>( t_{14,n} - 10n + 13 )</td>
<td>( D(2) )</td>
<td></td>
</tr>
<tr>
<td>( t_{14,n-1} )</td>
<td>( 2t_{14,n} + 2t_{14,n-1} - 4 )</td>
<td>( D(-6n^2 + 11n - 2) )</td>
<td></td>
</tr>
<tr>
<td>( 2t_{15,n} )</td>
<td>( 6t_{15,n} + 2t_{15,n-1} - 78n + 67 )</td>
<td>( D(-104n^2 + 296n - 10) )</td>
<td></td>
</tr>
<tr>
<td>( 8t_{15,n-2} + 104n - 197 )</td>
<td>( D(70) )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 2t_{15,n-1} )</td>
<td>( 8t_{14,n} + 4t_{4,n} - 56n - 37 )</td>
<td>( D(-13n^2 + 24n + 12) )</td>
<td></td>
</tr>
</tbody>
</table>
Construction of Dio 3-tuples for Centered Hendecagonal number:

Let \( a = 2ct_{11,n} \), \( b = 2ct_{11,n-2} \) be Centered Hendecagonal number of rank \( n \) and \( n-2 \) respectively such that \( ab + (a+b) + 352n^2 - 352n - 10 \) is a perfect square say \( \gamma^2 \).

Let \( c \) be any non zero integer such that:

\[
ac + (a+c) + 352n^2 - 352n - 10 = \alpha^2 \tag{8}
\]

\[
bc + (b+c) + 352n^2 - 352n - 10 = \beta^2 \tag{9}
\]

On solving equations (8) and (9), we get:

\[
(b+1)\alpha^2 - (a+1)\beta^2 = (a-b) + (352n^2 - 352n - 10)(b-\alpha) \tag{10}
\]

Assume \( \alpha = x + (a+1)T \) and \( \beta = x + (b+1)T \), in (10) and it reduces to:

\[
x^2 = (b+1)(a+1)T^2 + (352n^2 - 352n - 11) \tag{11}
\]

The initial solution of equation (11) is given by:

\[
T_0 = 1 \text{ and } x_0 = 11n^2 - 11n + 8
\]

Therefore, \( \alpha = 22n^2 + 11 \) \( \tag{12} \)

On substuting the values of \( \alpha \) and \( a \) in equation (8), we get:

\[
c := 44n^2 - 44n + 43 = t_{90,n} - n + 43
\]

Therefore triple \( (2ct_{11,n}, 2ct_{11,n-2}, t_{90,n} - n + 43) \) is Dio 3-tuple with property \( D(352n^2 - 352n - 10) \).

For simplicity, we present below the Diophantine triples for polygonal numbers from \( ct_{11,n} \) to \( ct_{15,n} \) with suitable properties.

<table>
<thead>
<tr>
<th>( a )</th>
<th>( b )</th>
<th>( c )</th>
<th>( D(n) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2ct_{11,n} )</td>
<td>( 2ct_{11,n-2} )</td>
<td>( ct_{22,2n-1} - 22n + 10 )</td>
<td>( D(-10) )</td>
</tr>
<tr>
<td>( 2ct_{11,n-1} )</td>
<td>( t_{88,n} - 44n )</td>
<td>( D(11n^2 - 4) )</td>
<td></td>
</tr>
<tr>
<td>( ct_{12,n} )</td>
<td>( ct_{12,n-2} )</td>
<td>( 2ct_{12,n-2} + ct_{24,n-4} )</td>
<td>( D(96n^2 - 96n - 11) )</td>
</tr>
<tr>
<td>( )</td>
<td>( 6ct_{8,n} - 48n + 1 )</td>
<td>( D(-11) )</td>
<td></td>
</tr>
<tr>
<td>( )</td>
<td>( 4ct_{12,n} - 24n - 1 )</td>
<td>( D(12n^2 - 3) )</td>
<td></td>
</tr>
<tr>
<td>( 2ct_{13,n} )</td>
<td>( 8ct_{13,n-1} + 43 )</td>
<td>( D(520n^2 - 520n + 2) )</td>
<td></td>
</tr>
<tr>
<td>( )</td>
<td>( 8ct_{13,n} - 104n + 3 )</td>
<td>( D(14) )</td>
<td></td>
</tr>
<tr>
<td>( 2ct_{13,n-1} )</td>
<td>( 2ct_{13,n-2} + 39n - 29 )</td>
<td>( D(13n^2 + 1) )</td>
<td></td>
</tr>
<tr>
<td>( ct_{14,n} )</td>
<td>( ct_{14,n-2} )</td>
<td>( ct_{14,2n-1} - 14n + 26 )</td>
<td>( D(140n^2 - 140n - 6) )</td>
</tr>
<tr>
<td>( )</td>
<td>( 7ct_{8,n-1} )</td>
<td>( D(6) )</td>
<td></td>
</tr>
</tbody>
</table>
Construction of Dio 3 - tuples for Linear Polynomials

Case.1:
Let \( a = r - 1, b = r - 2 \) be two linear polynomials such that \( ab + a + b + (k^2 + 1 + (2k + 1)r) \) is a perfect square say \( \gamma^2 \).

Let \( c \) be any non zero integer such that
\[
ac + a + c + (k^2 + 1 + (2k + 1)r) = \alpha^2
\]
\[
bc + b + c + (k^2 + 1 + (2k + 1)r) = \beta^2
\]

On solving equation, (14) and (15), we get
\[
(b + 1)\alpha^2 - (a + 1)\beta^2 = (a-b) + (k^2 +1 + (2k + 1)r)(b-a)
\]

Assuming \( \alpha = x + (a + 1)T \) and \( \beta = x + (b + 1)T \) in (16), it reduces to
\[
x^2 = (b + 1)(a + 1)T^2 + (k^2 +1 + (2k + 1)r) - 1
\]

The initial solution of equation (17) is given by
\[
I_0 = 1 \text{ and } x_0 = r + k
\]

Therefore, \( \alpha = 2r + k \)

On substuting the values of \( \alpha \) and \( a \) in equation (14), we get
\[
c = 4r + 2k - 2
\]

Therefore, triple \( (r-1, r-2, 4r+2k-2) \) is Dio 3-tuple with property
\[
D\left(k^2 +1 + (2k + 1)r\right)
\]

Case.2:
Let \( a = r + 1, b = r + 2 \) be two linear polynomials such that \( ab + a + b + r + 4 \) is a perfect square say \( \gamma^2 \).

Let \( c \) be any non zero integer such that
\[
ac + a + c + r + 4 = \alpha^2
\]
\[
bc + b + c + r + 4 = \beta^2
\]

On solving equation, (20) and (21), we get
\[
(b + 1)\alpha^2 - (a + 1)\beta^2 = (a-b) + (r+4)(b-a)
\]

Assuming \( \alpha = x + (a + 1)T \) and \( \beta = x + (b + 1)T \) in (22), it reduces to
\[
x^2 = (b + 1)(a + 1)T^2 + (r+4) - 1
\]

The initial solution of equation (17) is given by
\[
I_0 = 1 \text{ and } x_0 = r + 3
\]
Therefore, \( \alpha = 2r + 5 \)  
(25)

On substituting the values of \( \alpha \) and \( a \) in equation (20), we get

\[ c = 4r + 10 \]

Therefore, triple \( (r+1, r+2, 4r + 10) \) is Dio 3-tuple with property \( D(r+4) \)

Construction of Dio 3 - tuples for Jacobasthal -Lucas number:

Let \( a = j_{2n}, b = j_{2n+2} \) be Jacobasthal -Lucas numbers of rank 2n and 2n+2 respectively, such that \( ab + a + b + 2.2^{2n} + 6 \) is a perfect square say \( \beta^2 \)

Let \( c \) be any non zero integer such that

\[ ac + a + c + 2.2^{2n} + 6 = \alpha^2 \]  
(26)

\[ bc + b + c + 2.2^{2n} + 6 = \beta^2 \]  
(27)

On solving equation, (26) and (27), we get

\[ (b+1)\alpha^2 - (a+1)\beta^2 = (a-b) + 2.2^{2n} + 6(b-a) \]  
(28)

Assuming \( \alpha = x + (a+1)T \) and \( \beta = x + (b+1)T \) in (28), it reduces to

\[ x^2 = (b+1)(a+1)T^2 + 2.2^{2n} + 5 \]  
(29)

The initial solution of equation (29) is given by

\[ T_0 = 1 \text{ and } x_0 = 2.2^{2n} + 3 \]  
(30)

Therefore, \( \alpha = 3.2^{2n} + 5 \)  
(31)

On substituting the values of \( \alpha \) and \( a \) in equation (26), we get

\[ c = 9.2^{2n} + 9 \]

Therefore, triple \( \left(j_{2n}, j_{2n+2}, 9j_{2n}\right) \) is Dio 3- tuple with property \( D(2,2^{2n} + 6) \)

In general, it is noted that the triple \( \left(j_{2n}, j_{2n+2}, 92^{2n} + 2k + 1\right) \) is a Dio 3-tuple with property

\[ D\left[(4k - 14)2^{2n} + k^2 - 2k - 2\right] \]

Conclusion:

In this paper we have presented a few examples of constructing a special Dio 3 tuples for Polygonal numbers, Centered polygonal numbers, linear polynomials and Jacobasthal-lucas numbers with suitable properties. To conclude one may search for Dio 3 – tuples for higher order polygonal numbers and centered polygonal numbers with their corresponding suitable properties.

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