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Generalization of Pythagorean triplets, Quadruple

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ABSTRACT The method of computing Pythagorean triples is well known. All though the classical formulas produce all primitive triples, which do not generate all possible triples, specially non-primitive triples. This paper presents a novel approach to produce all likely triples both primitive and non-primitive, Quadruple for any extent.

1. Introduction and intuitive idea.

The sides of a right triangle follow the Pythagorean Theorem,

$$a^2+ b^2=c^2 \tag{1}$$

where a & b are the lengths of the legs of the right triangle while c is the length of the hypotenuse. A right triangle with sides of lengths 3, 4 and 5 is a special right triangle in that all the sides have positive integer lengths. The three numbers 3, 4 and 5 forms a Pythagorean triplet or Pythagorean triple. A Pythagorean triplet is a set of three whole numbers where the sum of the squares of the first two is equal to the square of the third number. Below are examples of Pythagorean triplets: (3, 4, and 5), (5, 12, and 13), (7, 24, and 25) etc. [1]. To generate a list of Pythagorean triples m could be allowed to take values from its minimum possible value (2) up to some maximum value entered by the user. For each value of n from 1 up to m-1 the corresponding triple could then be evaluated and output. An algorithmic description for this is:

```
Enter and validate a maximum value for m.
For each value of m from 2 to maximum do
{
        for each value of n from 1 to m-1 do
        {
        evaluate and print a triple.
        }
}
```

This description uses a for loop, a looping construct that repeats the loop statement as some control variable takes a sequence of values. Also note that one for loop is **nested** inside another, this is very common. From this description the following
C++ program is easily produced:

```
void main()
{
int max,            // maximum value of m to be used m, n,
             // control variables for loops
 t1, t2, t3; // Pythagorean triple
// enter and validate max do
{
cout << "Enter a maximum value for m ( >1 ): ";
cin >> max;
}
while (max < 2);
// loop on m from 2 to max for
```

```
(m=2; m <= max; m++)
{
// now loop on n from 1 to m-1 for
(n=1; n < m; n++)
{
// evaluate and print triple t1 =
m*m-n*n;
t2 = 2*m*n;
t3 = m*m+n*n;
cout << t1
         << " " << t2
         << " " << t3
         << endl;
         }
    }
}
```

An effective way to produce Pythagorean triples is based on Euclid's formula, which is fundamental formula for generating Pythagorean triples for given an arbitrary pair of positive integers *m* and *n* with *m* > *n*. The formula states that the integers $a = m^2 - n^2$, $b = 2mn$ and $c = m^2 + n^2$ form a Pythagorean triple. The triple generated by Euclid's formula is primitive if and only if *m* and *n* are co-prime and m – *n* is odd. If both *m* and *n* are odd, then *a*, *b*, and *c* will be even, and so the triple will not be primitive; however, dividing *a*, *b*, and *c* by 2 will yield a primitive triple if *m* and *n* are co-prime [2]. Now we introduce our tactic, which will be the remark that the difference between b & c or c & a can have only certain distinct values depending on the given number either a or b.

Let us study, considering the value of c as our hypotenuse side. Then, obviously, c is larger than b as well as a.

$$c = b + \Omega \qquad (2)$$

Now, by the (1);

$$a^2+b^2=c^2$$

$$= (b + \Omega^2)$$

$$=b^2=2b\Omega+\Omega^2$$

$$a^2= 2b\,\Omega + \Omega^2 \qquad (3)$$

$$b=\frac{a^2-\Omega^2}{2\Omega} \qquad (4)$$

From (3), we see $\Omega | a^2$ and $a^2-\Omega^2>0$

$$(a - \Omega)(a + \Omega) > 0$$

Either $a > \Omega$ or $a > -\Omega$ (which is not possible).

$$\text{Consider } a > \Omega \qquad (5)$$

As we know that for odd value of c, there exist either a or b as odd. So, assume that *a* is an even number ($a = 2^m$, $m \varepsilon z_+$) then we see Ω is also even. Otherwise, for $a = 2m + 1$, we see Ω itself odd.

Now, we discuss the general case for $a = 2^m$ and $\Omega = 2^r$ for some $m, r \in Z^+$. Clearly, we see m > r by (5) and by (3), we have;

$$(2^m)^2= 2^r (2b + 2^r)$$

$$\Rightarrow b = 2^{r-1} (2^{2m-2r} - 1) \qquad (6)$$

Since a is being an even, the value of b should be odd iff r = 1. From (5), we see the value of b is $2^{2m-2}-1$.

Thus, the required triple will be (a, b, c) ≡ (2^m $2^{2m-2}-1$, $2^{2m-2}+1$)

$$\equiv \left(2^m, \frac{4^m}{4}-1, \frac{4^m}{4}+1\right)$$

2. Interpretations for introduction part.

For another, even if *a* is a power of 2, you have no warrant to assert that will necessarily be a power of 2 as well; at least, no warrant that you have given. Of course, you may *assume* that's the case, but then you are dealing with a rather restrictive subset of Pythagorean triples. For another, it is false that if *a* is even then *b* is necessarily odd, though it is true for *primitive* Pythagorean triples (ones where *a*, *b*, and *c* are pairwise relatively prime). But you never make that assumption.
And finally, a complete description of primitive Pythagorean triples is well known. You can deduce our formula from them rather easily. Explicitly, suppose that (*a*,*b*,*c*) is a primitive Pythagorean triple, and let us assume that *a* is even; we have

$$(c+b)(c-b) = c^2 - b^2 = a^2 ;$$

Since *a*, *b*, *c* are pairwise co-prime, and *a* is even, then *b* and *c* are odd, so $c+b$ and $c-b$ are both even. Any common divisor of $c+b$ and $c-b$ will divide $(c-b)+(c+b)=2c$, and also $(c+b)-(c-b)=2b$; since gcd $(2c, 2b) = 2$ gcd $(b, c) = 2$, the gcd of $c-b$ and $c+b$ is 2. Dividing through by 4 (which we can do since *a* is even, so a^2 is a multiple of 4, we get;

$$\left(\frac{c+b}{2}\right)^2 + \left(\frac{c-b}{2}\right)^2 = \left(\frac{a}{2}\right)^2$$

Since $\left(\frac{c+b}{2}\right)$ and $\left(\frac{c-b}{2}\right)$ are relatively prime, and their product is a square, they are each a square. So we can write $\left(\frac{c+b}{2}\right)^2 = s^2$, $\left(\frac{c-b}{2}\right)^2 = r^2$ and a/2= rs , with r and s positive $r > s$, relatively prime, and of opposite parity(since c is odd)
This gives us;

$$c = \left(\frac{c+b}{2}\right) + \left(\frac{c-b}{2}\right) = s^2 + r^2 \quad \text{and} \quad b = \left(\frac{c+b}{2}\right) - \left(\frac{c-b}{2}\right) = s^2 - r^2$$

In our case, we need a = 2^m since *s* and *r* are of opposite parity, one must be equal to 1; since $s > r >$ 0, we will have $r = 1$, $s = 2^{m-1}$ and this gives us;

$$(a, b, c) = \left(2^m, 4^{m-1} - 1, 4^{m-1} + 1\right)$$

This is our result. Note, however, that there are other (non-primitive) Pythagorean triples that have *a* as power of 2: namely,

$$(a, b, c) = \left(2^{m+k}, 2^{2m+k-2} - 2^k, 2^{2m+k-2} + 2^k\right)$$

is a Pythagorean triple for all $k \geq 0$.
If *a* is odd, then just exchange the roles of *a* and *b* above to conclude (i) if we take odd numbers for powers of some prime and (ii) if we take even numbers with prime powers.

3. Pythagorean Quadruple

We know that a set of four positive integers a, b, c and d such that $a^2 + b^2 + c^2 = d^2$ set of four positive integers a, b, c and d such that $a^2 + b^2 + c^2 = d^2$ is called a Pythagorean quadruple. The simple example is (1,2,2,3), since $1^2 + 2^2 + 2^2 = 3^2$. The next simplest (primitive) example is (2,3,6,7), since $2^2 + 3^2 + 6^2 = 7^2$.

Here ,

$$\begin{aligned} a &= m^2 + n^2 - p^2 - q^2 \\ b &= 2(mq + np) \\ c &= 2(nq - mp) \\ d &= m^2 + n^2 + p^2 + q^2 \end{aligned}$$

Where m,n,p,q are non-nigative integers and gcd(m,n,p,q)=1 and m+n+p+q ≡ 1(mod 2) . Thus all primitive Pythagorean quadruples are characterized by the all quadruples are given by the following formula

$$(m^2 + n^2 + p^2 + q^2)^2 = (2mq + 2np)^2 + (2nq - 2mp)^2 + (m^2 + n^2 - p^2 - q^2)^2.$$

A Pythagorean quadruple is an ordered quadruple of positive integers (a,b,c,d) such that

$$a^2 + b^2 + c^2 = d^2 \tag{1}$$

In this section, we shall show the all possible generations of Pythagorean quadruple for some set of integers(a,b)

Consider $a^2 + b^2 = m$ and $d = c + \Omega$

By (1), $m^2 + c^2 = (c + \Omega)^2$

$$\Rightarrow c = \frac{m - \Omega^2}{2\Omega} \tag{2}$$

We can make some observations by (2)

(a) If $m = 2^l (l \in z^+)$ then Ω must be even

(b) If $k = 2^l + 1(odd)$, then Ω should be odd for integral value of C.

(c) For positive integer value of c it is regard that $m > \Omega^2$ (3)

Now we will discuss the above cited cases individually and we conclude the generating of Pythagorean quadruples.

Case -1: If a is even and b is odd

If a is odd and b is even

NOTE: For odd value or m, Ω is odd

(i) Let us consider a and b have common factors $p_1, p_2, \ldots\ldots\ldots, p_n$ then m can be written as

$m = q_1^{s_1}, q_2^{s_2}, \ldots\ldots\ldots q_n^{s_n} \quad q_1^{l_1}, q_2^{l_2}, \ldots\ldots\ldots q_n^{l_n}$

$\Omega = p_1, p_2, \ldots\ldots\ldots p_n \quad q_1, q_2, \ldots\ldots\ldots q_n$ k_i, l_i, r_i and s_i all being integers

for all i then (2) becomes

$$\Omega = \prod_{j=1}^{N} q_j^{s_j} \quad q_1^{k_1-s_1} q_2^{k_2-s_2} q_3^{k_3-s_3} \ldots\ldots\ldots\ldots q_n^{k_n-s_n})$$

For primitive solution $r_i = 0$ or $r_i = k_i$ for $i = 0,1, 2, 3,\ldots..n$ and s_j can take all integer

Value from 0 to s_j by (3) so, $\Omega = \prod_{i=1}^{n} p_i^{r_i} \prod_{j=1}^{N} q_j^{s_j}$ write either $r_i = 0$ or $r_i = k_i$

Example : (1)

$a = 12$ and $b = 15$ then $m = 369 = 3^2 X41$ is not valid as $\Omega^2 > m$, by (3)

For $c = 184$ and $d = 185$ when $\Omega = 1$

$c = 16$ and $d = 25$ when $\Omega = 3^2$ so the primitive quadruples for $a = 12$ and $b = 15$ are (12,15,184,185) (12,15,16, 25) .

Example : (2)

For $a = 2X3X5X7 = 210$ and $b = 3^3 X5 = 135$ then $m = 3^2 X5^2 X277 = 62325$

$\Omega = 1, 3^2, 5^2, 3^2 X5^2$

$\therefore$ other combinations of Ω are not possible due to (3)
Interpretations for third sections:
We can assume that a.b,c,d are relatively prime (no prime dividing them all), which is in the case need not be the same as co-prime (no prime dividing any pair).
Since by homogeneity any other solution is a multiple of one that is relatively prime, of cource, this assumptions, such as $x^3 + y^3 = z^5$ etc. Now, considering the equation (mod8) , noting that $x^2 \equiv 1 (\mathrm{mod}\ 8)$ if x is odd, one concludes that exactly two of a,b,c must be even, say a,b so that c,d are both add and d-c and d+c are both even. So each bracketed term in the following is an integer.

$$\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2 = \left(\frac{d-c}{2}\right)\left(\frac{d+c}{2}\right)$$

Now, for any prime $\equiv 3 (\mathrm{mod}\ 4)$ dividing either of the terms on he RHS to even multiplicity (or else it would divide a,b,c,d, contrary to hypothesis)
This, combined with the fact that 2 and every prime $\equiv 1 (\mathrm{mod}\ 4)$ can be expressed as the sum of two squares of integers, and their product composed into the same form by repeated use of

$$(p^2 + q^2)(r^2 + s^2) = (pq - rs)^2 + (pr + qs)^2$$

$$\frac{d-c}{2} \text{ and } \frac{d+c}{2}$$

Should be the sum of two squares of integers, say $v^2 + w^2$ and $t^2 + u^2$ respectively.
Now we can think of a sum of two squares as the norm or a so-called Gaussian Integer, of the form $x + y\sqrt{-1}$, the norm being the product of their and its conjugate $x + y\sqrt{-1}$ to give us $x^2 + y^2$

The very integrating thing about Gaussian integers, also referred to as the ring $d = t^2 + u^2 + v^2 + w^2$, is that factorization into "primes" of the same form is unique, just like the rational integers. So treating the first equation as expressing a Gaussian integer as the product of two others,

$$\text{i.e } \frac{a}{2} + \frac{b}{2}\sqrt{-1} = (v + w\sqrt{-1})(t + u\sqrt{-1})$$

where the norms of the bracketed terms on the RHS correspond to $\frac{d-c}{2}$ and $\frac{d+c}{2}$ respectively.
Now, we can conclude by multiplying out and the RHS and equating coefficients of ($\sqrt{-1}$ and 1) that there must be some set of integers t,u,v,w for which

$$\frac{a}{2} = tv - uw$$

$$\frac{b}{2} = uv - tw$$

And taking norms gives:

$$\frac{d-c}{2} = v^2 + w^2$$

$$\frac{d+c}{2} = t^2 + u^2$$

$$\Rightarrow c = t^2 + u^2 - v^2 - w^2$$

$$d = t^2 + u^2 + v^2 + w^2$$

Conclusion

The major advantage of our approach is that it does not require any primitive set to start with and finding proper multipliers to obtain the desired tuple. An interesting fact is that just by factorizing we can forecast how many primitive and non-primitive cases are possible before actually computations.

References

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