

A Two Species Amensalism Model with Harvesting By Homotopy Analysis Method

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Abstract. In this paper a two species Amensalism model is taken up for analytical study. The model comprises an Amensal, which is harvested at a rate proportional to their population and an enemy. Moreover both the species are provided with limited resources. The series solution of the non-linear system was approximated by the Homotopy analysis method (HAM) and the results are supported by numerical simulations.

1. Introduction:

Symbioses are a broad class of interactions among organisms --Amensalism involves one organism affecting another negatively without any positive or negative benefit for itself e.g., avian botulism and "red tides" (caused by dino flagellates) are often extremely toxic for birds, marine mammals, humans, etc.

Amensalism may lead to the pre-emptive colonization of a habitat. Once an organism establishes itself within a habitat it may prevent other populations from surviving in that habitat. The production of lactic acid or similar low-molecular-weight fatty acids is inhibitory to many bacterial populations. Populations able to produce and tolerate high concentrations of lactic acids, for example, are able to modify the habitat so as to preclude the growth of other bacterial populations. *E. coli* is unable to grow in the rumen, probably because of the presence of volatile fatty acids produced there by anaerobic heterotrophic microbial populations. Fatty acids produced by microorganisms on skin surfaces are believed to prevent the colonization of these habitats by other microorganisms. Populations of yeasts on skin surfaces are maintained in low numbers by microbial populations producing fatty acids. Acids produced by microbial populations in the vaginal tract are probably responsible for preventing infection by pathogens such as *Candida albicans*

In the present investigation a two species Amensalism model with harvesting with limited resources for both the species was taken up for analytic study. The model is represented by coupled non-linear ordinary differential equations. The series solution of the non-linear system is approximated by Homotopy Analysis Method.

The governing equations of the system are as follows

$$\begin{aligned}\frac{dx}{dt} &= a_1(1-k)x(t) - \alpha_{11}x^2(t) - \alpha_{12}x(t)y(t) \\ \frac{dy}{dt} &= a_2y(t) - \alpha_{22}y^2(t)\end{aligned}\tag{1.1}$$

With the notation

$x(t), y(t)$: populations of the species 1 and species 2

a_1, a_2 : rates of natural growth of species 1 and species 2.

α_{11}, α_{22} : rates of decrease due to insufficient food of the species 1 and species 2.

α_{12} : rate of decrease of the species 1 due to inhibition by the species 2.

K: rate of harvesting proportional to the population size of species 1 ($0 < k < 1$)

2. Basic ideas of HAM:

In this paper, we apply the homotopy analysis method to the discussed problem. To show the basic idea, let us consider the following differential equation

$$N(u,t) = 0, \quad (2.1)$$

Where N is a nonlinear operator, t denote independent variable, $u(t)$ is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao constructs the so-called zero-order deformation equation

$$(1-p)L[\phi(t;p) - u_0(t)] = p\bar{h}H(t)N[\phi(t;p)], \quad (2.2)$$

Where $p \in [0,1]$ is the embedding parameter, \bar{h} is a nonzero auxiliary, parameter H is an auxiliary, parameter H is an auxiliary function L is an auxiliary linear operator, $u_0(t)$ is an initial guess of $u(t)$, $\phi(t;p)$ is a unknown function, respectively. It is important that one has great freedom to choose auxiliary things in HAM. Obviously, When

$$p=0 \text{ and } p=1, \text{ it holds } \phi(t,0) = u_0(t), \quad \phi(t,1) = u(t) \quad (2.3)$$

Respectively. Thus as p increases from 0 to 1, the solution $\phi(t;p)$ varies from the initial guesses $u_0(t)$ to the solution $u(t)$. Expanding $\phi(t;p)$ in Taylor series with respect to p , one has

$$\phi(t;p) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t)p^m \quad (2.4)$$

Where

$$u_m(t) = \frac{1}{m!} \left. \frac{\partial^m \phi(t;p)}{\partial p^m} \right|_{p=0} \quad (2.5)$$

If the initial guess (2.3), the auxiliary linear parameter L_i , the non-zero auxiliary parameter h_i and the auxiliary function H_i are properly chosen, so that the power series (2.4) converges at $p=1$.

Then we have under these assumptions the solution series

$$u(t) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t) \quad (2.6)$$

Which must be one of solution s of original non linear equation, as proved by Liao [15]. As $\bar{h} = -1$ and $H(t)=1$, Eq (2.2) becomes $(1-p)L[\phi(t;p) - u_0(t)] + pN[\phi(t;p)] = 0$, (2.7)

which is used mostly in the homotopy perturbation method, whereas the solution obtained directly, without using Taylor series which is explained by H.Jafari, M.Zabihi and M.Saidy [14] and J.H.He [15,16] and S.J.Liao [17] compare the HAM and HPM. According to the definition, the governing equation can be deduced from the zero-order deformation equation (2.7).

Define the vector $\vec{u}_k = \{u_0, u_1, \dots, u_k\}$ Differentiating Eq. (2.7) ,m times with respect to embedding parameter p and then setting p=0 and finally dividing them by m! , we have the so-called mth-order deformation equation

$$L[u_m(t) - \chi_m u_{m-1}(t)] = \bar{h}H(t)R_m(\vec{u}_{m-1}), \tag{2.8}$$

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(t; p)]}{\partial p^{m-1}} \Big|_{p=0},$$

Where

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{2.9}$$

3. Application:

Consider the nonlinear differential equation(1.1) with initial conditions .we assume the solution of the system(1.1) ,x(t),y(t) can be expressed by following set of base functions in the form

$$x(t) = \sum_{m=1}^{+\infty} a_m t^m, \quad y(t) = \sum_{m=1}^{+\infty} b_m t^m \tag{3.1}$$

Where a_m, b_m are coefficients to be determined. This provides us the so called rule of solution expression i.e., the solution of (1.1) must be expressed in the same form as (3.1) and the other expressions must be avoided. According to (1.1) and (3.1) we chose the linear operator. To solve the system of Eqs.(1.1),Homotopy analysis method is employed. We consider the following initial approximations

$$x_0(t) = x(t=0) = x_0 \quad y_0(t) = y(t=0) = y_0 \tag{3.2}$$

The linear and non-linear operators are denoted as follows.

$$L_1[x(t; p)] = \frac{dx(t; p)}{dt}, \quad L_2[y(t; p)] = \frac{dy(t; p)}{dt}, \tag{3.3}$$

$$N_1[x(t, p)] = \frac{dx(t; p)}{dt} - a_1(1-k)x(t; p) + \alpha_{11}x^2(t; p) + \alpha_{12}x(t; p)y(t; p) \tag{3.3}$$

$$N_2[y(t, p)] = \frac{dy(t; p)}{dt} - a_2y(t; p) + \alpha_{22}y^2(t; p) \tag{3.4}$$

Using above definition the zero order deformation equation can be constructed

$$\begin{aligned} (1-p)L_1[x(t; p) - x_0(t)] &= ph_1N_1[x, y], \\ (1-p)L_2[y(t; p) - y_0(t)] &= ph_2N_2[x, y], \end{aligned} \tag{3.5}$$

When p=0 and p=1 ,from the zero-deformation equations one has,

$$\begin{aligned} x(t; 0) &= x_0(t) \quad x(t; 1) = x(t) \\ y(t; 0) &= y_0(t) \quad y(t; 1) = y(t) \end{aligned} \tag{3.6}$$

And expanding $x(t;p)$ and $y(t;p)$ in Taylors series, with respect to embedding parameter p . one obtains

$$x(t; p) = x_0(t) + \sum_{m=1}^{+\infty} x_m(t) p^m \quad (3.7)$$

$$y(t; p) = y_0(t) + \sum_{m=1}^{+\infty} y_m(t) p^m$$

$$x_m(t) = \frac{1}{m!} \left. \frac{d^m x(t; p)}{dp^m} \right|_{p=0} \quad (3.8)$$

$$y_m(t) = \frac{1}{m!} \left. \frac{d^m y(t; p)}{dp^m} \right|_{p=0}$$

$$p = 1 \begin{cases} x_m(t) = x_0(t) + \sum_{m=1}^{+\infty} x_m(t) \\ y_m(t) = y_0(t) + \sum_{m=1}^{+\infty} y_m(t) \end{cases} \quad (3.9)$$

Define the vector

$$\begin{aligned} \bar{x}_m &= [x_0(t), x_1(t), \dots, x_m(t)] \\ \bar{y}_m &= [y_0(t), y_1(t), \dots, y_m(t)] \end{aligned} \quad (3.10)$$

And apply the procedure stated before. The following m^{th} -order deformation Eq will be achieved.

$$\begin{aligned} L_1[x_m(t) - \chi_m x_{m-1}(t)] &= \bar{h}_1 H_1(t) R_{1m}(\bar{x}_{m-1}, \bar{y}_{m-1}), \\ L_2[y_m(t) - \chi_m y_{m-1}(t)] &= \bar{h}_2 H_2(t) R_{2m}(\bar{x}_{m-1}, \bar{y}_{m-1}), \end{aligned} \quad (3.11)$$

Let us consider $H_1(t) = H_2(t) = 1$ and the initial conditions

$$x_0(t) = x(t=0) = x_0 \quad y_0(t) = y(t=0) = y_0 \quad \text{in above equations}$$

$$R_{1m}(x_{m-1}, y_{m-1}) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dp^{m-1}} N[x(t, p)] = \frac{d}{dt} x_{m-1}(t) - a_1(1-k)x_{m-1} + \alpha_{11} \sum_{n=1}^m x_n(t) x_{m-n-1}(t) + \alpha_{12} \sum_{n=0}^{m-1} x_n(t) y_{m-n-1}(t)$$

$$R_{2m}(x_{m-1}, y_{m-1}) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dp^{m-1}} N[y(t, p)] = \frac{d}{dt} y_{m-1}(t) - a_2 y_{m-1} + \alpha_{22} \sum_{n=1}^m y_n(t) y_{m-n-1}(t) \quad (3.12)$$

The following will be obtained successively

$$L_1(x_1(t) - \chi_1 x_0(t)) = h_1 \left[\frac{d}{dt} x_0(t) - a_1(1-k)x_0(t) + \alpha_{11} x_0^2(t) + \alpha_{12} x_0(t) y_0(t) \right]$$

$$\chi_m(t) = 0, m \leq 0$$

$$1, m > 0$$

$$L_1(x_1(t)) = h_1 \left[-a_1(1-k)x_0(t) + \alpha_{11} x_0^2(t) + \alpha_{12} x_0(t) y_0(t) \right]$$

$$x_1(t) = h_1 \left[-a_1(1-k)x_0 + \alpha_{11} x_0^2 + \alpha_{12} x_0 y_0 \right] t$$

$$L_1(y_1(t) - \chi_1 y_0(t)) = h_2 \left[\frac{d}{dt} y_0(t) - a_2 y_0(t) + \alpha_{22} y_0^2(t) \right]$$

$$L_1(y_1(t)) = h_2 \left[-a_2 y_0(t) + \alpha_{22} y_0^2(t) \right]$$

$$y_1(t) = h_2 \left[-a_2 y_0 + \alpha_{22} y_0^2 \right] t$$

$$L_1(x_2(t) - \chi_2 x_1(t)) = h_1 \left[\frac{d}{dt} x_1(t) - a_1(1-k)x_1(t) + \alpha_{11} \sum_{n=0}^1 x_n(t)x_{1-n}(t) + \alpha_{12} \sum_{n=0}^1 x_n(t)y_{1-n}(t) \right]$$

$$x_2(t) = (h_1 + h_1^2)M_1 t + \frac{t^2}{2} \left[-a_1(1-k)h_1^2 M_1 + 2\alpha_{11}x_0 h_1 M_1 + \alpha_{12}y_0 M_1 + h_2 a_2 \alpha_{12} x_0 \right]$$

where $M_1 = -a_1(1-k)x_0 + \alpha_{11}x_0^2 + \alpha_{12}x_0y_0$

$$L_1(y_2(t) - \chi_2 y_1(t)) = h_2 \left[\frac{d}{dt} y_1(t) - a_2 y_1(t) + \alpha_{22} \sum_{n=0}^1 y_n(t)y_{1-n}(t) \right]$$

$$y_2(t) = (h_2 + h_2^2) \left[-a_2 y_0 + \alpha_{22} y_0^2 \right] t + \frac{h_2^2 y_0}{2} \left[\left(a_2^2 - 3a_2 \alpha_{22} y_0 + 2\alpha_{22}^2 y_0^2 \right) \right] t^2$$

$$L_1(x_3(t) - \chi_3 x_2(t)) = h_1 \left[\frac{d}{dt} x_2(t) - a_1 x_2(t) + \alpha_{11} \sum_{n=0}^2 x_n(t)x_{2-n}(t) + \alpha_{12} \sum_{n=0}^2 x_n(t)y_{2-n}(t) \right]$$

$$x_3(t) = (1+h_1)(h_1+h_1^2)M_1 t + \left[(1+h_1)M_2 + h_1 \alpha_{12} y_0 (h_1+h_1^2)M_1 + \alpha_{12} h_1 x_0 (h_2+h_2^2)(-a_2 y_0 + \alpha_{22} y_0^2) \right] \frac{t^2}{2}$$

$$+ \left[-2a_1 h_1 M_2 + 2h_1 \alpha_{11} M_2 x_0 + \alpha_{11} h_1^3 M_1^2 + 2h_1 \alpha_{12} h_2^2 x_0 y_0^2 (a_2^2 - 3a_2 \alpha_{22} y_0 + 2\alpha_{22}^2 y_0^2) + h_1^2 h_2 \alpha_{12} M_1 (-a_2 y_0 + \alpha_{22} y_0^2) + 2h_1 \alpha_{12} M_2 y_0 \right] \frac{t^3}{3}$$

$$M_2 = \left[-a_1 h_1^2 (1-k)M_1 + 2h_1 \alpha_{11} x_0 M_1 + \alpha_{12} y_0 M_1 + h_2 \alpha_{12} a_2 x_0 y_0^2 - h_2 \alpha_{12} a_2 x_0 \right]$$

$$L_1(y_3(t) - \chi_3 y_2(t)) = h_2 \left[\frac{d}{dt} y_2(t) - a_2 y_2(t) + \alpha_{22} \sum_{n=0}^2 y_n(t)y_{2-n}(t) \right]$$

$$y_3(t) = (1+h_2)k_3 t + \left[(1+h_2)k_4 - a_2 h_2 k_3 + 2h_2 k_3 \alpha_{22} y_0 \right] \frac{t^2}{2} + \left[\left(-\frac{1}{2} a_2 h_2 k_4 + h_2 \alpha_{22} k_4 y_0 + \frac{\alpha_{22} h_2^3 k_3^2}{(h_2 + h_2^2)^2} \right) \right] \frac{t^3}{3}$$

Where

$$k_1 = \left[-a_1 x_0 + \alpha_{11} x_0^2 + \alpha_{12} x_0 y_0 \right]$$

$$k_2 = \frac{h_1 x_0}{2} \left[h_1 (-a_1 + \alpha_{11} x_0 + \alpha_{12} y_0) (-a_1 + 2\alpha_{11} x_0 + \alpha_{12} y_0) + h_2 \alpha_{12} (-a_2 y_0 + \alpha_{22} y_0^2) \right]$$

$$k_3 = (h_2 + h_2^2) \left[-a_2 y_0 + \alpha_{22} y_0^2 \right]$$

$$k_4 = \frac{h_2^2 y_0}{2} \left[\left(a_2^2 - 3a_2 \alpha_{22} y_0 + 2\alpha_{22}^2 y_0^2 \right) \right]$$

The three terms approximation to the solution will be considered as

$$x(t) \approx x_0 + x_1(t) + x_2(t) + x_3(t)$$

$$y(t) \approx y_0 + y_1(t) + y_2(t) + y_3(t)$$

4. Numerical example:

$x_0 = 5; y_0 = 10; a_1 = .5; a_2 = .09; a_{11} = .05; a_{12} = 2.6; a_{22} = 7; h_1 = 1; h_2 = 1;$
 $k = .05;$

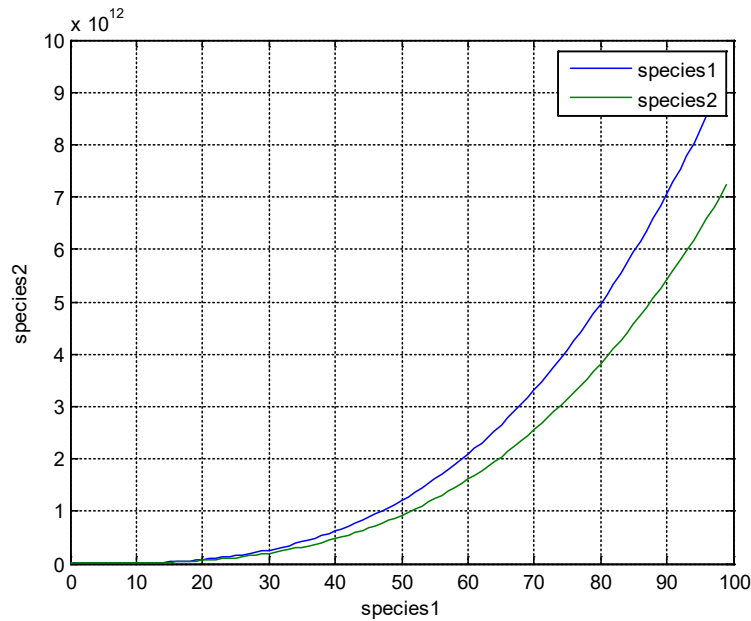


Fig 5.1: The Variation of x and y with respective Time (t) for system of Eq (1.1)

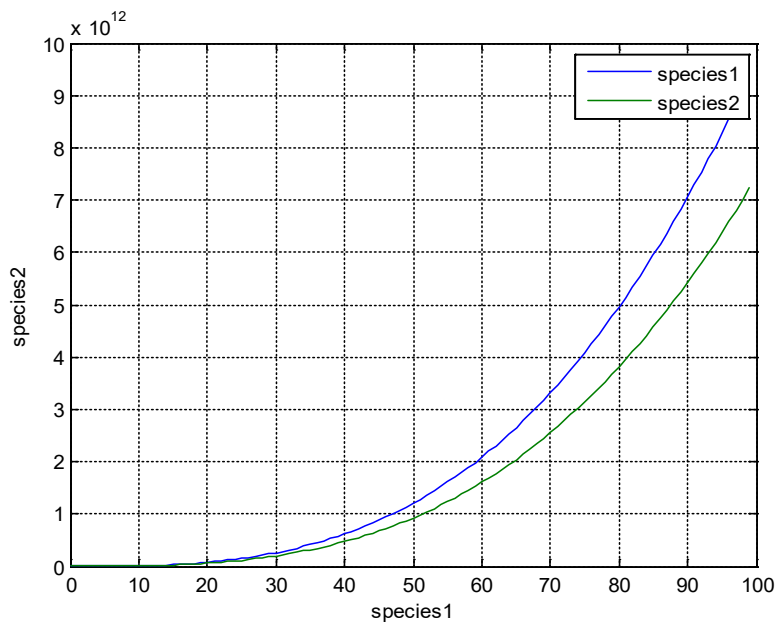


Fig 5.2: The Variation of x and y with respective Time (t) for system of Eq (1.1) with k=0

5. Conclusion:

In the analysis of considered two species Amensalism model with harvesting of the species 1, we discussed the series approximations of the model by homotopy analysis method and also discussed the effect of harvesting of the species 1. From the graphs shown in fig 5.1 and 5.2 it is evident that harvesting on species 1 does not have any influence on the stability.

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