

## A Three Species Ecological Model with A Prey, Predator and Competitor to the Predator and Optimal Harvesting of the Prey

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**Abstract:** The present paper is devoted to an analytical investigation of three species ecological model with a Prey ( $N_1$ ), a predator ( $N_2$ ) and a competitor ( $N_3$ ) to the Predator without effecting the prey ( $N_1$ ). In addition to that, the species are provided with alternative food. The model is characterized by a set of first order non-linear ordinary differential equations. All the eight equilibrium points of the model are identified and local and global stability criteria for the equilibrium states except fully washed out and single species existence are discussed. Further exact solutions of perturbed equations have been derived. The analytical stability criteria are supported by numerical simulations using mat lab. Further we discussed the effect of optimal harvesting on the stability.

### 1. Introduction:

Ecology relates to the study of living beings in relation to their living styles. Research in the area of theoretical ecology was initiated by Lotka [15] and by Volterra [16]. Since then many mathematicians and ecologists contributed to the growth of this area of knowledge as reported in the treatises of Paul Colinvaux [14], Freedman [2], Kapur [3,4] etc. Recently Archana Reddy [1] discussed the stability analysis of two interacting species with harvesting of both species. Lakshmi Narayan and Pattabhiramacharyulu [5, 6] and Shiva Reddy [7, 8] et al., Ravidra Reddy [9,10,11] et al. Shanker [12] et al and Papa Rao [13] have discussed different prey-predator models in detail. T.K.kar [17] studied the stability of several species models by incorporating the harvesting term. Inspired from that, we discussed a more general three species model. The model is characterized by a set of first order ordinary differential equations. All the eight equilibrium points of the model are identified and stability criteria for some equilibrium states are discussed. Further we discussed the effect of optimal harvesting on the stability.

### 2. Basic Equations:

The model equations for a three species Prey - Predator and competitor to the predator system is given by the following system of first order ordinary differential equations employing the following notation:

$$\begin{aligned}\frac{dN_1}{dt} &= a_1N_1 - \alpha_{11}N_1^2 - \alpha_{12}N_1N_2 - qEN_1 \\ \frac{dN_2}{dt} &= a_2N_2 - \alpha_{22}N_2^2 + \alpha_{21}N_1N_2 - \alpha_{23}N_2N_3 \\ \frac{dN_3}{dt} &= a_3N_3 - \alpha_{33}N_3^2 - \alpha_{32}N_2N_3\end{aligned}\tag{2.1}$$

Where  $N_1$ ,  $N_2$  and  $N_3$  are the populations of the prey and predator and a competitor to the predator with the natural growth rates  $a_1$ ,  $a_2$  and  $a_3$  respectively,

$\alpha_{11}$  is rate of decrease of the prey due to insufficient food

$\alpha_{12}$  is rate of decrease of the prey due to inhibition by the predator,

$\alpha_{21}$  is rate of increase of the predator due to successful attacks on the prey,

$\alpha_{22}$  is rate of decrease of the predator due to insufficient food other than the prey,

$\alpha_{23}$  is rate of decrease of the predator due to the competition with the third species

$\alpha_{33}$  is rate of decrease of the Competitor to the predator due to insufficient food

$\alpha_{32}$  is rate of decrease of the competitor to the predator due to the competition with the predator

$q$  Catchability coefficient of prey (N1),  $E$ : effort applied to the harvest of the prey

Throughout the analysis, we assume that  $(a_1 - qE) > 0$ .

### 3. Equilibrium states:

The system under investigation has eight equilibrium states. They are

I.E<sub>1</sub>: The fully washed out state  $\bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = 0$  (3.1)

II.E<sub>2</sub>: The state in which only the predator survives and the prey and competitor to the predator are washed out

$$\bar{N}_1 = 0, \bar{N}_2 = \frac{a_2}{\alpha_{22}}, \bar{N}_3 = 0 \quad (3.2)$$

III. E<sub>3</sub> : The state in which both the prey and the predators washed out and competitor to the predator survive

$$\bar{N}_1 = 0, \bar{N}_2 = 0, \bar{N}_3 = \frac{a_3}{\alpha_{33}} \quad (3.3)$$

IV. E<sub>4</sub>: The state in which both the predator and competitor to the predator washed out and prey survive

$$\bar{N}_1 = \frac{(a_1 - qE)}{\alpha_{11}}, \bar{N}_2 = 0, \bar{N}_3 = 0 \quad (3.4)$$

V. E<sub>5</sub> :The state in which both the prey and the predators exist and competitor to the predator washed out

$$\bar{N}_1 = \frac{((a_1 - qE)\alpha_{22} - a_2\alpha_{12})}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}}, \bar{N}_2 = \frac{(a_2\alpha_{11} + (a_1 - qE)\alpha_{21})}{\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}}, \bar{N}_3 = 0$$

This case arise only when  $(a_1 - qE)\alpha_{22} > a_2\alpha_{12}$  (3.5)

VI. E<sub>6</sub> :The state in which both prey and competitor to the predator exist and predator washed out

$$\bar{N}_1 = \frac{(a_1 - qE)}{\alpha_{11}}, \bar{N}_2 = 0, \bar{N}_3 = \frac{a_3}{\alpha_{33}} \quad (3.6)$$

VII. E<sub>7</sub>: The state in which both predator and competitor to the predator exist and prey washed out

$$\bar{N}_1 = 0, \bar{N}_2 = \frac{a_2\alpha_{33} - a_3\alpha_{23}}{\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}}, \bar{N}_3 = \frac{a_3\alpha_{22} - a_2\alpha_{32}}{\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}} \quad (3.7)$$

The equilibrium state exist only when  $a_2\alpha_{33} > a_3\alpha_{23}, a_3\alpha_{22} > a_2\alpha_{32}$  &  $\alpha_{22}\alpha_{33} > \alpha_{23}\alpha_{32}$

VIII.  $E_8$  : The state in which prey, predator and competitor to the predator exist

$$\begin{aligned} \overline{N}_1 &= \frac{(a_1 - qE)(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) - \alpha_{12}(a_2\alpha_{33} - a_3\alpha_{23})}{\alpha_{11}(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) + \alpha_{12}\alpha_{21}\alpha_{33}}, \overline{N}_2 = \frac{\alpha_{11}(a_2\alpha_{33} - a_3\alpha_{23}) + (a_1 - qE)\alpha_{21}\alpha_{33}}{\alpha_{11}(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) + \alpha_{12}\alpha_{21}\alpha_{33}} \\ \overline{N}_3 &= \frac{\alpha_{11}(a_3\alpha_{22} - a_2\alpha_{32}) + \alpha_{21}(a_3\alpha_{12} - (a_1 - qE)\alpha_{32})}{\alpha_{11}(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) + \alpha_{12}\alpha_{21}\alpha_{33}} \end{aligned} \quad (3.8)$$

The equilibrium state exist only when

$$\begin{aligned} a_3\alpha_{22} &> a_2\alpha_{32}, a_3\alpha_{12} > (a_1 - qE)\alpha_{32}, \\ (a_1 - qE)(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32}) &> \alpha_{12}(a_2\alpha_{33} - a_3\alpha_{23}), \\ (a_2\alpha_{33} > a_3\alpha_{23}) \\ &\&\alpha_{22}\alpha_{33} > \alpha_{23}\alpha_{32} \end{aligned}$$

#### 4 The stability of the equilibrium states:

$$\text{Let } N = (N_1, N_2, N_3)^T = \overline{N} + U \quad (4.1)$$

Where  $U = (u_1, u_2, u_3)^T$  is the perturbation over the equilibrium state.  $\overline{N} = (\overline{N}_1, \overline{N}_2, \overline{N}_3)^T$ . The basic equations (2.1), (2.2) and (2.3) are quasi-linearized to obtain the equations for the perturbed state.

$$\frac{dU}{dt} = AU \quad (4.2)$$

Where

$$A = \begin{bmatrix} a_1 - 2\alpha_{11}N_1 - \alpha_{12}N_2 - qE & -\alpha_{12}N_1 & 0 \\ -\alpha_{21}N_2 & a_2 - 2\alpha_{22}N_2 + \alpha_{21}N_1 - \alpha_{23}N_3 & -\alpha_{23}N_2 \\ 0 & -\alpha_{32}N_3 & a_3 - 2\alpha_{33}N_3 - \alpha_{32}N_2 \end{bmatrix}$$

$$\text{The characteristic equation for the system is } \det[A - \lambda I] = 0 \quad (4.3)$$

The equilibrium state is stable, if three roots of the equation (4.3) are negative in case they are real or the roots have negative real parts in case they are complex.

The local and global stability of the equilibrium states  $E_1, E_3$ , and  $E_4$  are found to be unstable. Remaining is stable. We restricted our study to the equilibrium states  $E_5, E_6, E_7$  and  $E_8$ .

#### 4.1. Stability of the equilibrium state $E_5$ :

One of the Eigen values of variational matrix A, is  $a_3 - \alpha_{32}\overline{N}_2$  and the other two are obtained from the quadratic equation

$$\lambda^2 + (\alpha_{11}\overline{N}_1 + \alpha_{22}\overline{N}_2)\lambda + (\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21})\overline{N}_1\overline{N}_2 = 0 \quad (4.1.1)$$

In (4.1.1), the sum of the roots,  $-(\alpha_{11}\overline{N}_1 + \alpha_{22}\overline{N}_2)$ , is negative and the product of the roots,

$(\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21})\overline{N}_1\overline{N}_2$ , is positive. Therefore the roots of (4.1.1) are real and negative or complex conjugates having negative real parts. Thus the state will be asymptotically stable when

$$a_3 < \alpha_{32}\overline{N}_2$$

The solution of the perturbed equations are:

$$u_1 = A_1e^{s_1t} + B_1e^{s_2t} + C_1e^{s_3t} \quad (4.1.2)$$

$$u_2 = A_2e^{s_1t} + B_2e^{s_2t} + C_2e^{s_3t} \quad (4.1.3)$$

$$u_3 = u_{30}e^{(a_3 - \alpha_{32}\overline{N}_2)t} \quad (4.1.4)$$

where  $s_1 = a_3 - \alpha_{32}\overline{N}_2$  and  $s_2, s_3$  are roots of equation (4.1.1)

$$A_1 = \left[ \frac{u_{10}\alpha_{22}\overline{N_2}}{(s_1-s_2)} + \frac{u_{10}s_1}{(s_1-s_2)} + \frac{u_{30}\alpha_{23}\alpha_{12}\overline{N_1}\overline{N_2}}{(s_1-s_2)(s_1-s_3)} \right], B_1 = \left[ \frac{u_{10}\alpha_{22}\overline{N_2}}{(s_2-s_1)} + \frac{u_{10}s_2}{(s_2-s_1)} + \frac{u_{30}\alpha_{23}\alpha_{12}\overline{N_1}\overline{N_2}}{(s_2-s_1)(s_2-s_3)} \right]$$

$$C_1 = \left[ \frac{u_{30}\alpha_{23}\alpha_{12}\overline{N_1}\overline{N_2}}{(s_3-s_1)(s_3-s_2)} \right], A_2 = \left[ \frac{-(u_{10}\alpha_{21}+u_{20}\alpha_{11})\overline{N_1}}{(s_1-s_2)} - \frac{u_{20}s_1}{(s_1-s_2)} + \frac{u_{30}\alpha_{11}\alpha_{23}\overline{N_1}\overline{N_2} + u_{30}\alpha_{23}\overline{N_2}s_1}{(s_1-s_2)(s_1-s_3)} \right]$$

$$B_2 = \left[ \frac{(u_{10}\alpha_{21}+u_{20}\alpha_{11})\overline{N_1}}{(s_1-s_2)} + \frac{u_{20}s_2}{(s_1-s_2)} + \frac{u_{30}\alpha_{11}\alpha_{23}\overline{N_1}\overline{N_2} + u_{30}\alpha_{23}\overline{N_2}s_2}{(s_2-s_1)(s_2-s_3)} \right], C_2 = \left[ \frac{u_{30}\alpha_{11}\alpha_{23}\overline{N_1}\overline{N_2} + u_{30}\alpha_{23}\overline{N_2}s_3}{(s_3-s_1)(s_3-s_2)} \right]$$

$u_{10}, u_{20}$  and  $u_{30}$  are the initial strengths of  $u_1, u_2$  and  $u_3$  respectively

#### 4.2. Stability of the equilibrium state $E_6$ :

The Eigen values of variational matrix A, are  $-(a_1 - qE), -a_3$  &  $a_2 + \frac{\alpha_{21}}{\alpha_{11}}(a_1 - qE) - \frac{\alpha_{23}}{\alpha_{33}}a_3$

the state will be asymptotically stable when  $\left( a_2 + \frac{\alpha_{21}}{\alpha_{11}}(a_1 - qE) \right) < \frac{\alpha_{23}}{\alpha_{33}}a_3$

The solution of the perturbed equations are:

$$u_1 = A_3 e^{\left( a_2 + \frac{\alpha_{21}}{\alpha_{11}}(a_1 - qE) - \frac{\alpha_{23}}{\alpha_{33}}a_3 \right)t} + B_3 e^{-(a_1 - qE)t} \quad (4.2.1)$$

$$u_2 = u_{20} e^{\left( a_2 + \frac{\alpha_{21}}{\alpha_{11}}(a_1 - qE) - \frac{\alpha_{23}}{\alpha_{33}}a_3 \right)t} \quad (4.2.2)$$

$$u_3 = A_4 e^{\left( a_2 + \frac{\alpha_{21}}{\alpha_{11}}(a_1 - qE) - \frac{\alpha_{23}}{\alpha_{33}}a_3 \right)t} + B_4 e^{-a_3 t} \quad (4.2.3)$$

Where

$$A_3 = \frac{-u_{20}(a_1 - qE)\alpha_{33}\alpha_{12}}{a_2\alpha_{11}\alpha_{33} + \alpha_{33}(\alpha_{11} + \alpha_{21})(a_1 - qE) - \alpha_{11}\alpha_{23}a_3},$$

$$B_3 = u_{10} + \frac{u_{20}(a_1 - qE)\alpha_{33}\alpha_{12}}{a_2\alpha_{11}\alpha_{33} + \alpha_{33}(\alpha_{11} + \alpha_{21})(a_1 - qE) - \alpha_{11}\alpha_{23}a_3}$$

$$A_4 = \frac{-u_{20}a_3\alpha_{32}\alpha_{11}}{a_2\alpha_{11}\alpha_{33} + \alpha_{21}\alpha_{33}(a_1 - qE) + (\alpha_{33} - \alpha_{23})\alpha_{11}a_3},$$

$$B_4 = u_{30} + \frac{u_{20}a_3\alpha_{32}\alpha_{11}}{a_2\alpha_{11}\alpha_{33} + \alpha_{21}\alpha_{33}(a_1 - qE) + (\alpha_{33} - \alpha_{23})\alpha_{11}a_3}$$

where  $u_{10}, u_{20}$  and  $u_{30}$  are the initial strengths of  $u_1, u_2$  and  $u_3$  respectively

#### 4.3. Stability of the equilibrium state $E_7$ :

One of the Eigen values of variational matrix A, is  $a_1 - qE - \alpha_{12}\overline{N_2}$  and the other two are obtained from the quadratic equation

$$\lambda^2 + (\alpha_{22}\overline{N_2} + \alpha_{33}\overline{N_3})\lambda + (\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32})\overline{N_2}\overline{N_3} = 0 \quad (4.3.1)$$

In (4.3.1), the sum of the roots,  $-(\alpha_{22}\overline{N_2} + \alpha_{33}\overline{N_3})$ , is negative and the product of the roots,  $(\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32})\overline{N_2}\overline{N_3}$ , is positive. Therefore the roots of (4.3.1) are real and negative or complex conjugates having negative real parts. Thus the state will be asymptotically stable when  $(a_1 - qE) < \alpha_{12}\overline{N_2}$

The solution of the perturbed equations are:

$$u_1 = u_{10}e^{s_1 t} \tag{4.3.2}$$

$$u_2 = A_5 e^{s_1 t} + B_5 e^{s_2 t} + C_5 e^{s_3 t} \tag{4.3.3}$$

$$u_3 = A_6 e^{s_1 t} + B_6 e^{s_2 t} + C_6 e^{s_3 t} \tag{4.3.4}$$

Where  $s_1 = a_1 - qE - \alpha_{12}\overline{N_2}$  and  $s_2, s_3$  are roots of equation (4.3.1)

$$A_5 = \left[ \frac{(u_{20}\alpha_{33}\overline{N_3} - u_{30}\alpha_{23}\overline{N_2})}{(s_1 - s_2)} + \frac{u_{20}s_1}{(s_1 - s_2)} + \frac{u_{10}\alpha_{21}\alpha_{23}\overline{N_3}\overline{N_2} + u_{10}\alpha_{21}\overline{N_2}s_1}{(s_1 - s_2)(s_1 - s_3)} \right]$$

$$A_6 = \left[ \frac{(u_{20}\alpha_{32}\overline{N_3} - u_{30}\alpha_{22}\overline{N_2})}{(s_1 - s_2)} - \frac{u_{30}s_1}{(s_1 - s_2)} + \frac{u_{10}\alpha_{21}\alpha_{32}\overline{N_3}\overline{N_2}}{(s_1 - s_2)(s_1 - s_3)} \right]$$

$$B_5 = \left[ \frac{(u_{30}\alpha_{23}\overline{N_2} - u_{20}\alpha_{33}\overline{N_3})}{(s_1 - s_2)} + \frac{u_{20}s_2}{(s_2 - s_1)} + \frac{u_{10}\alpha_{21}\alpha_{23}\overline{N_3}\overline{N_2} + u_{10}\alpha_{21}\overline{N_2}s_2}{(s_2 - s_1)(s_2 - s_3)} \right]$$

$$B_6 = \left[ \frac{(u_{30}\alpha_{22}\overline{N_2} - u_{20}\alpha_{32}\overline{N_3})}{(s_1 - s_2)} - \frac{u_{30}s_2}{(s_2 - s_1)} + \frac{u_{10}\alpha_{21}\alpha_{32}\overline{N_3}\overline{N_2}}{(s_2 - s_1)(s_2 - s_3)} \right]$$

$$C_5 = \left[ \frac{u_{10}\alpha_{21}\alpha_{23}\overline{N_3}\overline{N_2} + u_{10}\alpha_{21}\overline{N_2}s_3}{(s_3 - s_1)(s_3 - s_2)} \right] \text{ and } C_6 = \left[ \frac{u_{10}\alpha_{21}\alpha_{32}\overline{N_3}\overline{N_2}}{(s_3 - s_1)(s_3 - s_2)} \right]$$

where  $u_{10}, u_{20}$  and  $u_{30}$  are the initial strengths of  $u_1, u_2$  and  $u_3$  respectively

#### 4.4. Stability of the equilibrium state $E_8$ :

In this case the characteristic equation of co-existing state is

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0 \tag{4.4.1}$$

where

$$b_1 = \alpha_{11}\overline{N_1} + \alpha_{22}\overline{N_2} + \alpha_{33}\overline{N_3}, \quad b_2 = \alpha_{11}\alpha_{33}\overline{N_1}\overline{N_3} + (\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32})\overline{N_2}\overline{N_3} + (\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21})\overline{N_1}\overline{N_2} \text{ and}$$

$$b_3 = (\alpha_{12}\alpha_{21}\alpha_{33} + \alpha_{22}\alpha_{11}\alpha_{33} - \alpha_{11}\alpha_{23}\alpha_{32})\overline{N_1}\overline{N_2}\overline{N_3}$$

By Routh-Hurwitz criteria, when

$$D_1 = b_1 = \alpha_{11}\overline{N_1} + \alpha_{22}\overline{N_2} + \alpha_{33}\overline{N_3} > 0, \quad D_2 = b_1b_2 - b_3 > 0 \text{ and } D_3 = b_3(b_1b_2 - b_3) > 0$$

Therefore the roots of (4.4.1) are real and negative or complex conjugates having negative real parts. Thus the state will be asymptotically stable

The solution of the perturbed equations are:

$$u_1 = A_7 e^{s_1 t} + B_7 e^{s_2 t} + C_7 e^{s_3 t} \tag{4.4.2}$$

$$u_2 = A_8 e^{s_1 t} + B_8 e^{s_2 t} + C_8 e^{s_3 t} \tag{4.4.3}$$

$$u_3 = A_9 e^{s_1 t} + B_9 e^{s_2 t} + C_9 e^{s_3 t} \tag{4.4.4}$$

Where  $s_1, s_2$  and  $s_3$  are roots of equation (4.4.1)

$$A_7 = \left[ \frac{u_{10}(s_1 + \alpha_{22}\bar{N}_2) + (s_1 + \alpha_{33}\bar{N}_3) - u_{20}\alpha_{12}\bar{N}_1(s_1 + \alpha_{33}\bar{N}_3) - [u_{30}\alpha_{12}\alpha_{23}\bar{N}_1\bar{N}_2 + u_{10}\alpha_{23}\alpha_{32}\bar{N}_2\bar{N}_3]}{(s_1 - s_2)(s_1 - s_3)} \right]$$

$$B_7 = \left[ \frac{u_{10}(s_2 + \alpha_{22}\bar{N}_2) + (s_2 + \alpha_{33}\bar{N}_3) - u_{20}\alpha_{12}\bar{N}_1(s_2 + \alpha_{33}\bar{N}_3) - [u_{30}\alpha_{12}\alpha_{23}\bar{N}_1\bar{N}_2 + u_{10}\alpha_{23}\alpha_{32}\bar{N}_2\bar{N}_3]}{(s_2 - s_1)(s_2 - s_3)} \right]$$

$$C_7 = \left[ \frac{u_{10}(s_3 + \alpha_{22}\bar{N}_2) + (s_3 + \alpha_{33}\bar{N}_3) - u_{20}\alpha_{12}\bar{N}_1(s_3 + \alpha_{33}\bar{N}_3) - [u_{30}\alpha_{12}\alpha_{23}\bar{N}_1\bar{N}_2 + u_{10}\alpha_{23}\alpha_{32}\bar{N}_2\bar{N}_3]}{(s_3 - s_1)(s_3 - s_2)} \right]$$

$$A_8 = \left[ \frac{u_{20}(s_1 + \alpha_{11}\bar{N}_1) + (s_1 + \alpha_{33}\bar{N}_3) - u_{30}\alpha_{23}\bar{N}_2(s_1 + \alpha_{11}\bar{N}_1) + u_{10}\alpha_{21}\bar{N}_2(s_1 + \alpha_{33}\bar{N}_3)}{(s_1 - s_2)(s_1 - s_3)} \right]$$

$$B_8 = \left[ \frac{u_{20}(s_2 + \alpha_{11}\bar{N}_1) + (s_2 + \alpha_{33}\bar{N}_3) - u_{30}\alpha_{23}\bar{N}_2(s_2 + \alpha_{11}\bar{N}_1) + u_{10}\alpha_{21}\bar{N}_2(s_2 + \alpha_{33}\bar{N}_3)}{(s_2 - s_1)(s_2 - s_3)} \right]$$

$$C_8 = \left[ \frac{u_{20}(s_3 + \alpha_{11}\bar{N}_1) + (s_3 + \alpha_{33}\bar{N}_3) - u_{30}\alpha_{23}\bar{N}_2(s_3 + \alpha_{11}\bar{N}_1) + u_{10}\alpha_{21}\bar{N}_2(s_3 + \alpha_{33}\bar{N}_3)}{(s_3 - s_1)(s_3 - s_2)} \right]$$

$$A_9 = \left[ \frac{u_{30}(s_1 + \alpha_{11}\bar{N}_1) + (s_1 + \alpha_{22}\bar{N}_2) - u_{20}\alpha_{32}\bar{N}_3(s_1 + \alpha_{11}\bar{N}_1) + u_{30}\alpha_{12}\alpha_{21}\bar{N}_1\bar{N}_2 - u_{10}\alpha_{21}\alpha_{32}\bar{N}_2\bar{N}_3}{(s_1 - s_2)(s_1 - s_3)} \right]$$

$$B_9 = \left[ \frac{u_{30}(s_2 + \alpha_{11}\bar{N}_1) + (s_2 + \alpha_{22}\bar{N}_2) - u_{20}\alpha_{32}\bar{N}_3(s_2 + \alpha_{11}\bar{N}_1) + u_{30}\alpha_{12}\alpha_{21}\bar{N}_1\bar{N}_2 - u_{10}\alpha_{21}\alpha_{32}\bar{N}_2\bar{N}_3}{(s_2 - s_1)(s_2 - s_3)} \right]$$

$$C_9 = \left[ \frac{u_{30}(s_3 + \alpha_{11}\bar{N}_1) + (s_3 + \alpha_{22}\bar{N}_2) - u_{20}\alpha_{32}\bar{N}_3(s_3 + \alpha_{11}\bar{N}_1) + u_{30}\alpha_{12}\alpha_{21}\bar{N}_1\bar{N}_2 - u_{10}\alpha_{21}\alpha_{32}\bar{N}_2\bar{N}_3}{(s_3 - s_1)(s_3 - s_2)} \right]$$

Where  $u_{10}$ ,  $u_{20}$  and  $u_{30}$  are the initial strengths of  $u_1$ ,  $u_2$  and  $u_3$  respectively

## 5. Global Stability

### 5.1. Theorem

Let the Lapnouv function for the case  $E_5$  is:

$$V(\bar{N}_1, \bar{N}_2) = \left\{ N_1 - \bar{N}_1 - \bar{N}_1 \ln \left[ \frac{N_1}{\bar{N}_1} \right] \right\} + \left\{ N_2 - \bar{N}_2 - \bar{N}_2 \ln \left[ \frac{N_2}{\bar{N}_2} \right] \right\} \quad (5.1.1)$$

Differentiating  $V$  w.r.to 't' we get

$$\frac{dV}{dt} = \left( \frac{N_1 - \bar{N}_1}{N_1} \right) \frac{dN_1}{dt} + \left( \frac{N_2 - \bar{N}_2}{N_2} \right) \frac{dN_2}{dt} \quad (5.1.2)$$

$$\frac{dV}{dt} = - \left( \alpha_{11} + \frac{1}{2} [\alpha_{12} - \alpha_{21}] \right) [N_1 - \bar{N}_1]^2 - \left( \alpha_{22} + \frac{1}{2} [\alpha_{12} - \alpha_{21}] \right) [N_2 - \bar{N}_2]^2 \quad (5.1.3)$$

$$\frac{dV}{dt} < 0$$

Therefore,  $E_5(\bar{N}_1, \bar{N}_2)$  is globally asymptotically stable

**5.2. Theorem**

Let the Lapnouv function for the case E<sub>6</sub> is:

$$V(\bar{N}_1, \bar{N}_3) = (N_1 - \bar{N}_1) - \bar{N}_1 \ln \left[ \frac{N_1}{\bar{N}_1} \right] + (N_3 - \bar{N}_3) - \bar{N}_3 \ln \left[ \frac{N_3}{\bar{N}_3} \right] \tag{5.2.1}$$

Differentiating V w.r.to ‘t’ we get

$$\frac{dV}{dt} = \left( \frac{N_1 - \bar{N}_1}{N_1} \right) \frac{dN_1}{dt} + \left( \frac{N_3 - \bar{N}_3}{N_3} \right) \frac{dN_3}{dt} \tag{5.2.2}$$

$$\frac{dV}{dt} = -\alpha_{11} [N_1 - \bar{N}_1]^2 - \alpha_{33} [N_3 - \bar{N}_3]^2 \tag{5.2.3}$$

$$\frac{dV}{dt} < 0$$

Therefore , E<sub>6</sub>( $\bar{N}_1, \bar{N}_3$ ) is globally asymptotically stable

**5.3. Theorem**

Let the Lapnouv function for the case E<sub>7</sub> is:

$$V(\bar{N}_2, \bar{N}_3) = (N_2 - \bar{N}_2) - \bar{N}_2 \ln \left[ \frac{N_2}{\bar{N}_2} \right] + (N_3 - \bar{N}_3) - \bar{N}_3 \ln \left[ \frac{N_3}{\bar{N}_3} \right] \tag{5.3.1}$$

Differentiating V w.r.to ‘t’ we get

$$\frac{dV}{dt} = \left( \frac{N_2 - \bar{N}_2}{N_2} \right) \frac{dN_2}{dt} + \left( \frac{N_3 - \bar{N}_3}{N_3} \right) \frac{dN_3}{dt} \tag{5.3.2}$$

$$\frac{dV}{dt} < -\left( \alpha_{22} + \frac{1}{2} [\alpha_{23} + \alpha_{32}] \right) [N_2 - \bar{N}_2]^2 - \left( \alpha_{33} + \frac{1}{2} [\alpha_{23} + \alpha_{32}] \right) [N_3 - \bar{N}_3]^2 \tag{5.3.3}$$

$$\frac{dV}{dt} < 0$$

Therefore , E<sub>7</sub>( $\bar{N}_2, \bar{N}_3$ ) is globally asymptotically stable

**5.4. Theorem**

The Equilibrium point E<sub>8</sub>( $\bar{N}_1, \bar{N}_2, \bar{N}_3$ ) is globally asymptotically stable.

Proof: Let us consider the following Lyapunov function

$$V(\bar{N}_1, \bar{N}_2, \bar{N}_3) = \left\{ N_1 - \bar{N}_1 - \bar{N}_1 \ln \left[ \frac{N_1}{\bar{N}_1} \right] \right\} + \left\{ N_2 - \bar{N}_2 - \bar{N}_2 \ln \left[ \frac{N_2}{\bar{N}_2} \right] \right\} + \left\{ N_3 - \bar{N}_3 - \bar{N}_3 \ln \left[ \frac{N_3}{\bar{N}_3} \right] \right\} \tag{5.4.1}$$

Differentiating V w.r.to ‘t’ we get

$$\frac{dV}{dt} = \left( \frac{N_1 - \bar{N}_1}{N_1} \right) \frac{dN_1}{dt} + \left( \frac{N_2 - \bar{N}_2}{N_2} \right) \frac{dN_2}{dt} + \left( \frac{N_3 - \bar{N}_3}{N_3} \right) \frac{dN_3}{dt} \tag{5.4.2}$$

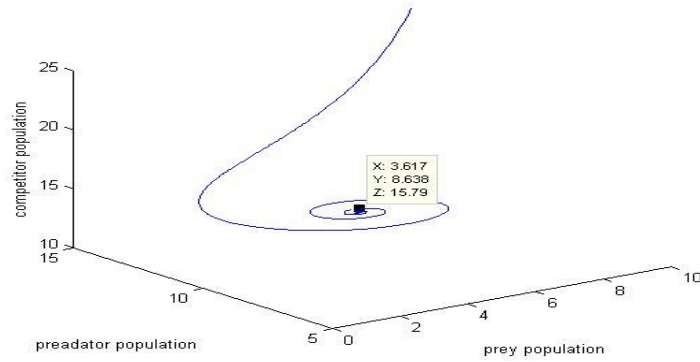
$$\frac{dV}{dt} < -\alpha_{11} [N_1 - \bar{N}_1]^2 - \left( \alpha_{22} + \frac{1}{2} [\alpha_{12} - \alpha_{21} + \alpha_{23} + \alpha_{32}] \right) [N_2 - \bar{N}_2]^2 - \left( \alpha_{33} + \frac{1}{2} (\alpha_{32} + \alpha_{23}) \right) [N_3 - \bar{N}_3]^2 \tag{5.4.3}$$

$$\frac{dV}{dt} < 0$$

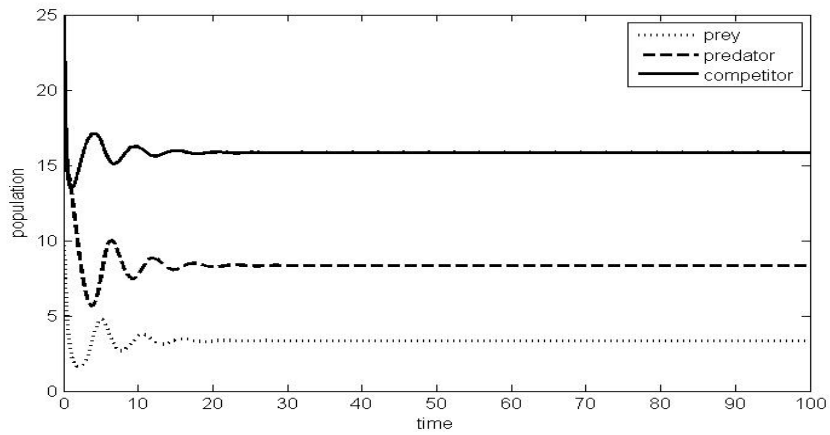
Therefore , E<sub>8</sub>( $\bar{N}_1, \bar{N}_2, \bar{N}_3$ ) is globally asymptotically stable.

**6. Numerical example:**

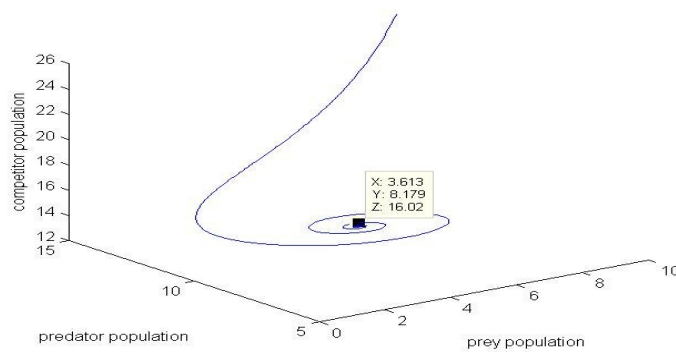
1. Let  $a_1=2$ ;  $a_2=3$ ;  $a_3=4$ ;  $\alpha_{11}=0.1$ ;  $\alpha_{12}=0.2$ ;  $\alpha_{22}=0.1$ ;  $\alpha_{21}=0.3$ ;  $\alpha_{23}=0.2$ ;  $\alpha_{33}=0.2$ ;  $\alpha_{32}=0.1$



**Fig 6.1.1: The Phage diagram of N1, N2, N3 for system of Eq (2.1)  $qE=0$**

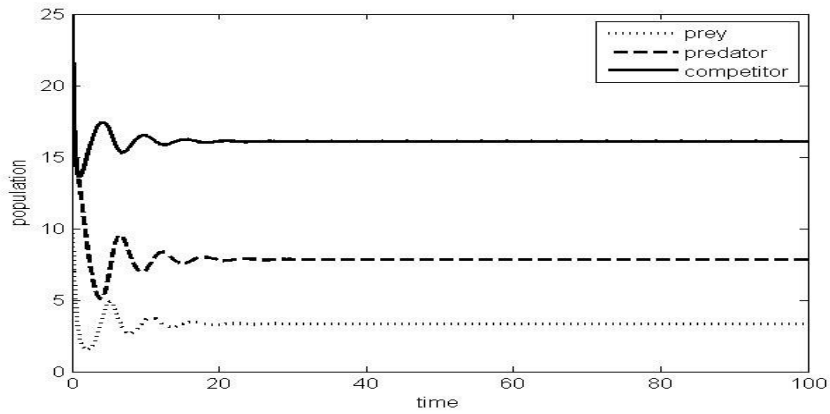


**Fig 6.1.2: The Variation of N1, N2 & N3 with respective Time (t) for system of Eq (2.1) For  $qE = 0$**

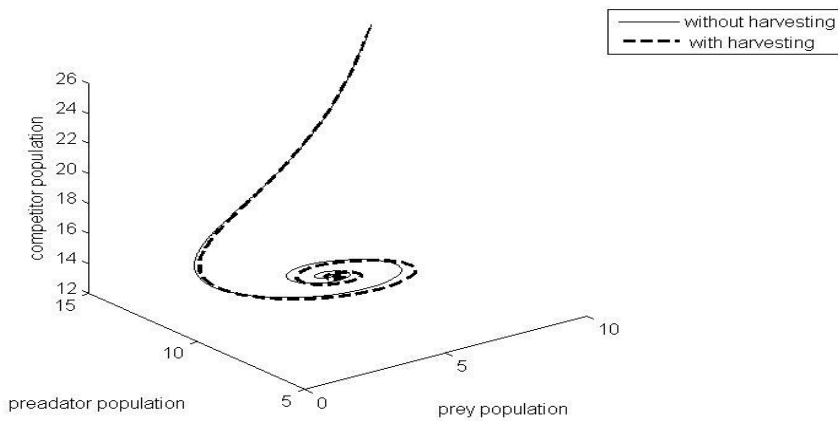


**Fig 6.1.3: The Phage diagram of N1, N2, N3 for system of Eq (2.1)**





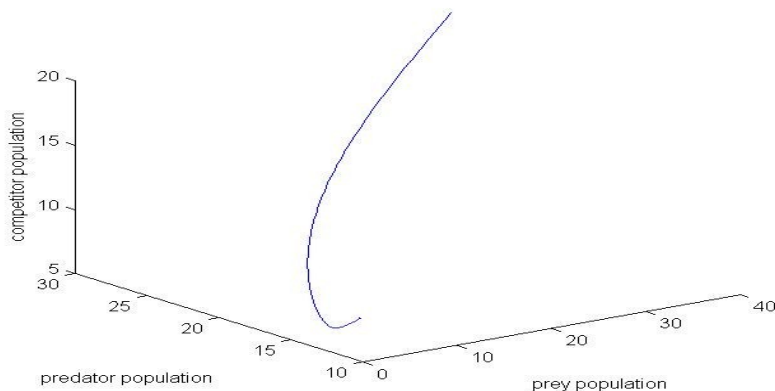
**Fig 6.1.4: The Variation of N1, N2 & N3 with respective Time (t) for system of Eq( 2.1)**



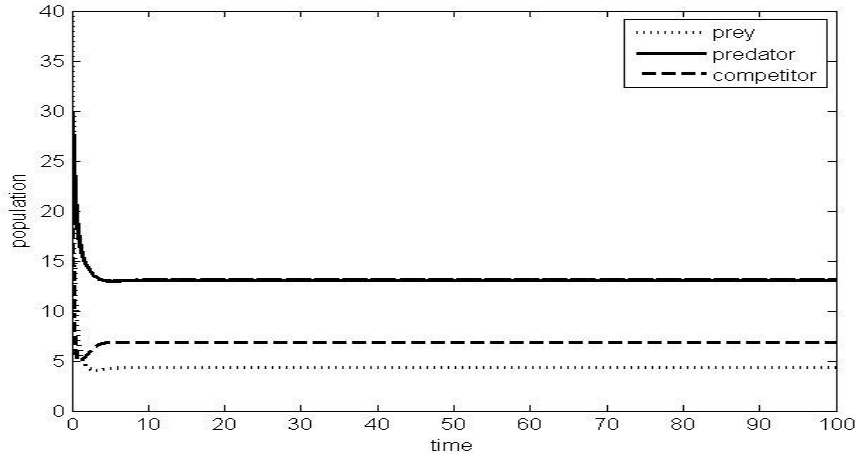
**Fig 6.1.5: Comparative phage diagram for Eq (2.1) with  $qE \neq 0$  and  $qE = 0$**

The above graph shows the variation with initial strengths 10, 15, 25 of prey, predator and competitor populations respectively

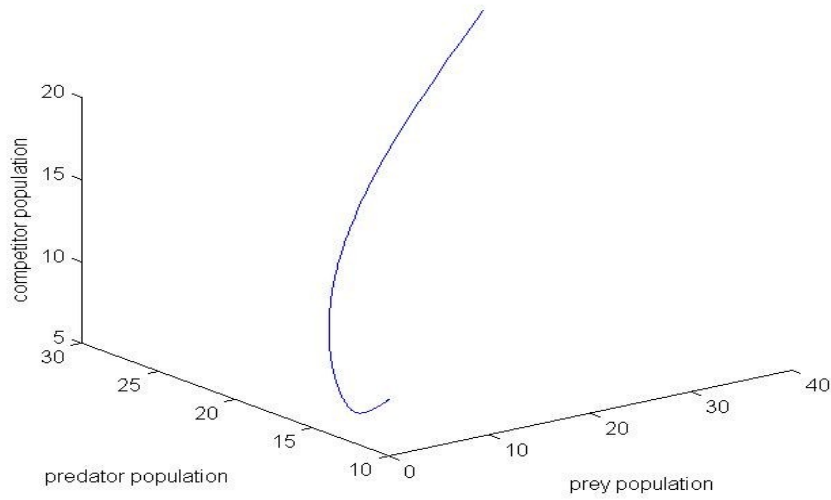
2. Let  $a_1=2$ ;  $a_2=3$ ;  $a_3=4$ ;  $\alpha_{11}=0.1$ ;  $\alpha_{12}=0.12$ ;  $\alpha_{22}=0.2$ ;  $\alpha_{21}=0.13$ ;  $\alpha_{23}=0.14$ ;  $\alpha_{33}=0.3$ ;  $\alpha_{32}=0.15$



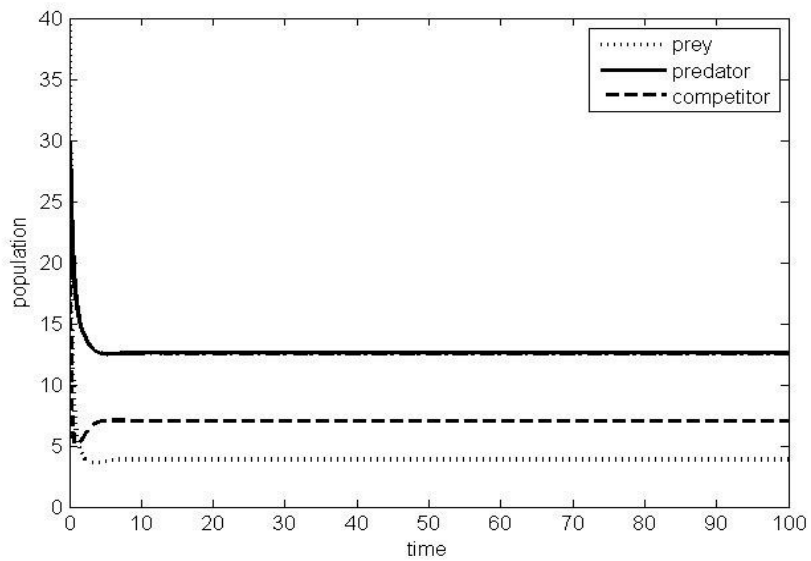
**Fig 6.2.1: The Phage diagram of N1, N2, N3 for system of Eq (2.1) for  $qE = 0$**



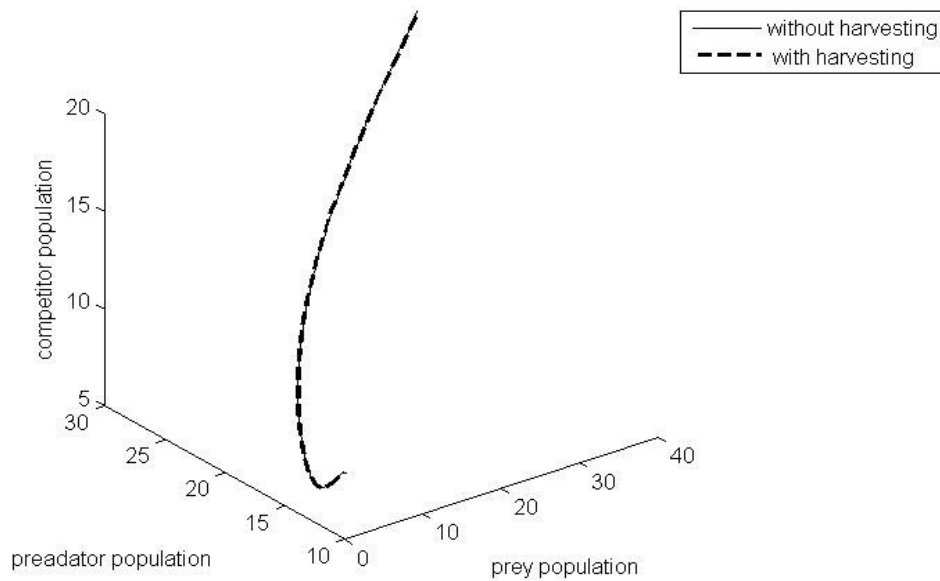
**Fig 6.2.2: The Variation of  $N_1, N_2$  &  $N_3$  with respective Time ( $t$ ) for system of Eq (2.1) For  $qE = 0$**



**Fig 6.2.3: The Phage diagram of  $N_1, N_2, N_3$  for system of Eq (2.1)**



**Fig 6.2.4: The Variation of  $N_1, N_2$  &  $N_3$  with respective Time ( $t$ ) for system of Eq (2.1)**



**Fig 6.2.5: Comparative phase diagram for Eq (2.1) with  $qE \neq 0$  and  $qE = 0$**

The above graph shows the variation with initial strengths 40, 30, 20 of prey, predator and competitor populations respectively

## 7. Conclusion:

In the analysis of the considered prey, predator and a competitor to the predator and optimal harvesting of the prey model, we discussed the local, global stability of the model and exact solutions of perturbed equations have been derived for stable cases. Two set of Numerical examples are studied for which first example with complex roots and second example with real roots. And also study the stability of the system (2.1) with harvesting ( $qE \neq 0$ ) and without harvesting ( $qE = 0$ ). From the graphs shown fig 6.1.5 and 6.2.5 it is evident that the harvesting of the prey does not have any influence on the stability.

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