

## ON CONFORMAL MAPPING IN THE QUASI- EINSTEIN MANIFOLDS

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**Abstract:** In this paper we have shown the existence of conformal mapping from quasi-Einstein Manifold to Einstein manifold.

### 1. Introduction

A rule of correspondence between two Riemannian manifolds  $(M^n, g)$  and  $(\bar{M}^n, \bar{g})$  is said to be conformal [5] if the metric tensors  $g$  and  $\bar{g}$  are connected by the following relation:

$$\bar{g}(X, Y) = e^{2\sigma} g(X, Y), \text{ where } \sigma \text{ is a scalar.} \quad (1.1)$$

The geometric significance of the conformal mapping are as follows: if the relation (1.1) holds then the lengths of the vectors at the same points and with the same components in the manifolds  $M^n$  and  $\bar{M}^n$  are differed by a factor depending only on the particular points and also the angles between two corresponding directions at corresponding points of that manifolds are equal.

A Riemannian manifold  $(M^n, g)$  is said to be Einstein manifold [1] if its Ricci tensor  $S$  satisfies the following relation:  $S(X, Y) = \kappa g(X, Y)$ , where  $\kappa$  is a constant, which reduces to

$$S(X, Y) = (r/n) g(X, Y), \text{ where } r \text{ being the scalar curvature.} \quad (1.2)$$

A Riemannian manifold  $(M^n, g)$  is said to be quasi-Einstein manifold ([2], [3], [4]) if its Ricci tensor  $S$  is not identically zero and satisfies the following relation:

$$S(X, Y) = \alpha g(X, Y) + \beta L(X)(L(Y)), \text{ where } \alpha, \beta (\neq 0) \quad (1.3)$$

are the scalars and  $L$  be the non-zero 1- form, which is defined by  $L(X) = g(X, \rho)$ ;  $\rho$  being the associate unit vector. Such an  $n$ - dimensional manifold is denoted by  $(QE)_n$ .

Now we shall discuss about two special type of operators. Let  $\{e_i, i = 1, 2, \dots, n\}$  be an orthonormal basis of the tangent space at any point of a manifold  $(M^n, g)$  and  $\phi$  be a scalar then Beltrami operator  $\Delta_1$  of first kind is defined by  $\Delta_1 \phi = \sum_{i=1}^n d\phi(e_i)d\phi(e_i)$  and the Beltrami operator  $\Delta_2$  of second kind is defined by  $\Delta_2 \phi = \sum_{i=1}^n (\Delta e_i d\phi)(e_i)$ .

In section 2 we study about the conformal mapping of Riemannian manifolds and some basic relations between two manifolds due to conformal mapping. The section 3 is devoted to the study of the conformal mapping of an Einstein manifold on a quasi-Einstein manifold. The section 4 is concerned about the conformal mapping of a quasi-Einstein manifold on an Einstein manifold and the existence of such a mapping is ensured by a non-trivial example. In the last section we discuss about the conformal mapping of a quasi-Einstein manifold on a quasi-Einstein manifold.

### 2. Conformal Mapping of Riemannian Manifolds

In this section we deal with the conformal mapping of Riemannian manifolds.

Using the relation (1.1), we can easily get the corresponding relations between the curvature tensor, Ricci tensor and scalar curvature as follows:

$$R(\bar{X}, Y, U, V) = e^{2\sigma} [R(X, Y, U, V) + g(X, V)\omega(Y, U) + g(Y, U)\omega(X, V) - g(X, U)\omega(Y, V) - g(Y, V)\omega(X, U) + \Delta_1\sigma \{g(X, V)g(Y, U) - g(X, U)g(Y, V)\}] \tag{2.1}$$

$$\bar{S}(X, Y) = S(X, Y) + (n - 2)\omega(X, Y) + [\Delta_2\sigma + (n - 2)\Delta_1\sigma]g(X, Y) \tag{2.2}$$

and

$$\bar{r} = e^{-2\sigma} [r + 2(n - 1)\Delta_2\sigma + (n - 1)(n - 2)\Delta_1\sigma], \tag{2.3}$$

where  $\omega(X, Y) = (\nabla_Y d\sigma)(X) + d\sigma(X)d\sigma(Y)$ ;  $R, S, r$  denote the curvature tensor, Ricci tensor, scalar curvature of the manifold  $M^n$  respectively and  $\bar{R}, \bar{S}, \bar{r}$  denote the curvature tensor, Ricci tensor, scalar curvature of the manifold  $M^n$  respectively.

### 3. Conformal Mapping of Einstein Manifolds on Quasi-Einstein Manifolds

This section is concerned about the conformal mapping of an Einstein manifold on a quasi-Einstein manifold. Let us choose a Riemannian manifold  $(M^n, g)$  as an Einstein manifold *i.e.*, its Ricci tensor  $S$  satisfies the following relation:

$$S(X, Y) = (r/n)g(X, Y), \text{ where } r \text{ being the scalar curvature.} \tag{3.1}$$

Now we construct a conformal transformation from the Einstein manifold  $(M^n, g)$  to a Riemannian manifold  $(\bar{M}^n, \bar{g})$ . By virtue of the equations (1.1), (2.2), (2.3) and (3.1) we have the Ricci tensor  $\bar{S}$  and the scalar curvature  $\bar{r}$  of  $\bar{M}^n$  as follows:

$$\bar{S}(X, Y) = \left[ \frac{r}{n} + \Delta_2\sigma + (n - 2)\Delta_1\sigma \right] g(X, Y) + (n - 2)\omega(X, Y) \tag{3.2}$$

and

$$\bar{r} = e^{-2\sigma} [r + 2(n - 1)\Delta_2\sigma + (n - 1)(n - 2)\Delta_1\sigma]. \tag{3.3}$$

If the manifold  $(\bar{M}^n, \bar{g})$  be a quasi-Einstein manifold then in view of (1.3) it follows that the Ricci tensor  $\bar{S}$  of  $\bar{M}^n$  is not identically zero and satisfies the following relation:

$$\bar{S}(X, Y) = \bar{\alpha} \bar{g}(X, Y) + \bar{\beta} \bar{L}(X)\bar{L}(Y). \tag{3.4}$$

From the above equation we have

$$\bar{r} = n\bar{\alpha} + \bar{\beta}. \tag{3.5}$$

From the equations (3.2) and (3.4) we have

$$e^{2\sigma} \bar{\alpha} g(X, Y) + e^{4\sigma} \bar{\beta} L(X)L(Y) = \left[ \frac{r}{n} + \Delta_2\sigma + (n - 2)\Delta_1\sigma \right] g(X, Y) + (n - 2)\omega(X, Y) \tag{3.6}$$

where

$L = e^{-2\sigma} \bar{L}$ . Taking contraction of (3.6) with respect to X, Y we get

$$(n - 1)[2\Delta_2\sigma + (n - 2)\Delta_1\sigma] = ne^{2\sigma} \bar{\alpha} + e^{4\sigma} \bar{\beta} - r. \tag{3.7}$$

In the view of (3.5) and (3.7), the relation (3.3) yields

$$(e^{2\sigma} - 1)\bar{\beta} = 0.$$

Since  $\bar{\beta} \neq 0$ , so the above relation implies

$$e^{2\sigma} - 1 = 0 \text{ i.e., } \sigma = 0,$$

which is not possible. Therefore it is clear that there does not exist any conformal mapping between the manifolds  $(M^n, g)$  and  $(\bar{M}^n, \bar{g})$ . So we have the following theorem:

*Theorem 3.1.* An Einstein manifold can not be mapped conformally on to a quasi-Einstein manifold.

#### 4. Conformal Mapping of Quasi-Einstein Manifolds on Einstein Manifolds

In this section we study about the conformal mapping of a quasi-Einstein manifold on an Einstein manifold. Let us consider a Riemannian manifold  $(M^n, g)$  as a quasi-Einstein manifold. So by virtue of (1.3) we have the following relation:

$$S(X, Y) = \alpha g(X, Y) + \beta L(X)L(Y) \quad (4.1)$$

Contraction of the above equation yields

$$(4.2) \quad r = n\alpha + \beta.$$

Now we construct a conformal transformation from the quasi-Einstein manifold  $(M^n, g)$  to a Riemannian manifold  $(\bar{M}^n, \bar{g})$ . In view of the equations (1.1), (2.2) and (4.1) the Ricci tensor  $\bar{S}$  of  $\bar{M}^n$  takes the following form:

$$S(X, Y) = [\alpha + \Delta_2\sigma + (n-2)\Delta_1\sigma]g(X, Y) + \beta L(X)L(Y) + (n-2)\omega(X, Y). \quad (4.3)$$

Contracting the last relation and in view of the equations (1.1), (2.3) and (4.2) we have the scalar curvature  $\bar{r}$  of  $\bar{M}^n$  as follows

$$(4.4) \quad \bar{r} = e^{-2\sigma} [n\alpha + \beta + 2(n-1)\Delta_2\sigma + (n-1)(n-2)\Delta_1\sigma].$$

Let us consider that the manifold  $(\bar{M}^n, \bar{g})$  be an Einstein manifold i.e., its Ricci tensor  $\bar{S}$  satisfies the following relation:

$$\bar{S}(X, Y) = (\bar{r}/n)\bar{g}(X, Y). \quad (4.5)$$

By virtue of the equations (4.3), (4.4) and (4.5) we get

$$n\beta L(X)L(Y) + n(n-2)\omega(X, Y) = [\beta + (n-2)\{\Delta_2\sigma - \Delta_1\sigma\}]g(X, Y). \quad (4.6)$$

So it is clear that the manifold  $(M^n, g)$  maps conformally to  $(\bar{M}^n, \bar{g})$  with the conformal mapping (1.1), provided the relation (4.6) holds.

Again let  $\sigma$  satisfies the relation (4.6). Then using the equations (4.6), (4.4) and (4.3) we can easily prove that the manifold  $(\bar{M}^n, \bar{g})$  is an Einstein manifold. Hence we can state the following result:

*Theorem 4.1.* A quasi-Einstein manifold  $(M^n, g)$  can be mapped conformally on to an Einstein manifold  $(\bar{M}^n, \bar{g})$  with the conformal mapping defined by (1.1) if and only if the scalar  $\sigma$  satisfies the relation (4.6).

Now we shall verify the above theorem by a suitable example:

**Example:** Let  $M^n = \{(x^1, x^2, \dots, x^n) \in R^n\}$  be an open subset of  $R^n$  endowed with the metric

$$ds^2 = g_{ij} dx^i dx^j = \phi(dx^1)^2 + \psi \sum_{p=2}^n (dx^p)^2 \tag{4.7}$$

where  $\phi$  and  $\psi$ , being the functions of  $x^1$  only, are connected by the relation

$$\frac{n-2}{2\psi} \left[ \psi_{11} - \frac{\phi_1 \psi_1}{2\phi} - \frac{\psi_1^2}{\psi} \right] + \frac{\phi_1 \psi + \phi \psi_1}{x^1 + k} = 0, \tag{4.8}$$

and  $k$  is an arbitrary constant and the subscript denotes the partial derivative with respect to  $x^1$ . Then the only non-vanishing components of the curvature tensor, the Ricci tensor and the scalar curvature are given by

$$\begin{cases} R_{i1i1} = \frac{1}{2} \left[ \psi_{11} - \frac{\phi_1 \psi_1}{2\phi} - \frac{\psi_1^2}{2\psi} \right], & R_{iji} = \frac{\psi_1^2}{4\phi}, & S_{11} = \frac{n-1}{2\psi} \left[ \psi_{11} - \frac{\phi_1 \psi_1}{2\phi} - \frac{\psi_1^2}{2\psi} \right], \\ S_{ii} = \frac{1}{2\phi} \left[ \psi_{11} - \frac{\phi_1 \psi_1}{2\phi} + \frac{(n-3)\psi_1^2}{2\psi} \right], & r = \frac{n-1}{\phi\psi} \left[ \psi_{11} - \frac{\phi_1 \psi_1}{2\phi} + \frac{(n-4)\psi_1^2}{4\psi} \right], \end{cases} \tag{4.9}$$

where  $i, j$  run from 2 to  $n$  and  $i \neq j$ . Now we shall verify that our considered manifold  $M^n$  is a  $(QE)_n$ . For this verification let us consider the 1-form  $L$  and associated scalars  $\alpha, \beta$  as follows:

$$L\left(\frac{\partial}{\partial x^i}\right) = L_i = \begin{cases} \sqrt{\phi}, & i = 1 \\ 0, & i \neq 1 \end{cases}, \tag{4.10}$$

$$\alpha = \frac{1}{2\phi\psi} \left[ \psi_{11} - \frac{\phi_1 \psi_1}{2\phi} + \frac{(n-3)\psi_1^2}{2\psi} \right], \quad \beta = \frac{n-2}{2\phi\psi} \left[ \psi_{11} - \frac{\phi_1 \psi_1}{2\phi} - \frac{\psi_1^2}{\psi} \right] \tag{4.11}$$

According to our  $M^n$  the equation (4.1) reduces to following equations

$$S_{ii} = g_{ii} + L_i L_i, \quad i = 1, 2, \dots, n, \tag{4.12}$$

since for the cases other than (4.12), the components of (4.1) vanishes identically and the relation

(4.1) holds trivially. By virtue of (4.7), (4.9), (4.10) and (4.11), it follows that:

$$\begin{aligned} &= \frac{1}{2\phi\psi} \left[ \psi_{11} - \frac{\phi_1 \psi_1}{2\phi} + \frac{(n-3)\psi_1^2}{2\psi} \right] g_{11} + \frac{n-2}{2\phi\psi} \left[ \psi_{11} - \frac{\phi_1 \psi_1}{2\phi} - \frac{\psi_1^2}{\psi} \right] L_1 L_1 \\ &= \frac{n-1}{2\psi} \left[ \psi_{11} - \frac{\phi_1 \psi_1}{2\phi} - \frac{\psi_1^2}{2\psi} \right] \\ &= \text{left hand side of (4.12), for } i = 1. \end{aligned} \tag{4.12}$$

By similar argument it can be easily shown that the relations in (4.12) hold for  $i=2, \dots, n$ . Therefore our  $(M^n, g)$  is a  $(QE)_n$ .

Let us choose a conformal mapping from  $(M^n, g)$  to a Riemannian manifold  $(\bar{M}, \bar{g})$  as follows

$$\bar{g} = e^{2\sigma} g \text{ and } \sigma = -\log(x^1 + k). \quad (4.13)$$

Therefore the non-vanishing components of  $\omega$ , which is locally defined by  $\omega_{ij} = \sigma_{,ij} - \sigma_{,i}\sigma_{,j}$ , are

$$\omega_{11} = \frac{\phi_1}{2(x^1 + k)\phi} \text{ and } \omega_{pp} = \frac{\psi_1}{2(x^1 + k)\phi}, \quad (p = 2, 3, 4) \quad (4.14)$$

and the values of  $\sigma$  with respect to the Beltrami operators are

$$\Delta_1 \sigma = \frac{1}{(x^1 + k)^2 \phi}, \quad \Delta_2 \sigma = \frac{1}{2(x^1 + k)\phi} \left[ \frac{\phi_1}{\phi} - (n-1) \frac{\psi_1}{\psi} + \frac{2}{x^1 + k} \right] \quad (4.15)$$

We claim that the relation (4.6) is valid in our manifold  $(QE)_n$ . With respect to our  $(QE)_n$  the equation (4.6) reduces to the following equations:

$$n\beta L_i L_i + n(n-2)\omega_{ii} = [\beta + (n-2)(\Delta_2 \sigma - \Delta_1 \sigma)] g_{ii}, \text{ for } i = 1, 2, \dots, n, \quad (4.16)$$

since for the cases other than (4.16), the components of (4.6) vanishes identically and the relation (4.6) holds trivially. We only check the validity of the equation (4.16) for the case  $i = 1$ , because the other cases can be checked easily in a similar manner. By virtue of (4.7), (4.8), (4.10), (4.11), (4.14) and (4.15), it follows that

$$\begin{aligned} \text{left hand side of (4.16)} &= \frac{n-2}{2\phi\psi} \left[ \psi_{11} - \frac{\phi_1\psi_1}{2\phi} - \frac{\psi_1^2}{\psi} \right] L_1 L_1 + (n-2)\omega_{11} = -\frac{(n-2)\psi_1}{2(x^1 + k)\psi} \\ &= -\frac{(n-2)\psi_1}{2(x^1 + k)\phi\psi} g_{11} = \text{right hand side of (4.16) for } i = 1. \end{aligned}$$

According to the conformal mapping (4.13) and relations (4.8), (4.3), (4.4); the non-vanishing components of Ricci tensor and the scalar curvature of the manifold  $(\bar{M}^n, \bar{g})$  are

$$\begin{cases} \bar{S}_{11} = \frac{1}{2\psi} \left[ \psi_{11} - \frac{\phi_1\psi_1}{2\phi} + \frac{(n-3)\psi_1^2}{2\psi} \right] + \frac{1}{2(x^1 + k)} \left[ \frac{\phi_1}{\phi} - \frac{(2n-3)\psi_1}{\psi} + \frac{2(n-1)}{x^1 + k} \right], \\ \bar{S}_{ii} = \frac{1}{2\phi} \left[ \psi_{11} - \frac{\phi_1\psi_1}{2\phi} + \frac{(n-3)\psi_1^2}{2\psi} \right] + \frac{\psi}{2(x^1 + k)\phi} \left[ \frac{\phi_1}{\phi} - \frac{(2n-3)\psi_1}{\psi} + \frac{2(n-1)}{x^1 + k} \right] \end{cases} \quad (4.17)$$

$$\bar{r} = \frac{n(x^1 + k)^2}{2\phi\psi} \left[ \psi_{11} - \frac{\phi_1\psi_1}{2\phi} + \frac{(n-3)\psi_1^2}{2\psi} \right] + \frac{n(x^1 + k)}{2\phi} \left[ \frac{\phi_1}{\phi} - \frac{(2n-3)\psi_1}{\psi} + \frac{2(n-1)}{x^1 + k} \right], \quad (4.18)$$

Where  $i=2$

$\dots, n$ . In view of (4.13), (4.17) and (4.18) we can easily prove that the relation (4.5) holds in the manifold  $(\bar{M}^n, \bar{g})$ . So by the definition of Einstein manifold it can be said that the manifold  $(\bar{M}^n, \bar{g})$  is an Einstein manifold endowed with the metric

$$ds^2 = \bar{g}_{ij} dx^i dx^j = (x^1 + k)^2 [\phi(dx^1)^2 + \psi \sum_{p=2}^n (dx^p)^2] \text{ where } i, j \text{ run from } 1 \text{ to } n.$$

Thus we can state the following:

**Theorem 4.2.** The manifold  $(M^n, g)$  endowed with the metric given by (4.7), is a quasi-Einstein manifold and under the conformal mapping (4.13), it maps conformally to  $(\bar{M}^n, \bar{g})$ , which is an Einstein manifold endowed with the metric

$$ds^2 = \bar{g}_{ij} dx^i dx^j = (x^1 + k)^2 \left[ \phi(dx^1)^2 + \psi \sum_{p=2}^n (dx^p)^2 \right]$$

where  $i, j$  run from 1 to  $n$ . Hence the theorem (4.1) is verified.

### 5. Conformal Mapping of Quasi-Einstein Manifolds on Quasi-Einstein Manifolds

This section is concerned with the conformal mapping of quasi-Einstein manifold on quasi-Einstein manifold. Let us take the manifold  $(M^n, g)$  as a quasi-Einstein manifold. Therefore in view of (1.3) we get the following relation:

$$S(X, Y) = \alpha g(X, Y) + \beta L(X)L(Y). \tag{5.1}$$

Contracting the relation (5.1) with respect to  $X$  and  $Y$  we get

$$r = n\alpha + \beta. \tag{5.2}$$

Now we construct a conformal transformation from the quasi-Einstein manifold  $(M^n, g)$  to a

Riemannian manifold  $(\bar{M}^n, \bar{g})$ . By virtue of the equations (1.1), (2.2), (2.3) and (5.1) we have the Ricci tensor  $\bar{S}$  and the scalar curvature  $\bar{r}$  of  $\bar{M}^n$  as follows:

$$\bar{S}(X, Y) = [\alpha + \Delta_2 \sigma + (n - 2)\Delta_1 \sigma] g(X, Y) + \beta L(X)L(Y) + (n - 2)\omega(X, Y) \tag{5.3}$$

and

$$\bar{r} = e^{-2\sigma} [n\alpha + \beta + 2(n - 1)\Delta_2 \sigma + (n - 1)(n - 2)\Delta_1 \sigma]. \tag{5.4}$$

Let us consider that the manifold  $(\bar{M}^n, \bar{g})$  is an quasi-Einstein manifold. Then in view of (1.3) we have the following relation:

$$\bar{S}(X, Y) = \bar{\alpha} \bar{g}(X, Y) + \bar{\beta} L(X)L(Y), \tag{5.5}$$

where  $\bar{L} = e^{2\sigma} L$ . The above equation implies

$$\bar{r} = n\bar{\alpha} + \bar{\beta}. \tag{5.6}$$

Again in view of the equation (5.3), the relation (5.5) reduces to

$$[\alpha + \Delta_2 \sigma + (n - 2)\Delta_1 \sigma - e^{2\sigma} \bar{\alpha}] g(X, Y) + [\beta - e^{4\sigma} \bar{\beta}] L(X)L(Y) + (n - 2)\omega(X, Y) = 0. \tag{5.7}$$

Setting  $X = Y = e_i$  in (5.7) and taking summation over  $i$ ,  $1 \leq i \leq n$ , it follows that

$$(n - 1)[2\Delta_2 \sigma + (n - 2)\Delta_1 \sigma] = n(e^{2\sigma} \bar{\alpha} - \alpha) + e^{4\sigma} \bar{\beta} - \beta. \tag{5.8}$$

Using (5.6), (5.8) in (5.4) we have

$$(e^{2\sigma} - 1)\bar{\beta} = 0.$$

Since  $\bar{\beta} \neq 0$ , so from the above relation we obtain

$$e^{2\sigma} - 1 = 0 \text{ i.e., } \sigma = 0,$$

which is not possible. Hence it is clear that there does not exist any conformal mapping between the manifolds  $(M^n, g)$  and  $(\bar{M}^n, \bar{g})$ . This leads to the following theorem:

*Theorem 5.1.* A quasi-Einstein manifold can not be mapped conformally to a quasi-Einstein manifold.

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