Strongly Connectedness in Closure Space

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Abstract: A Čech closure space \((X, u)\) is a set \(X\) with Čech closure operator \(u: P(X) \to P(X)\) where \(P(X)\) is a power set of \(X\), which satisfies \(u \phi = \phi\), \(A \subseteq uA\) for every \(A \subseteq X\), \(u(A \cup B) = uA \cup uB\), for all \(A, B \subseteq X\). Many properties which hold in topological space hold in closure space as well. A topological space \(X\) is strongly connected if and only if it is not a disjoint union of countably many but more than one closed set. If \(X\) is strongly connected, and \(E_i\)'s are nonempty disjoint closed subsets of \(X\), then \(X \neq E_1 \cup E_2 \cup \ldots\). We further extend the concept of strongly connectedness in closure space.

The aim of this paper is to introduce and study the concept of strongly connectedness in closure space.

1. Introduction:

Čech closure space was introduced by Čech E. [1] in 1963. The modern notion of connectedness was proposed by Jorden (1893) and Schoenflies, and put on firm footing by Riesz [7] with the use of subspace topology. Many mathematicians such as EissaD.Habil, Khalid A. Elzenati[4], EissaD. Habil [5], Stadler B.M.R. and Stadler P.F. [6] have extended various concepts of strongly connectedness in topological space.

In this paper, we introduce strongly connectedness in closure space and study some of their properties.

2. Preliminaries:

Definition 2.1[2]: An operator \(u: P(X) \to P(X)\) defined on the power set \(P(X)\) of a set \(X\) satisfying the axioms:

1. \(u \phi = \phi\),
2. \(A \subseteq uA\) , for every \(A \subseteq X\),
3. \(u(A \cup B) = uA \cup uB\) , for all \(A, B \subseteq X\).

is called a Čech closure operator and the pair \((X, u)\) is a Čech closure space.

Definition 2.2 [8]:- A closure space \((X, u)\) is connected if and only if there exists a continuous function from \(X\) to the discrete space \(\{0, 1\}\) is constant. A subset \(A\) in a closure space \((X, u)\) is said to be connected if \(A\) with the subspace topology is a connected space.

Definition 2.3[3]: A topological space \(X\) is strongly connected if and only if it is not a disjoint union of countably many but more than one closed set.

If \(X\) is strongly connected, and \(E_i\)'s are nonempty disjoint closed subsets of \(X\), then \(X \neq E_1 \cup E_2 \cup \ldots\).
3. Strongly connectedness in closure space:

**Definition 3.1:** A connected closure space \((X, u)\) is said to be strongly connected if and only if it cannot be expressed as a disjoint union of countably many but more than one closed sets.

In connected closure space \((X, u)\), let \(E_1\) and \(E_2\) are two nonempty disjoint closed subsets of \(X\) then 
\[X \neq E_1 \cup E_2.\]

In strongly connected closure space \((X, u)\), \(E_i, s\) are nonempty disjoint closed subsets of \(X\) then 
\[X \neq E_1 \cup E_2 \cup \ldots \ldots \ldots.\]

**Example 3.2:** Let \(X = \{a, b, c\}\), 
Define a closure function \(u: P(X) \rightarrow P(X)\) such that 
\[u\{a\} = \{a, b\}, u\{b\} = u\{c\} = \{b, c\}, u\{a, b\} = u\{a, c\} = u(X) = X, u(\emptyset) = \emptyset.\]

Hence \((X, u)\) is a closure space.
Define a continuous function \(f: X \rightarrow \{0, 1\}\) such that 
\[f\{a\} = f\{b\} = f\{c\} = f\{a, b\} = f\{a, c\} = f\{b, c\} = 1, f(\emptyset) = \phi.\]

Hence \((X, u)\) is a connected closure space.
Here \((X, u)\) is strongly connected closure space because we cannot find nonempty disjoint closed subsets \(E_i\) of \(X\) such that 
\[X = E_1 \cup E_2 \cup \ldots \ldots \ldots.\]

**Definition 3.3:** A connected closure space \((X, u)\) is said to be strongly disconnected if and only if it can be expressed as a disjoint union of countably many but more than one closed sets.

**Theorem 3.4:** A continuous image of a strongly connected closure space is strongly connected closure space.

**Proof:** Let \((X, u)\) be a strongly connected closure space. Suppose \(f(X)\) is not strongly connected closure space then by definition it can be expressed as a disjoint union of countably many but more than one closed sets. Since \(f\) is continuous and the inverse image of closed set is still closed, so that \(X\) can be expressed as a disjoint union of countably many but more than one closed sets. Hence \((X, u)\) is a strongly disconnected closure space which is a contradiction. Therefore \(f(X)\) is strongly connected closure space.

**Definition 3.5[9]:** Let \((Z, u)\) be a closure space with more than one point. A closure space \((X, u)\) is called \(z\)-connected closure space if and only if any continuous map from \(X\) to \(Z\) is constant.

**Theorem 3.6:** A connected closure space \((X, u)\) is strongly connected if and only if it is \(z\)-connected closure space.

**Proof:** - **Necessary condition:** Suppose \((X, u)\) is strongly connected closure space.
Let \(f: X \rightarrow \{0, 1\} = Z\) be any continuous map. By proposition 3.4, \(f(X)\) is strongly connected. The only strongly connected subsets of \(Z\) are the one point spaces. Hence \(f\) is constant; i.e. \(X\) is \(z\)-connected closure space.

**Sufficient condition:** Suppose connected closure space \((X, u)\) is \(z\)-connected. If possible let \((X, u)\) is not strongly connected closure space so that \(X\) can be expressed as a disjoint union of countably many but more than one closed sets i.e. \(X = \bigcup E_i\). Then define a function \(f: X \rightarrow Z = \{0, 1\}\) by taking 
\[f(x) = i\] whenever \(x \in E_i\). This \(f\) is continuous but not constant. Hence \(X\) is not \(z\)-connected closure space which is a contradiction. Therefore, \(X\) is strongly connected closure space.

From the above theorem, strongly connected closure space is a special case of \(z\)-connectedness in closure space. Thus, all the properties proved for \(z\)-connected closure space are applicable to strongly connected closure space.

**Theorem 3.7:** A strongly connected closure space is connected. But converse is not true as the following example shows.
Example 3.8:- Let \( X= \{a, b, c\} \) and define \( u: P(X) \rightarrow P(X) \) as follows

\[
u(\emptyset)=\emptyset, u(a)=\{a\}, u(b)=\{b\}, u(c)=\{c\}, u(a, b)=u(a, c)=u(b, c)=u(X)=X.\]

Hence \((X, u)\) is a closure space.

We define a function \( f: X \rightarrow \{0, 1\} \) such that \( f(a)=f(b)=f(c)=f(a, b)=f(a, c)=f(b, c)=f(X)=X, f(\emptyset)=0.\)

Hence \( f \) is constant so that \((X, u)\) is a connected closure space. However, \( X \) is not strongly connected because there exists pair wise semi separated sets \( \{a\}, \{b\}, \{c\} \) such that \( X= \{a\} \cup \{b\} \cup \{c\} \).

Theorem 3.9:- The union of any family of strongly connected subsets of strongly connected closure space with a common point is strongly connected closure space.

Proof:- Let \((X, u)\) is a strongly connected closure space. Let each \( \{E_i: i \in \Lambda\} \) is strongly connected subset of strongly connected closure space \((X, u)\) and common point is \( y_0 \). Let \( C= \{\bigcup E_i: E_i \subseteq X\}, y_0 \in \bigcap E_i. \)

For any continuous function \( f: C \rightarrow \{0, 1\} \). Let \( ia: E_i \rightarrow C \) be the inclusion function. Each \( E_i \) is strongly connected, so that \( f:\bigcap ia: \bigcup E_i \rightarrow \{0, 1\} \) is continuous and constant and \( \bigcap E_i \neq \emptyset \), so there exists a \( y_0 \) such that \( y_0 \in \bigcap E_i \), i.e. \( f \) ia is constant and equal to \( f(y_0) \). Therefore \( f \) is constant and \( \bigcup E_i \) is strongly connected.

Theorem 3.10:- Let \( A \) and \( B \) are subsets of a strongly connected closure space \((X, u)\) such that \( A \subseteq B \subseteq A \), where \( A \) is the closure of \( A \). If \( A \) is strongly connected, then is strongly connected in closure space.

References


