

Strongly Connectedness in Closure Space

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Abstract: A Čech closure space (X, u) is a set X with Čech closure operator $u: P(X) \rightarrow P(X)$ where $P(X)$ is a power set of X , which satisfies $u \phi = \phi$, $A \subseteq uA$ for every $A \subseteq X$, $u(A \cup B) = uA \cup uB$, for all $A, B \subseteq X$. Many properties which hold in topological space hold in closure space as well. A topological space X is strongly connected if and only if it is not a disjoint union of countably many but more than one closed set. If X is strongly connected, and E_i 's are nonempty disjoint closed subsets of X , then $X \neq E_1 \cup E_2 \cup \dots$. We further extend the concept of strongly connectedness in closure space.

The aim of this paper is to introduce and study the concept of **strongly connectedness in closure space**.

1. Introduction:

Čech closure space was introduced by Čech E. [1] in 1963. The modern notion of connectedness was proposed by Jordan (1893) and Schoenfliesz, and put on firm footing by Riesz [7] with the use of subspace topology. Many mathematicians such as Eissa D. Habil, Khalid A. Elzenati [4], Eissa D. Habil [5], Stadler B.M.R. and Stadler P.F. [6] have extended various concepts of strongly connectedness in topological space.

In this paper, we introduce **strongly connectedness in closure space** and study some of their properties.

2. Preliminaries:

Definition 2.1[2]:- An operator $u: P(X) \rightarrow P(X)$ defined on the power set $P(X)$ of a set X satisfying the axioms:

1. $u \phi = \phi$,
2. $A \subseteq uA$, for every $A \subseteq X$,
3. $u(A \cup B) = uA \cup uB$, for all $A, B \subseteq X$.

is called a Čech closure operator and the pair (X, u) is a Čech closure space.

Definition 2.2 [8]:- A closure space (X, u) is connected if and only if there exists a continuous function from X to the discrete space $\{0, 1\}$ is constant. A subset A in a closure space (X, u) is said to be connected if A with the subspace topology is a connected space.

Definition 2.3[3]: A topological space X is strongly connected if and only if it is not a disjoint union of countably many but more than one closed set.

If X is strongly connected, and E_i 's are nonempty disjoint closed subsets of X , then

$$X \neq E_1 \cup E_2 \cup \dots$$

3. Strongly connectedness in closure space:

Definition 3.1: A connected closure space (X, u) is said to be strongly connected if and only if it cannot be expressed as a disjoint union of countably many but more than one closed sets.

In connected closure space (X, u) , let E_1 and E_2 are two nonempty disjoint closed subsets of X then $X \neq E_1 \cup E_2$.

In strongly connected closure space (X, u) , E_i, s are nonempty disjoint closed subsets of X then $X \neq E_1 \cup E_2 \cup \dots$

Example 3.2: Let $X = \{a, b, c\}$,

Define a closure function $u: P(X) \rightarrow P(X)$ such that

$$u\{a\} = \{a, b\}, u\{b\} = u\{c\} = u\{b, c\} = \{b, c\}, u\{a, b\} = u\{a, c\} = u(X) = X, u(\emptyset) = \emptyset.$$

Hence (X, u) is a closure space.

Define a continuous function $f: X \rightarrow \{0, 1\}$ such that $f\{a\} = f\{b\} = f\{c\} = f\{a, b\} = f\{a, c\} = f\{b, c\} = 1$, $f\{\emptyset\} = \emptyset$.

Hence (X, u) is a connected closure space.

Here (X, u) is strongly connected closure space because we cannot find nonempty disjoint closed subsets E_i 's of X such that $X = E_1 \cup E_2 \cup \dots$

Definition 3.3:- A connected closure space (X, u) is said to be strongly disconnected if and only if it can be expressed as a disjoint union of countably many but more than one closed sets.

Theorem 3.4:- A continuous image of a strongly connected closure space is strongly connected closure space.

Proof: Let (X, u) is a strongly connected closure space. Suppose $f(X)$ is not strongly connected closure space then by definition it can be expressed as a disjoint union of countably many but more than one closed sets. Since f is continuous and the inverse image of closed set is still closed, so that X can be expressed as a disjoint union of countably many but more than one closed sets. Hence (X, u) is a strongly disconnected closure space which is a contradiction. Therefore $f(X)$ is strongly connected closure space.

Definition 3.5[9]:- Let (Z, u) be a closure space with more than one point. A closure space (X, u) is called z -connected closure space if and only if any continuous map from X to Z is constant.

Theorem 3.6:- A connected closure space (X, u) is strongly connected if and only if it is z -connected closure space.

Proof: - Necessary condition: Suppose (X, u) is strongly connected closure space.

Let $f: X \rightarrow \{0, 1\} = Z$ be any continuous map. By proposition 3.4, $f(X)$ is strongly connected. The only strongly connected subsets of Z are the one point spaces. Hence f is constant; i.e. X is z -connected closure space.

Sufficient condition: Suppose connected closure space (X, u) is z -connected. If possible let (X, u) is not strongly connected closure space so that X can be expressed as a disjoint union of countably many but more than one closed sets i. e. $X = \bigcup E_i$. Then define a function $f: X \rightarrow Z = \{0, 1\}$ by taking $f(x) = i$ whenever $x \in E_i$. This f is continuous but not constant. Hence X is not z -connected closure space which is a contradiction. Therefore, X is strongly connected closure space.

From the above theorem, strongly connected closure space is a special case of z -connectedness in closure space. Thus all the properties proved for z -connected closure space are applicable to strongly connected closure space.

Theorem 3.7:- A strongly connected closure space is connected. But converse is not true as the following example shows.

Example 3.8:- Let $X = \{a, b, c\}$ and define $u: P(X) \rightarrow P(X)$ as follows

$u(\emptyset) = \emptyset, u\{a\} = \{a\}, u\{b\} = \{b\}, u\{c\} = \{c\}, u\{a, b\} = u\{a, c\} = u\{b, c\} = u\{X\} = X$. Hence (X, u) is a closure space.

We define a function $f: X \rightarrow \{0, 1\}$ such that $f\{a\} = f\{b\} = f\{c\} = f\{a, b\} = f\{a, c\} = f\{c, a\} = f\{X\} = X$, $f\{\emptyset\} = 0$.

Hence f is constant so that (X, u) is a connected closure space. However, X is not strongly connected because there exists pair wise semi separated sets $\{a\}, \{b\}, \{c\}$ such that

$$X = \{a\} \cup \{b\} \cup \{c\}.$$

Theorem 3.9:- The union of any family of strongly connected subsets of strongly connected closure space with a common point is strongly connected closure space.

Proof:- Let (X, u) is a strongly connected closure space. Let each $\{E_i: i \in \Lambda\}$ is strongly connected subset of strongly connected closure space (X, u) and common point is y_0 . Let $C = \{U E_i: E_i \subseteq X\}$, $y_0 \in \cap E_i$. For any continuous function $f: C \rightarrow \{0, 1\}$. Let $i_a: E_i \rightarrow C$ be the inclusion function. Each E_i is strongly connected, so that $f \circ i_a: E_i \rightarrow \{0, 1\}$ is continuous and constant and $\cap E_i \neq \emptyset$, so there exists a y_0 such that $y_0 \in \cap E_i$, i. e. $f \circ i_a$ is constant and equal to $f(y_0)$. Therefore f is constant and $\cup E_i$ is strongly connected.

Theorem 3.10:- Let A and B are subsets of a strongly connected closure space (X, u) such that $A \subseteq B \subseteq \bar{A}$, where \bar{A} is the closure of A . If A is strongly connected, then B is strongly connected in closure space

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