Magic Graphoidal on Class of Trees

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Abstract: B.D.Acharya and E. Sampathkumar [1] defined Graphoidal cover as partition of edge set of a graph G into internally disjoint paths (not necessarily open). The minimum cardinality of such cover is known as graphoidal covering number of G.

Let G = {V, E} be a graph and let ψ be a graphoidal cover of G. Define f:

\[ V \cup E \rightarrow \{1, 2, \ldots, p+q\} \]

such that for every path \( P = (v_0, v_1, v_2, \ldots, v_n) \) in ψ with \( f^*(P) = f(v_0) + f(v_n) + \sum_{i=1}^{n} f(v_{i-1}v_i) = k \), a constant, where \( f^* \) is the induced labeling on ψ. Then, we say that G admits ψ - magic graphoidal total labeling of G.

A graph G is called magic graphoidal if there exists a minimum graphoidal cover ψ of G such that G admits ψ - magic graphoidal total labeling.

In this paper, we proved that \([P_n:S_1]\), \([P_n:S_2]\), \(T_n\), \(P_m \oplus K_{1,3}\), \(P_m \oplus 2K_1\) and \(K_{1,n}\) are magic graphoidal.

1. INTRODUCTION

By a graph, we mean a finite simple and undirected graph. The vertex set and edge set of a graph G are denoted by V(G) and E(G) respectively. Terms and notations not used here are as in [3].

1.1. Definition: Let \( S_1 = (v_0, v_1) \) be a star and let \([P_n; S_1]\) be the graph obtained from n copies of \( S_1 \) and the path \( P_n = (u_1, u_2, \ldots, u_n) \) by joining \( u_j \) with the vertex \( v_0 \) of the \( j^{th} \) copy of \( S_1 \) by means of an edge, for \( 1 \leq j \leq n \).

1.2. Definition: Let \( S_2 = (v_0, v_1, v_2) \) be a star and let \([P_n; S_2]\) be the graph obtained from n copies of \( S_2 \) and the path \( P_n = (u_1, u_2, \ldots, u_n) \) by joining \( u_j \) with the vertex \( v_0 \) of the \( j^{th} \) copy of \( S_2 \) by means of an edge, for \( 1 \leq j \leq n \).

1.3. Definition: Let T be any Tree. Denote the tree, obtained from T by considering two copies of T by adding an edge between them, by \( T(2) \) and in general the graph obtained from \( T(n-1) \) and T by adding an edge between them is denoted by \( T(n) \).

1.4. Result [11]: For a Tree T, \( \gamma(T) = n-1 \) where n is the number of pendent vertices of G.

2. PRELIMINARIES

Let G = {V, E} be a graph with p vertices and q edges. A graphoidal cover \( \psi \) of G is a collection of (open) paths such that

(i) every edge is in exactly one path of \( \psi \)

(ii) every vertex is an interval vertex of at most one path in \( \psi \).

We define \( \gamma(G) = \min_{\psi \in \zeta} |\psi| \),

where \( \zeta \) is the collection of graphoidal covers \( \psi \) of G and \( \gamma \) is graphoidal covering number of G.
Let $\psi$ be a graphoidal cover of $G$. Then we say that $G$ admits $\psi$ - magic graphoidal total labeling of $G$ if there exists a bijection $f: V \cup E \rightarrow \{1, 2, \ldots, p+q\}$ such that for every path $P = (v_0v_1v_2 \ldots v_n)$ in $\psi$, then, $f^*(P) = f(v_0) + f(v_n) + \sum_{i=1}^{n} f(v_{i-1}v_i) = k$, a constant, where $f^*$ is the induced labeling of $\psi$. A graph $G$ is called magic graphoidal if there exists a minimum graphoidal cover $\psi$ of $G$ such that $G$ admits $\psi$ - magic graphoidal total labeling.

3. Magical Graphoidal on Trees

3.1. Theorem : $[P_n; S_1]$, $(n\text{-}even)$ is magic graphoidal.

Proof: Let $G = [P_n; S_1]$

Let $V(G) = \{u_i, v_i, w_i: 1 \leq i \leq n\}$ and
$E(G) = \{(u_iw_i) \cup (v_iw_i) : 1 \leq i \leq n\} \cup [(u_iu_{i+1}) : 1 \leq i \leq n-1]\}$

Define $f: V \cup E \rightarrow \{1, 2, \ldots, p+q\}$ by

$$
\begin{align*}
f(w_i) & = 1 \\
f(v_iw_i) & = 2 \\
f(u_i) & = 3 \\
f(u_{i+1}) & = 6 + i \quad 1 \leq i \leq n-2 \\
f(w_{i+1}) & = 6n - i \quad 1 \leq i \leq n-1 \\
f(v_{i+1}w_{i+1}) & = 4n + 1 + i \quad 1 \leq i \leq n-1 \\
f(u_{i+1}v_{i+1}) & = 3n + 3 - 2i \quad 1 \leq i \leq (n/2) - 1 \\
f\left(\frac{u_{n+1}}{2}, \frac{v_{n+1}}{2}\right) & = 3n + 4 - 2i \quad 1 \leq i \leq n/2 \\
f(u_iu_{i+1}) & = \frac{3n}{2} + 4 + i \quad 1 \leq i \leq n/2 - 1 \\
f\left(\frac{u_{n+1}}{2}, \frac{u_{n+1}}{2}\right) & = n + 4 + i \quad 1 \leq i \leq n/2
\end{align*}
$$

Let $\psi = \{P_1 = (w_1, v_1, u_1, u_2, v_2, w_2), P_2 = (u_i, u_{i+1}, v_{i+1}, w_{i+1}) : 2 \leq i \leq n-1\}$

$$
\begin{align*}
f^*[P_1] & = f(w_1) + f(w_2) + f(v_1v_1) + f(v_1u_1) + f(u_1u_2) + f(u_2v_2) + f(v_2w_2) \\
& = 1 + 6n - 1 + 2 + 3 + \frac{3n}{2} + 4 + 1 + 3n + 3 - 2 + 4n + 2 \\
& = 13n + \frac{3n}{2} + 13 \quad \text{(A)}
\end{align*}
$$

For $2 \leq i \leq (n/2)-1,$

$$
\begin{align*}
f^*[P_2] & = f(u_i) + f(w_{i+1}) + f(u_iu_{i+1}) + f(u_{i+1}v_{i+1}) + f(v_{i+1}w_{i+1}) \\
& = 6 + i - 1 + 6n - i + (3n/2) + 4 + i + 3n + 3 - 2i + 4n + 1 + i \\
& = 13n + (3n/2) + 13 \quad \text{(B)}
\end{align*}
$$

For $(n/2) \leq i \leq n-1,$
\[ f[P_2] = f(u_i) + f(w_{i+1}) + f(u_{i-1}v_i) + f(v_{i+1}w_{i+1}) \]

\[ = 6 + i - 1 + 6n - i + n + 4 + i - (n/2) + 1 + 3n + 4 - 2(i + 1 - (n/2)) + 4n + 1 + i \]

\[ = 13n + (3n/2) + 13 \quad \text{(C)} \]

From (A), (B) and (C), we conclude that \( G \) admits \( \psi \) - magic graphoidal total labeling. Hence, \([P_n; S_1], \) (n - even) is magic graphoidal.

For example, consider the graph \([P_6; S_1]\) shown in figure 1.

![Figure 1 [P6 ; S1]](image)

Clearly, \( \psi = \{(w_1, v_1, u_1, u_2, v_2, w_2), (u_2, u_3, v_3, w_3), (u_3, u_4, v_4, w_4), (u_4, u_5, v_5, w_5), (u_5, u_6, v_6, w_6)\} \) is a minimum graphoidal cover and \([P_6 ; S_1]\) is magic graphoidal. Here the constant \( K = 100 \)

**3.2. Theorem:** \([P_n ; S_1], \) (n - odd) is magic graphoidal.

**Proof:** Let \( G = [P_n ; S_1] \)

Let \( V(G) = \{u_i, v_i, w_i: 1 \leq i \leq n\} \) and

\[ E(G) = \{ (u_i, v_i) \cup (v_i, w_i): 1 \leq i \leq n \} \cup [(u_{i+1}, v_i) : 1 \leq i \leq n-1] \}

Define \( f: V \cup E \to \{1, 2, ..., p+q\} \) by

\[ f(w_1) = 1 \]

\[ f(w_1v_1) = 2 \]

\[ f(v_1u_1) = 3 \]

\[ f(u_{i+1}) = 5 + 2i \quad 1 \leq i \leq (n-1)/2 \]

\[ f(v_{i+1}w_{i+1}) = 4n + 1 + i \quad 1 \leq i \leq n-1 \]

\[ f(u_1u_2) = n + 5 \]

\[ f(u_{i+1}u_{i+2}) = \frac{3n + 1 + i}{2} \quad 1 \leq i \leq \frac{n-1}{2} \]

\[ f(u_{n+1}, u_{n+2}) = \frac{3n + 9 + i}{2} \quad 1 \leq i \leq \frac{n-3}{2} \]

\[ f(u_{i+2}v_{i+2}) = 3n + 2 - i \quad 1 \leq i \leq n-2 \]

\[ f(u_2v_2) = \frac{7n + 3}{2} \]
Let $\psi = \{P_1 = (w_1, v_1, u_1, u_2, v_2, w_2), P_2 = (u_i, u_{i+1}, v_{i+1}, w_{i+1}) : 2 \leq i \leq n - 1\}$

$f^*[P_1] = f(w_1) + f(w_2) + f(v_1u_1) + f(u_1u_2) + f(u_2v_2) + f(v_2w_2)$

$= 1 + 6n - 1 + 2 + 3 + n + 5 + \frac{7n + 3}{2} + 4n + 1 + 1$

$= 14n + \frac{n + 1}{2} + 13 \quad \text{(A)}$

For $2 \leq i < \frac{n+1}{2}$

$f^*[P_2] = f(u_i) + f(w_{i+1}) + f(u_{i+1}v_{i+1}) + f(v_{i+1}w_{i+1})$

$= 5 + 2(i-1) + 6n - i + \frac{3n+11}{2} - 3n + 2 - (i-1) + 4n + 1 + i$

$= 14n + \frac{n + 1}{2} + 13 \quad \text{(B)}$

For $\frac{n+1}{2} \leq i \leq n - 1$,

$f^*[P_2] = f(u_i) + f(v_i) + f(u_{i+1}) + f(u_{i+1}v_{i+1}) + f(v_{i+1}w_{i+1})$

$= 6 + 2 \left( i - \frac{n+1}{2} \right) + 6n - i + \frac{3n+9}{2} + (n-i) + 3n + 2 - (i-1) + 4n + 1 + i$

$= 14n + \frac{n + 1}{2} + 13 \quad \text{(C)}$

From (A), (B) and (C), we conclude that $\psi$ is minimum magic graphoidal cover. Hence, $[P_n; S_i]$, $(n$ - even) is magic graphoidal.

For example, consider the graph $[P_7; S_1]$ shown in figure 2.

\[ \text{Figure 2} \ [P_7; S_1] \]

Clearly, $\psi = \{(w_1, v_1, u_1, u_2, v_2, w_2), (u_2, u_3, v_3, w_3), (u_3, u_4, v_4, w_4), (u_4, u_5, v_5, w_5), (u_5, u_6, v_6, w_6), (u_6, u_7, v_7, w_7)\}$ is a minimum graphoidal cover and $[P_7; S_1]$ is magic graphoidal. Here the constant $K = 115$.

3.3. Theorem : $[P_n; S_2]$ is magic graphoidal.

Proof: Let $G = [P_n; S_2]$

$V(G) = \{(u_i, v_i) : 1 \leq i \leq n\}$
\[ E(G) = \{ (u_iu_{i+1}) : 1 \leq i \leq n-1 \} \cup \{ (u_iv_i) : 1 \leq i \leq n \} \cup \{ (v_iv_{i1}) \cup (v_iv_{i2}) : 1 \leq i \leq n \} \}

Define \( f : V \cup E \rightarrow \{ 1, 2, ..., p+q \} \) by

\[
\begin{align*}
    f(v_i) &= 1 \\
f(u_iv_i) &= 2f \\
f(u_i) &= 3 \\
f(u_{i+1}) &= 3 + i \quad 1 \leq i \leq n-2 \\
f(u_a) &= p + q \\
f(v_i) &= n + 1 + i \quad 1 \leq i \leq n \\
f(v_{i+1}) &= 8n - 1 - i \quad 1 \leq i \leq n - 1 \\
f(v_{i2}) &= 7n - i \quad 1 \leq i \leq n \\
f(v_{i1}) &= 5n - 1 + i \quad 1 \leq i \leq n \\
f(u_{i+1}v_{i+1}) &= 4n + i \quad 1 \leq i \leq n - 1 \\
f(u_iv_{i+1}) &= 4n + 1 - i \quad 1 \leq i \leq n - 1 \\
f(v_{i1}) &= 3n + 2 - i \quad 1 \leq i \leq n
\end{align*}
\]

Let \( \psi = \{ P_1 = [(v_{i1},v_i,v_{i2}) : 1 \leq i \leq n], P_2 = (v_1,u_1,u_2,v_2), P_3 = [(u_i,u_{i+1},v_{i+1}) : 2 \leq i \leq n-1] \} \)

For \( 1 \leq i \leq n \),

\[
\begin{align*}
    f^*_{[P_1]} &= f(v_{i1}) + f(v_{i2}) + f(v_{i1}v_i) + f(v_{i1}) \\
    &= n + 1 + i + 7n - i + 3n + 2 - i + 5n - 1 + i \\
    &= 16n + 2 \quad \text{--------- (A)}
\end{align*}
\]

\[
\begin{align*}
    f^*_{[P_2]} &= f(v_1) + f(v_2) + f(v_1u_1) + f(u_1u_2) + f(u_2v_2) \\
    &= 1 + 8n - 2 + 2 + 4n + 4n + 1 \\
    &= 16n + 2 \quad \text{--------- (B)}
\end{align*}
\]

For \( 2 \leq i \leq n - 1 \),

\[
\begin{align*}
    f^*_{[P_3]} &= f(u_i) + f(u_{i+1}) + f(u_{i+1}v_{i+1}) \\
    &= 3 + i - 1 + 8n - 1 - i + 4n + 1 - i + 4n + i \\
    &= 16n + 2 \quad \text{--------- (C)}
\end{align*}
\]

From (A), (B) and (C), we conclude that \( G \) admits \( \psi \) - magic graphoidal total labeling. Hence, \( [P_n : S_2] \) is magic graphoidal.

For example, consider the graphs \([P_3 : S_2]\) and \([P_4 : S_2]\) shown in figure 3.1 and 3.2.

![Figure 3.1 [P_3 ; S_2] ](image)

Clearly, \( \psi = \{ (v_{11},v_1,v_{12}), (v_{21},v_2,v_{22}), (v_{31},v_3,v_{32}), (v_1,u_1,u_2,v_2), (u_2,u_3,v_3) \} \) is a minimum graphoidal cover and \([P_3 ; S_2]\) is magic graphoidal. Here the constant \( K = 50 \)
Clearly, $\psi = \{(v_{11}, v_{1}, v_{12}), (v_{21}, v_{2}, v_{22}), (v_{31}, v_{3}, v_{32}), (v_{41}, v_{4}, v_{42}), (v_{1}, u_{1}, u_{2}, v_{2}), \ldots (A), (B) \text{ and } (C)),$ we conclude that $G$ admits $\psi$ - magic graphoidal total labeling. Hence, $T(n)$ is magic graphoidal.

3.4. Theorem: For a Tree, $T_{(a)}$ is magic graphoidal.

Proof:

Let $T_{(a)}$ be a graph such that

$V[T_{(a)}] = \{u_{i1}, u_{i2}, u_{i3}, u_{i4}, u_{i5} : 1 \leq i \leq n \}$ and

$E[T_{(a)}] = \{[(u_{i1}u_{i2}), (u_{i2}u_{i3}), (u_{i3}u_{i4})], (u_{i4}u_{i5}) : 1 \leq i \leq n \} \cup \{(u_{in}u_{i+1n}) : 1 \leq i \leq n-1\}$

Define $f: V \cup E \to \{1, 2, \ldots, p+q\}$ by

- $f(u_{i1}) = i \quad 1 \leq i \leq n$
- $f(u_{i3}) = i \quad 1 \leq i \leq n$
- $f(u_{i3}u_{i4}) = n + 2$
- $f(u_{i4}u_{i5}) = 2n + 3 - i \quad 1 \leq i \leq n$
- $f(u_{i1}u_{i2}) = 5n - 3 - 2i \quad 1 \leq i \leq n - 3$
- $f(u_{i2}u_{i3}) = 4n - 1 + i \quad n - 2 \leq i \leq n - 1$
- $f(u_{i3}u_{i4}) = 5n - 2 + i \quad 1 \leq i \leq n$
- $f(u_{i4}u_{i5}) = 6n - 2 + i \quad 1 \leq i \leq n$
- $f(u_{i1}u_{i2}) = 5n - 2 + i \quad 1 \leq i \leq n - 1$
- $f(u_{i3}u_{i4}) = 7n - 2 + i \quad 1 \leq i \leq n - 1$
- $f(u_{i4}) = 8n - 2 + 2(i-1) \quad 1 \leq i \leq n$
- $f(u_{i1}u_{i2}) = 8n - 1 + 2(i-1) \quad 1 \leq i \leq n - 1$

Let $\psi = \{P_1 = (u_{i1}u_{i2}u_{i3}u_{i4}), P_2 = (u_{i3}u_{i4}u_{i5}u_{i2}), P_3 = [(u_{i5}u_{i+15}, u_{i+13}) : 2 \leq i \leq n - 1]\}$

- $f^*[P_1] = f(u_{i1}) + f(u_{i4}) + f(u_{i1}u_{i2}) + f(u_{i2}u_{i3}) + f(u_{i3}u_{i4})$
  $= i + 8n - 2 + 2(i-1) + 2n + 3 - i + 5n - 2 + (n + 1 - i) + 6n - 2 + (n + 1 - i)$
  $= 23n - 3 \quad \text{-------- (A)}$

- $f^*[P_2] = f(u_{i3}) + f(u_{i1}u_{i2}) + f(u_{i3}u_{i4}) + f(u_{i1}u_{i2}u_{i3})$
  $= 2n + 2 + i - 1 + 8n - 1 + 2(i - 1) + 5n - 3 - 2(i - 2) + 7n - 2 + n - i$
  $= 23n - 3 \quad \text{-------- (B)}$

- $f^*[P_3] = f(u_{i3}) + f(u_{i2}u_{i3}) + f(u_{i3}u_{i4}) + f(u_{i1}u_{i2}u_{i3}) + f(u_{i2}u_{i3}u_{i4})$
  $= n + 1 + 8n - 1 + 2(i - 1) + 5n - 3 - 2(i - 2) + 7n - 2 + n - i$
  $= 23n - 3 \quad \text{-------- (C)}$

From (A), (B) and (C), we conclude that $G$ admits $\psi$ - magic graphoidal total labeling. Hence, $T_{(a)}$ is magic graphoidal.
For example, consider the graph $T(3)$ shown in figure 4.

Clearly, $\psi = \{(u11,u12,u13,u14), \ldots + f(u_{i}) + f(u_{ui}) = 2n + 3 + 3n + 4 + i - 2 + 4n + 2 + n + 1 - i = 10n + 8 \quad \text{---------- (B)}$

For $1 \leq i \leq n$,

3.5. Theorem: The graph Double Crowned Star $K_{1,n} \circ K_{1}$ is magic graphoidal. Here the constant $K = 66$

Proof: Let $G = K_{1,n} \circ 2K_{1}$

$V(G) = \{u, \{u_i : 1 \leq i \leq n\}, \{(u_{i}, u_{i+1}) : 1 \leq i \leq n\}\} \text{ and }$ $E(G) = \{(u_{ui}) : 1 \leq i \leq n\} \cup \{(u_{ui}) \cup (u_{ui+1}) : 1 \leq i \leq n \}$

Define $f: V \cup E \rightarrow \{1, 2, \ldots, p+q\}$ by

\[
\begin{align*}
    f(u) &= 2n + 3 \\
    f(u_{i}) &= i \quad 1 \leq i \leq n \\
    f(u_{1}) &= n + 1 \\
    f(u_{2}) &= n + 2 \\
    f(u_{n+1-i,2}) &= n + 2 + i \quad 1 \leq i \leq n \\
    f(u_{n+1-i,1}) &= 2n + 3 + i \quad 1 \leq i \leq n \\
    f(u_{ui}) &= 3n + 4 \\
    f(u_{2+i}) &= 3n + 4 + i \quad 1 \leq i \leq n - 2 \\
    f(u_{n+1-i}) &= 4n + 2 + i \quad 1 \leq i \leq n - 1 \\
    f(u_{n+1-i, u_{n+1-i,2}}) &= 5n + 1 + i \quad 1 \leq i \leq n
\end{align*}
\]

Let $\psi = \{P_1 = (u_1, u_2, u_3), P_2 = [(u_i, u_{i+1}) : 3 \leq i \leq n], P_3 = [(u_{i+1}, u_{n+1}, u_{n+i,2}) : 1 \leq i \leq n]\}$

\[
\begin{align*}
f'[P_1] &= f(u_1) + f(u_2) + f(u_{i+1}) + f(u_{i+2}) \\
&= n + 1 + n + 2 + 3n + 4 + 5n + 1 \\
&= 10n + 8 \quad \text{---------- (A)}
\end{align*}
\]

For $3 \leq i \leq n$,

\[
\begin{align*}
f'[P_2] &= f(u) + f(u_{i}) + f(u_{ui}) \\
&= 2n + 3 + 3n + 4 + i - 2 + 4n + 2 + n + 1 - i \\
&= 10n + 8 \quad \text{---------- (B)}
\end{align*}
\]

For $1 \leq i \leq n$, 

Figure 4 $T(3)$
\[ f[P_3] = f(u_{i1}) + f(u_{i2}) + f(u_{i1}u_i) + f(u_{i1}u_{i2}) \]
\[ = i + n + 2 + n + 1 - i + 2n + 3 + i + 5n + 1 + n + 1 - i \]
\[ = 10n + 8 \quad (C) \]

From (A), (B) and (C), we conclude that G admits \( \psi \) - magic graphoidal total labeling. Hence, Double Crowned Star \( K_{1,n} \odot 2K_1 \) is magic graphoidal. For example, consider the graph \( K_{1,5} \odot 2K_1 \) shown in figure 5.

Clearly, \( \psi = \{ (u_{i1}, u_{i1}u_{i2}), (u_{i2}, u_{i2}), (u_{i1}, u_{i3}, u_{i3}), (u_{i4}, u_{i4}, u_{i4}), (u_{i5}, u_{i5}, u_{i5}) \} \) is a minimum graphoidal cover and \( K_{1,5} \odot 2K_1 \) is magic graphoidal. Here the constant \( K = 58 \).

3.6. Theorem: \( P_m \odot 2K_1 \) is magical graphoidal.

Proof: Let \( G = P_m \odot 2K_1 \)

\[ V(G) = \{ (u_i : 1 \leq i \leq m), (u_{ij} : 1 \leq i \leq m, 1 \leq j \leq 2) \} \text{ and} \]
\[ E(G) = \{ [(u_{i1}, u_{i2}) : 1 \leq i \leq m-1] \cup [(u_{i1}, u_{i2}) : 1 \leq i \leq m, 1 \leq j \leq 2] \} \]

Let \( \psi = \{ P_1 = [(u_{i1}, u_{i1}u_{i2}) : 1 \leq i \leq m], P_2 = [(u_{i1}, u_{i1}) : 1 \leq i \leq m-1] \} \)

Define \( f : V \rightarrow \{ 0, 1, 2, \ldots, 6m-1 \} \) by

\[ f(u_{i1}) = i \quad 1 \leq i \leq m \]
\[ f(u_{i2}) = 2m+1-i \quad 1 \leq i \leq m \]
\[ f(u_{i2}) = \begin{cases} 
2m+i & 1 \leq i \leq \frac{m}{2} \\
2m+i & 1 \leq i \leq \frac{m-1}{2} 
\end{cases} \text{ if } m \text{ is even} \]
\[ \text{if } m \text{ is odd} \]

\[ \text{Figure 5. } K_{1,5} \odot 2K_1 \]
\[
\begin{align*}
\text{Case (i):} \quad &\text{when } m \text{ is odd} \\
&\text{For } 1 \leq i \leq m, \\
&f'(P_1) = f(u_{i1}) + f(u_{i2}) + f(u_{i1}u_{i}) + f(u_{i1}u_{i2}) \\
&= \frac{5m-1}{2} + i + 5m - 1 + m + 1 - i + 2m + 1 - i \\
&= \frac{21m + 1}{2} \quad \text{(A)} \\
\text{For } 1 \leq i \leq m - 1, \ i \equiv 1 \mod 2 \\
&f'(P_2) = f(u_{i}) + f(u_{i+1}) + f(u_{i}u_{i+1}) \\
&= \frac{7m-1}{2} + \frac{i+1}{2} + 2m + \frac{i+1}{2} + 4m + m - i \\
&= \frac{21m + 1}{2} \quad \text{(B)} \\
\text{For } 1 \leq i \leq m - 1, \ i \equiv 0 \mod 2 \\
&f'(P_2) = f(u_{i}) + f(u_{i+1}) + f(u_{i}u_{i+1}) \\
&= 2m + \frac{i}{2} + \frac{7m-1}{2} + \frac{i+2}{2} + 4m + m - i \\
&= \frac{21m + 1}{2} \quad \text{(C)} \\
\end{align*}
\]

From (A), (B) and (C), we conclude that G admits $\psi$ - magic graphoidal total labeling. Hence, $P_m \boxtimes 2K_1 \ (m\text{-odd})$ is magic graphoidal.

For example, consider the graph $P_5 \boxtimes 2K_1$ shown in figure 6.1.

\[\text{Case (ii):} \quad \text{when } m \text{ is even}\]

For $1 \leq i \leq m$,

\[
f'(P_1) = f(u_{i1}) + f(u_{i2}) + f(u_{i1}u_{i}) + f(u_{i1}u_{i2})
\]
\[
\frac{5m}{2} = i + 5m - 1 + m + 1 - i + i + 2m + 1 - i \\
= \frac{21m + 2}{2} \quad \text{(A)}
\]

For \(1 \leq i \leq m - 1, \ i \equiv 1 \mod 2\)

\[
f'[P_2] = f(u_i) + f(u_{i+1}) + f(u_{i+2})
= \frac{7m}{2} + \frac{i+1}{2} + \frac{2m}{2} + \frac{i+1}{2} + 4m + m - i
= \frac{21m + 2}{2} \quad \text{(B)}
\]

For \(1 \leq i \leq m - 1, \ i \equiv 0 \mod 2\)

\[
f'[P_2] = f(u_i) + f(u_{i+1}) + f(u_{i+2})
= \frac{2m}{2} + \frac{i}{2} + \frac{7m}{2} + \frac{i+2}{2} + 4m + m - i
= \frac{21m + 2}{2} \quad \text{(C)}
\]

From (A), (B) and (C), we conclude that \(G\) admits \(\psi\) - magic graphoidal total labeling. Hence, \(P_m \bigcirc 2K_1\) (m-even) is magic graphoidal.

For example, consider the graph \(P_4 \bigcirc 2K_1\) shown in figure 6.2.

![Figure 6.1 P_5 \bigcirc 2K_1](image)

Clearly, \(\psi = \{u_{11}, u_{12}, (u_{21}, u_{22}), (u_{31}, u_{32}), (u_{41}, u_{42}), (u_{51}, u_{52}), (u_i, u_{i+1}), (u_{i+1}, u_{i+2}), (u_{i+2}, u_{i+3}), (u_{i+3}, u_{i+4})\}\) is a minimum graphoidal cover and \(P_4 \bigcirc 2K_1\) is magic graphoidal. Here the constant \(K = 53\).
Clearly, \( \psi = \{(u_1, u_1, u_2, u_3, u_4) : 1 \leq i \leq m\} \) is a minimum graphoidal cover and \( P_4 \otimes 2K_1 \) is magic graphoidal. Here the constant \( K = 43 \).

### 3.7. Theorem

\( P_m \otimes K_{1,3} \) is magic graphoidal.

**Proof:** Let \( G = P_m \otimes K_{1,3} \).

\[
\begin{align*}
V(G) &= \{[(u_i, v_i), 1 \leq i \leq m], [(v_j), 1 \leq i \leq m, 1 \leq i \leq 3]\} \\
E(G) &= \{[(u_i, u_{i+1}), 1 \leq i \leq m-1] \cup [(v_i, v_{i+1}), 1 \leq i \leq m, 1 \leq j \leq 3]\}
\end{align*}
\]

with \( u_i = u_{i+1}, 1 \leq i \leq m \).

Let \( \psi = [(v_1, v_1, v_2) : 1 \leq i \leq m], P_2 = [(v_i, u_i, u_{i+1}) : 1 \leq i \leq m-2], P_3 = (v_{m-1}, u_{m-1}, u_m, v_m) \).

Define \( f : V \cup E \rightarrow \{1, 2, \ldots, 8m-1\} \) by

\[
\begin{align*}
f(v_m) &= 1 \\
f(v_1, v_1) &= i+1 \quad 1 \leq i \leq m \\\nf(u_i, v_i) &= m+1+i \quad 1 \leq i \leq m-1 \\
f(u_{m+i}, u_{m+i}) &= 2m+i \quad 1 \leq i \leq m-1 \\
f(v_{m+i}, v_{m+i}) &= 3m-1+i \quad 1 \leq i \leq m \\
f(v_1) &= 4m-1+i \quad 1 \leq i \leq m \\
f(v_i) &= 5m-1+i \quad 1 \leq i \leq m-1 \\
f(u_m) &= 6m-1 \\
f(u_{m-i}) &= 6m+i \quad 1 \leq i \leq m-2 \\
f(v_{m+i}) &= 7m-2+i \quad 1 \leq i \leq m
\end{align*}
\]

For \( 1 \leq i \leq m \),

\[
\begin{align*}
f'[P_1] &= f(v_1) + f(v_2) + f(v_1, v_1) + f(v_1, v_2) \\
&= 4m-1+i + 7m-2 + m + 1 + i + 1 + 3m-1 + m + 1 - i \\
&= 16m - 1 \quad \text{(A)}
\end{align*}
\]

For \( 1 \leq i \leq m-2 \),

\[
\begin{align*}
f'[P_2] &= f(v_i) + f(u_{i+1}) + f(v_i, u_i) + f(u_i, u_{i+1}) \\
&= 5m-1+i + 6m + (m - i - 1) + m + 1 + i + 2m + m - i \\
&= 16m - 1 \quad \text{(B)}
\end{align*}
\]
\[ f'[P_3] = f(v_{m-1}) + f(v_m) + f(v_{m-1}u_{m-1}) + f(u_{m-1}u_m) + f(u_mv_m) \]
\[ = 6m - 2 + 1 + 2m + 2 + 1 + 6m - 1 \]
\[ = 16m - 1 \quad \text{(C)} \]

From (A), (B) and (C), we conclude that G admits \( \psi \) - magic graphoidal total labeling. Hence, \( P_m \square K_{1,3} \) is magic graphoidal.

For example, consider the graph \( P_4 \square K_{1,3} \) shown in figure 7.

![Figure 7. P_4 \square K_{1,3}](image)

Clearly, \( \psi = \{(v_{11},v_1,v_{12}), (v_{21},v_2,v_{22}), (v_{31},v_3,v_{32}), (v_{41},v_4,v_{42}), (v_1,u_1,u_2), (v_1,u_2,u_3), (v_3,u_3,u_4,v_4)\} \) is magic graphoidal. Here the constant \( K = 63 \).

REFERENCES