Magic Graphoidal on Class of Trees

A. Nellai Murugan

Department of Mathematics, V.O.Chidambaram College, Tuticorin, Tamilnadu.

**Keywords:** Graphoidal Cover, Magic Graphoidal, Graphoidal Constant.

**Abstract:** B.D.Acharya and E. Sampathkumar [1] defined Graphoidal cover as partition of edge set of a graph $G$ into internally disjoint paths (not necessarily open). The minimum cardinality of such cover is known as graphoidal covering number of $G$.

Let $G = \{V, E\}$ be a graph and let $\psi$ be a graphoidal cover of $G$. Define $f$:

$$V \cup E \to \{1, 2, \ldots, p+q\}$$

such that for every path $P = (v_0, v_1, v_2, \ldots, v_n)$ in $\psi$ with $f^*(P) = f(v_0) + f(v_n) + \sum_{i=1}^{n} f(v_{i-1}v_i) = k$, a constant, where $f^*$ is the induced labeling on $\psi$. Then, we say that $G$ admits $\psi$ - magic graphoidal total labeling of $G$.

A graph $G$ is called magic graphoidal if there exists a minimum graphoidal cover $\psi$ of $G$ such that $G$ admits $\psi$ - magic graphoidal total labeling.

In this paper, we proved that $[P_n; S_1]$, $[P_n; S_2]$, $T(n)$, $P_m \bowtie K_{1,3}$, $P_m \bowtie 2K_1$ and $K_{1,n}$ are magic graphoidal.

1. **INTRODUCTION**

By a graph, we mean a finite simple and undirected graph. The vertex set and edge set of a graph $G$ are denoted by $V(G)$ and $E(G)$ respectively. Terms and notations not used here are as in [3].

1.1. **Definition:** Let $S_1 = (v_0, v_1)$ be a star and let $[P_n; S_1]$ be the graph obtained from $n$ copies of $S_1$ and the path $P_n = (u_1, u_2, \ldots, u_n)$ by joining $u_j$ with the vertex $v_0$ of the $j^{th}$ copy of $S_1$ by means of an edge for $1 \leq j \leq n$.

1.2. **Definition:** Let $S_2 = (v_0, v_1, v_2)$ be a star and let $[P_n; S_2]$ be the graph obtained from $n$ copies of $S_2$ and the path $P_n = (u_1, u_2, \ldots, u_n)$ by joining $u_j$ with the vertex $v_0$ of the $j^{th}$ copy of $S_2$ by means of an edge, for $1 \leq j \leq n$.

1.3. **Definition:** Let $T$ be any Tree. Denote the tree, obtained from $T$ by considering two copies of $T$ by adding an edge between them, by $T(2)$ and in general the graph obtained from $T_{n-1}$ and $T$ by adding an edge between them is denoted by $T_n$.

1.4. **Result** [11]: For a Tree $T$, $\gamma(T) = n-1$ where $n$ is the number of pendent vertices of $G$.

2. **PRELIMINARIES**

Let $G = \{V, E\}$ be a graph with $p$ vertices and $q$ edges. A graphoidal cover $\psi$ of $G$ is a collection of (open) paths such that

(i) every edge is in exactly one path of $\psi$

(ii) every vertex is an interval vertex of atmost one path in $\psi$.

We define $\gamma(G) = \min_{\psi \in \zeta} |\psi|$

where $\zeta$ is the collection of graphoidal covers $\psi$ of $G$ and $\gamma$ is graphoidal covering number of $G$. 
Let $\psi$ be a graphoidal cover of $G$. Then we say that $G$ admits $\psi$ - magic graphoidal total labeling of $G$ if there exists a bijection $f: V \cup E \rightarrow \{1, 2, ..., p+q\}$ such that for every path $P = (v_0v_1v_2 ... v_n)$ in $\psi$, then, $f^*(P) = f(v_0) + f(v_n) + \sum_{i=1}^{n} f(v_{i-1}v_i) = k$, a constant, where $f^*$ is the induced labeling of $\psi$. A graph $G$ is called magic graphoidal if there exists a minimum graphoidal cover $\psi$ of $G$ such that $G$ admits $\psi$ - magic graphoidal total labeling.

3. Magical Graphoidal on Trees

3.1. Theorem: $[P_n ; S_1]$, (n - even) is magic graphoidal.

Proof: Let $G = [P_n ; S_1]$

Let $V(G) = \{u_i, v_i, w_i: 1 \leq i \leq n\}$ and $E(G) = \{(u_iw_i) : 1 \leq i \leq n \} \cup \{(u_iu_{i+1}) : 1 \leq i \leq n-1\}$

Define $f: V \cup E \rightarrow \{1, 2, ..., p+q\}$ by

\[
\begin{align*}
    f(w_1) &= 1 \\
    f(w_1v_1) &= 2 \\
    f(v_1u_1) &= 3 \\
    f(u_{i+1}) &= 6 + i \quad 1 \leq i \leq n-2 \\
    f(w_{i+1}) &= 6n - i \quad 1 \leq i \leq n-1 \\
    f(v_{i+1}w_{i+1}) &= 4n + 1 + i \quad 1 \leq i \leq n-1 \\
    f(u_{i+1}v_{i+1}) &= 3n + 3 - 2i \quad 1 \leq i \leq (n^2) - 1
\end{align*}
\]

\[
\begin{align*}
    f\left(\begin{pmatrix} u_n \\ v_n \end{pmatrix}\right) &= 3n + 4 - 2i \quad 1 \leq i \leq \frac{n}{2} \\
    f(u_nu_{i+1}) &= \frac{3n}{2} + 4 + i \quad 1 \leq i \leq \frac{n}{2} - 1 \\
    f\left(\begin{pmatrix} u_{n-1} \\ v_{n-1} \end{pmatrix}\right) &= n + 4 + i \quad 1 \leq i \leq \frac{n}{2}
\end{align*}
\]

Let $\psi = \{P_1=(w_1,v_1,u_1,u_2,v_2), P_2=(u_iu_{i+1},v_{i+1}w_{i+1}) : 2 \leq i \leq n-1\}$

\[
\begin{align*}
    f^*[P_1] &= f(w_1) + f(w_2) + f(w_1v_1) + f(v_1u_1) + f(u_1u_2) + f(u_2v_2) + f(v_2w_2) \\
    &= 1 + 6n - 1 + 2 + 3 + \frac{3n}{2} + 4 + 1 + 3n + 3 - 2 + 4n + 2 \\
    &= 13n + \frac{3n}{2} + 13 \quad (A)
\end{align*}
\]

For $2 \leq i \leq (n^2)-1$,

\[
\begin{align*}
    f^*[P_2] &= f(u_i) + f(w_{i+1}) + f(u_iu_{i+1}) + f(u_{i+1}v_{i+1}) + f(v_{i+1}w_{i+1}) \\
    &= 6 + i - 1 + 6n - i + (3n^2) + 4 + i + 3n + 3 - 2i + 4n + 1 + i \\
    &= 13n + (3n^2) + 13 \quad (B)
\end{align*}
\]

For $(n^2) \leq i \leq n - 1$, 

...
\[ f(P_2) = f(u_i) + f(v_{i+1}) + f(u_i u_{i+1}) + f(u_{i+1} v_{i+1}) + f(v_{i+1} w_{i+1}) \]

\[ = 6 + i - 1 + 6n - i + n + 4 + i - (n/2) + 1 + 3n + 4 - 2(i + 1 - (n/2)) + 4n - 1 + i \]

\[ = 13n + (3n/2) + 13 \quad \text{(C)} \]

From (A), (B) and (C), we conclude that G admits \( \varphi \) - magic graphoidal total labeling. Hence, \([P_n; S_1], \text{ (n - even)}\) is magic graphoidal.

For example, consider the graph \([P_6; S_1]\) shown in figure 1.

![Figure 1 [P6 ; S1]](image)

Clearly, \( \varphi = \{(w_1, v_1, u_1, u_2, v_2, w_2), (u_2, u_3, v_2, w_3), (u_3, u_4, v_3, w_4), (u_4, u_5, v_4, w_5), (u_5, u_6, v_5, w_6)\} \) is a minimum graphoidal cover and \([P_6; S_1]\) is magic graphoidal. Here the constant \( K = 100 \)

3.2. **Theorem** : \([P_n; S_1], \text{ (n - odd)}\) is magic graphoidal.

**Proof**: Let \( G = [P_n; S_1] \)

Let \( V(G) = \{u_i, v_i, w_i: 1 \leq i \leq n\} \) and

\[ E(G) = \{(u_i v_i) \cup (v_i w_i): 1 \leq i \leq n\} \cup \{(u_i u_{i+1}): 1 \leq i \leq n-1\} \]

Define \( f: V \cup E \to \{1, 2, ..., p+q\} \) by

\[
\begin{align*}
    f(w_1) &= 1 \\
    f(w_1 v_1) &= 2 \\
    f(v_1 u_1) &= 3 \\
    f(u_{i+1}) &= 5 + 2i, \quad 1 \leq i \leq (n-1)/2 \\
    f(w_{i+1}) &= 6n - i, \quad 1 \leq i \leq n - 1 \\
    f(v_{i+1} w_{i+1}) &= 4n + 1 + i, \quad 1 \leq i \leq n - 1 \\
    f(u_i u_2) &= n + 5 \\
    f(u_{i+1} u_{i+2}) &= \frac{3n + 1}{2} - i, \quad 1 \leq i \leq \frac{n-1}{2} \\
    f(u_{n+1}, u_{n+3}) &= \frac{3n + 9}{2} + i, \quad 1 \leq i \leq \frac{n-3}{2} \\
    f(u_{i+2} v_{i+2}) &= 3n + 2 - i, \quad 1 \leq i \leq n - 2 \\
    f(u_i v_2) &= \frac{7n + 3}{2}
\end{align*}
\]
Let \( \psi = \{P_1 = (w_1,v_1,u_1,u_2,v_2,w_2), P_2 = (u_i,u_{i+1},v_{i+1},w_{i+1}) : 2 \leq i \leq n-1\} \)

\[
f' [P_1] = f(w_1) + f(w_2) + f(v_1u_1) + f(u_1u_2) + f(u_2v_2) + f(v_2w_2)
= 1 + 6n - 1 + 2 + 3 + n + 5 + \frac{7n + 3}{2} + 4n + 1 + 1
= 14n + \frac{n+1}{2} + 13 \quad \text{(A)}
\]

For \(2 \leq i < \frac{n+1}{2}\)

\[
f' [P_2] = f(u_i) + f(w_{i+1}) + f(u_{i+1}v_{i+1}) + f(v_{i+1}w_{i+1})
= 5 + 2(i - 1) + 6n - i + \frac{3n + 11}{2} - (i - 1) + 3n + 2 - (i - 1) + 4n + 1 + 1
= 14n + \frac{n+1}{2} + 13 \quad \text{(B)}
\]

For \( \frac{n+1}{2} \leq i \leq n - 1\).

\[
f' [P_2] = f(u_i) + f(w_{i+1}) + f(u_{i+1}v_{i+1}) + f(v_{i+1}w_{i+1})
= 6 + 2\left( i - \frac{n+1}{2} \right) + 6n - i + \frac{3n + 9}{2} + (n - i) + 3n + 2 - (i - 1) + 4n + 1 + 1
= 14n + \frac{n+1}{2} + 13 \quad \text{(C)}
\]

From (A), (B) and (C), we conclude that \( \psi \) is minimum magic graphoidal cover. Hence, \([P_n ; S_1]\), (n - even) is magic graphoidal.

For example, consider the graph \([P_7 ; S_1]\) shown in figure 2.

\[\text{Figure 2 } [P_7 ; S_1]\]

Clearly, \( \psi = \{(w_1,v_1,u_1,u_2,v_2,w_2), (u_2,u_3,v_3,w_3), (u_3,u_4,v_4,w_4), (u_4,u_5,v_5,w_5), (u_5,u_6,v_6,w_6), (u_6,u_7,v_7,w_7)\} \) is a minimum magic graphoidal cover and \([P_7 ; S_1]\) is magic graphoidal. Here the constant \( K = 115 \).

3.3. Theorem: \([P_n ; S_2]\) is magic graphoidal.

Proof: Let \( G = [P_n ; S_2]\)

\( V(G) = \{(u_i, v_i) : 1 \leq i \leq n\} \)
E(G) = \{ [ (u_i u_{i+1}) : 1 \leq i \leq n-1] \cup [(u_i v_i) : 1 \leq i \leq n] \\
\cup [(v_i v_{ii}) \cup (v_i v_{ii2}) : 1 \leq i \leq n] \}

Define \( f : V \cup E \to \{1, 2, ..., p+q\} \) by

\[
\begin{align*}
f(v_i) &= 1 \\
f(u_i v_i) &= 2f \\
(u_i) &= 3 \\
f(u_{i+1}) &= 3 + i \quad 1 \leq i \leq n-2 \\
f(u_n) &= p + q \\
f(v_{ii}) &= n + 1 + i \quad 1 \leq i \leq n \\
f(v_{i+1}) &= 8n - 1 - i \quad 1 \leq i \leq n-1 \\
f(v_{ii2}) &= 7n - i \quad 1 \leq i \leq n \\
f(v_{ii3}) &= 5n - 1 + i \quad 1 \leq i \leq n \\
f(u_{i+1} v_{ii}) &= 4n + i \quad 1 \leq i \leq n-1 \\
f(u_i u_{i+1}) &= 4n + 1 - i \quad 1 \leq i \leq n-1 \\
f(v_i v_{ii}) &= 3n + 2 - i \quad 1 \leq i \leq n 
\end{align*}
\]

Let \( \psi = \{P_1 = [(v_{i1}, v_{i}, v_{i2}) : 1 \leq i \leq n], P_2 = (v_1, u_1, u_2, v_2), P_3 = [(u_i, u_{i+1}, v_{i+1}) : 2 \leq i \leq n-1]\} \)

For \(1 \leq i \leq n,\)

\[
f^* [P_1] = f(v_{i1}) + f(v_{i2}) + f(v_{i1} v_{i}) + f(v_{i} v_{i2})
= n + 1 + i + 7n - i + 3n + 2 - i + 5n - 1 + i
= 16n + 2 \quad \text{(A)}
\]

\[
f^* [P_2] = f(v_1) + f(v_2) + f(v_1 u_1) + f(u_1 u_2) + f(u_2 v_2)
= 1 + 8n - 2 + 2 + 4n + 4n + 1
= 16n + 2 \quad \text{(B)}
\]

For \(2 \leq i \leq n-1,\)

\[
f^* [P_3] = f(u_i) + f(u_{i+1}) + f(u_i u_{i+1}) + f(u_{i+1} v_{i+1})
= 3 + i - 1 + 8n - 1 - i + 4n + 1 - i + 4n + i
= 16n + 2 \quad \text{(C)}
\]

From (A), (B) and (C), we conclude that \( G \) admits \( \psi \) - magic graphoidal total labeling. Hence, \([P_n : S_2]\) is magic graphoidal.

For example, consider the graphs \([P_3 : S_2]\) and \([P_4 : S_2]\) shown in figure 3.1 and 3.2.

![Figure 3.1](image_url)

Clearly, \( \psi = \{(v_{11}, v_{11}, v_{12}), (v_{21}, v_{22}, v_{23}), (v_{31}, v_{31}, v_{32}), (v_1, u_1, u_2, v_2), (u_2, u_3, v_3)\} \) is a minimum graphoidal cover and \([P_3 : S_2]\) is magic graphoidal. Here the constant \( K = 50 \)
Clearly, $\psi = \{(v_1, v_1, v_1), (v_2, v_2, v_2), (v_3, v_3, v_3), (v_4, v_4, v_4), (v_1, u_1, u_2), \ldots\}$ is a minimum graphoidal cover and $[P_4 ; S_2]$ is magic graphoidal. Here the constant $K = 66$.

### 3.4. Theorem:
For a Tree, $T(n)$ is magic graphoidal.

**Proof:**

Let $T(n)$ be a graph such that

$V[T(n)] = \{u_1, u_2, u_3, u_4, u_5 : 1 \leq i \leq n\}$ and

$E[T(n)] = \{[(u_i, u_2), (u_2, u_3), (u_3, u_4), (u_4, u_5) : 1 \leq i \leq n] \cup [(u_n, u_{i+1}) : 1 \leq i \leq n-1)]\}$

Define $f: V \cup E \rightarrow \{1, 2, \ldots, p+q\}$ by

- $f(u_1) = i, 1 \leq i \leq n$
- $f(u_3) = n + 1$
- $f(u_4) = n + 2$
- $f(u_{12}) = 2n + 3 - i, 1 \leq i \leq n$
- $f(u_{12}) = 5n - 3 - 2i, 1 \leq i \leq n - 3$
- $f(u_{n+1-i}) = 4n - 1 - i, n - 2 \leq i \leq n - 1$
- $f(u_{n+1-i}) = 5n - 2 + i, 1 \leq i \leq n$
- $f(u_{n+1-i}) = 6n - 2 + i, 1 \leq i \leq n$
- $f(u_{n+1-i}) = 7n - 2 + i, 1 \leq i \leq n - 1$
- $f(u_4) = 8n - 2 + 2(i-1), 1 \leq i \leq n$
- $f(u_{i+1}) = 8n - 1 + 2(i-1), 1 \leq i \leq n - 1$

Let $\psi = \{P_1 = (u_1, u_2, u_3, u_4), P_2 = (u_3, u_5, u_7, u_9), P_3 = [(u_5, u_{i+15}, u_{i+13}) : 2 \leq i \leq n - 1]\}$

$f'(P_1) = f(u_1) + f(u_4) + f(u_1u_2) + f(u_2u_3) + f(u_4u_4)
= i + 8n - 2 + 2(i-1) + 2n + 3 - i + 5n - 2 + (n + 1 - i) + 6n - 2 + (n + 1 - i)
= 23n - 3 \quad \text{-------- (A)}$

$f'(P_2) = f(u_3) + f(u_{i+1}) + f(u_5, u_{i+13}) + f(u_{i+15}, u_{i+13})
= 2n + 2 + i + 8n - 1 + 2(i - 1) + 5n - 3 - 2(i - 2) + 7n - 2 + n - i
= 23n - 3 \quad \text{-------- (B)}$

$f'(P_3) = f(u_5) + f(u_23) + f(u_3u_15) + f(u_{i+15}u_{23}) + f(u_{25}u_{23})
= n + 1 + 8n - 1 + 2(i - 1) + 5n - 3 - 2(i - 2) + 7n - 2 + n - i
= 23n - 3 \quad \text{-------- (C)}$

From (A), (B) and (C), we conclude that $G$ admits $\psi$ - magic graphoidal total labeling. Hence, $T(n)$ is magic graphoidal.
For example, consider the graph $T(3)$ shown in figure 4.

Figure 4 $T(3)$

Clearly, $\psi = \{(u_{11},u_{12},u_{13},u_{14}), ..., + f(u_i) + f(u_{ui}) \}

= 2n + 3 + 3n + 4 + i - 2 + 4n + 2 + n + 1 - i

= 10n + 8 \quad \text{---------- (B)}$

For $1 \leq i \leq n$,

3.5. Theorem: The graph Double Crowned Star $K_{1,n} \odot K_1$ is magic graphoidal.

Proof: Let $G = K_{1,n} \odot 2K_1$

$V(G) = \{u, [u_1 : 1 \leq i \leq n], [(u_{i1},u_{i2}) : 1 \leq i \leq n] \}$ and

$E(G) = \{ [(u_1) : 1 \leq i \leq n] \cup [(u_{u_1}) \cup (u_{u_2}) : 1 \leq i \leq n] \}$

Define $f : V \cup E \rightarrow \{1, 2, ..., p+q\}$ by

$f(u) = 2n + 3$

$f(u_i) = i \quad \text{1 \leq i \leq n}$

$f(u_{1i}) = n + 1$

$f(u_{2i}) = n + 2$

$f(u_{n+1,i,2}) = n + 2 + i \quad \text{1 \leq i \leq n}$

$f(u_{u_1i}) = 2n + 3 + i \quad \text{1 \leq i \leq n}$

$f(u_{u_2i}) = 3n + 4$

$f(u_{u_{2+i}}) = 3n + 4 + i \quad \text{1 \leq i \leq n - 2}$

$f(u_{u_{n+1,i}}) = 4n + 2 + i \quad \text{1 \leq i \leq n - 1}$

$f(u_{u_{n+1,i}}) = 5n + 1 + i \quad \text{1 \leq i \leq n}$

Let $\psi = \{P_1 = (u_1,u_{u_2}), P_2 = [(u_{u_1}) : 3 \leq i \leq n], P_3 = [(u_{u_1},u_{u_2}) : 1 \leq i \leq n] \}$

$f'[P_1] = f(u_1) + f(u_2) + f(u_{u_1}) + f(u_{u_2})$

$= n + 1 + n + 2 + 3n + 4 + 5n + 1$

$= 10n + 8 \quad \text{---------- (A)}$

For $3 \leq i \leq n$,

$f'[P_2] = f(u) + f(u_i) + f(u_{ui})$

$= 2n + 3 + 3n + 4 + i - 2 + 4n + 2 + n + 1 - i$

$= 10n + 8 \quad \text{---------- (B)}$

For $1 \leq i \leq n$, 


\[ f[\text{P}_3] = f(u_{i1}) + f(u_{i2}) + f(u_{i1}u_i) + f(u_{i1}u_{i2}) \]
\[ = i + n + 2 + n + 1 - i + 2n + 3 + i + 5n + 1 + n + 1 - i \]
\[ = 10n + 8 \quad \square \ 
\]

From (A), (B) and (C), we conclude that G admits ψ - magic graphoidal total labeling. Hence, Double Crowned Star \( K_{1,n} \odot 2K_1 \) is magic graphoidal. For example, consider the graph \( K_{1,5} \odot 2K_1 \) shown in figure 5.

![Figure 5. \( K_{1,5} \odot 2K_1 \)](image)

Clearly, \( \psi = \{ (u_{11},u_{i1}u_{i2}), (u_{21},u_{i2}), (u_{i1},u_{i3},u_{i5}), (u_{41},u_{i4},u_{i2}), (u_{i},u,v,u_{i2}), (u,u_1), (u,u_4), (u,u_5) \} \) is a minimum graphoidal cover and \( K_{1,5} \odot 2K_1 \) is magic graphoidal. Here the constant \( K = 58 \).

### 3.6. Theorem: \( P_m \odot 2K_1 \) is magical graphoidal.

**Proof:** Let \( G = P_m \odot 2K_1 \)

- \( V(G) = \{ u_i: 1 \leq i \leq m \} \) \( \cup \{ u_j: 1 \leq j \leq m-1 \} \) and
- \( E(G) = \{ [(u_i, u_{i+1}): 1 \leq i \leq m-1] \cup [(u_{i1}, u_{i2}): 1 \leq i \leq m, 1 \leq j \leq 2] \} \)

Let \( \psi = \{ \mathcal{P}_1 = [(u_{11},u_{i1}u_{i2}): 1 \leq i \leq m], \mathcal{P}_2 = [(u_{i},u_{i+1}): 1 \leq i \leq m-1] \} \)

Define \( f: V \rightarrow \{0, 1, 2, ..., 6m-1\} \) by

- \( f(u_{i1}) = i \quad 1 \leq i \leq m \)
- \( f(u_{i2}) = 2m+1-i \quad 1 \leq i \leq m \)

\[
    f(u_{2i}) = \begin{cases} 
        2m+i & \text{if } m \text{ is even} \\
        2m+i & \text{if } m \text{ is odd} \\
    \end{cases} 
\]

- \( 1 \leq i \leq \frac{m}{2} \)
- \( 1 \leq i \leq \frac{m-1}{2} \)
\[
f(u_1) = \begin{cases} 
\frac{5m}{2} + i & 1 \leq i \leq m \text{ if } m \text{ is even} \\
\frac{5m-1}{2} + i & 1 \leq i \leq m \text{ if } m \text{ is odd} 
\end{cases}
\]

\[
f(u_{2i+1}) = \begin{cases} 
\frac{7m}{2} + i & 1 \leq i \leq \frac{m}{2} \text{ if } m \text{ is even} \\
\frac{7m-1}{2} + i & 1 \leq i \leq \frac{m+1}{2} \text{ if } m \text{ is odd} 
\end{cases}
\]

\[
f(u_{m+1-i, 2}) = \begin{cases} 
4m + i & 1 \leq i \leq m - 1 \\
5m - 1 + i & 1 \leq i \leq m
\end{cases}
\]

**Case (i) :** when \( m \) is odd

For \( 1 \leq i \leq m \),

\[
f^*(P_1) = f(u_1) + f(u_2) + f(u_{i1}u_i) + f(u_{i1}u_{i2})
\]

\[
= \frac{5m-1}{2} + i + 5m - 1 + m + 1 - i + 2m + 1 - i
\]

\[
= \frac{21m+1}{2} \quad \text{(A)}
\]

For \( 1 \leq i \leq m - 1, \ i \equiv 1 \mod 2 \)

\[
f^*[P_2] = f(u_i) + f(u_{i+1}) + f(u_{ui+1})
\]

\[
= \frac{7m-1}{2} + \frac{i+1}{2} + 2m + \frac{i+1}{2} + 4m + m - i
\]

\[
= \frac{21m+1}{2} \quad \text{(B)}
\]

For \( 1 \leq i \leq m - 1, \ i \equiv 0 \mod 2 \)

\[
f^*[P_2] = f(u_i) + f(u_{i+1}) + f(u_{ui+1})
\]

\[
= 2m + \frac{i}{2} + \frac{7m-1}{2} + \frac{i+2}{2} + 4m + m - i
\]

\[
= \frac{21m+1}{2} \quad \text{(C)}
\]

From (A), (B) and (C), we conclude that \( G \) admits \( \psi \) - magic graphoidal total labeling. Hence, \( P_m \cong 2K_1 \) (m-odd) is magic graphoidal.

For example, consider the graph \( P_5 \cong 2K_1 \) shown in figure 6.1.

**Case (ii) :** when \( m \) is even

For \( 1 \leq i \leq m \),

\[
f^*[P_1] = f(u_1) + f(u_2) + f(u_{i1}u_i) + f(u_{i1}u_{i2})
\]
\[
\frac{5m}{2} = i + 5m - 1 + m + 1 - i + i + 2m + 1 - i \\
= \frac{21m + 2}{2} \quad \text{(A)}
\]

For \(1 \leq i \leq m - 1\), \(i \equiv 1 \mod 2\)
\[f'[P_2] = f(u_i) + f(u_{i+1}) + f(u_{i+1})
\]
\[= \frac{7m + \frac{i + 1}{2} + 2m + \frac{i + 1}{2} + 4m + m - i}{2}
\]
\[= \frac{21m + 2}{2} \quad \text{(B)}
\]

For \(1 \leq i \leq m - 1\), \(i \equiv 0 \mod 2\)
\[f'[P_2] = f(u_i) + f(u_{i+1}) + f(u_{i+1})
\]
\[= \frac{2m + \frac{i}{2} + 7m + \frac{i}{2} + 4m + m - i}{2}
\]
\[= \frac{21m + 2}{2} \quad \text{(C)}
\]

From (A), (B) and (C), we conclude that \(G\) admits \(\psi\) - magic graphoidal total labeling. Hence, \(P_m \bigotimes 2K_1\) (m-even) is magic graphoidal.

For example, consider the graph \(P_4 \bigotimes 2K_1\) shown in figure 6.2.

![Figure 6.1 P_5 \bigotimes 2K_1](image)

Clearly, \(\psi = \{(u_{11}, u_1, u_{12}), (u_{21}, u_2, u_{22}), (u_{31}, u_3, u_{32}), (u_{41}, u_4, u_{42}), (u_{51}, u_5, u_{52}), (u_1, u_2), (u_2, u_3), (u_3, u_4), (u_4, u_5)\}\) is a minimum graphoidal cover and \(P_4 \bigotimes 2K_1\) is magic graphoidal. Here the constant \(K = 53\).
Clearly, \( \psi = \{ (u_{i1}, u_{i2}), (u_{21}, u_{22}), (u_{31}, u_{32}), (u_{41}, u_{42}), (u_{1}, u_{2}), (u_{2}, u_{3}), (u_{3}, u_{4}) \} \) is a minimum graphoidal cover and \( P_4 \square 2K_1 \) is magic graphoidal. Here the constant \( K = 43 \).

### 3.7. Theorem

\( P_m \square K_{1,3} \) is magic graphoidal.

#### Proof

Let \( G = P_m \square K_{1,3} \)

\[
V(G) = \{(u_i, v_j) : 1 \leq i \leq m, 1 \leq j \leq 3 \}
\]

\[
E(G) = \{(u_i, u_{i+1}) : 1 \leq i \leq m-1 \} \cup \{(v_j, v_{j+1}) : 1 \leq j \leq 3 \}
\]

with \( u_i = u_{i+1}, 1 \leq i \leq m \)

Let \( \psi = \{ (v_{i1}, v_{i2}, v_{i3}) : 1 \leq i \leq m \}, \)

\[
P_1 = \{ (v_i, v_{i+1}) : 1 \leq i \leq m-2 \},
\]

\[
P_2 = \{ (v_{m-1}, u_{m-1}, u_m, v_m) \}
\]

Define \( f : V \cup E \rightarrow \{1, 2, ..., 8m-1\} \) by

- \( f(v_m) = 1 \)
- \( f(v_{i1}) = i+1 \) \( 1 \leq i \leq m \)
- \( f(u_i, v_i) = m+1+i \) \( 1 \leq i \leq m-1 \)
- \( f(u_{m+1}, u_{m+i}) = 2m+i \) \( 1 \leq i \leq m-1 \)
- \( f(v_{m+1}, v_{m+i}) = 3m-1+i \) \( 1 \leq i \leq m \)
- \( f(v_{i1}) = 4m-1+i \) \( 1 \leq i \leq m \)
- \( f(v_i) = 5m-1+i \) \( 1 \leq i \leq m-1 \)
- \( f(u_{m}, v_m) = 6m-1 \)
- \( f(u_{m-i}) = 6m+i \) \( 1 \leq i \leq m-2 \)
- \( f(v_{m+1}, v_{m+i}) = 7m-2+i \) \( 1 \leq i \leq m \)

For \( 1 \leq i \leq m \),

\[
f'[P_1] = f(v_{i1}) + f(v_{i2}) + f(v_{i1}, v_i) + f(v_i, v_{i2})
\]

\[
= 4m-1+i + 7m-2+m+1-i + i+1+3m-1+m+1-i
\]

\[
= 16m-1 \quad \text{(A)}
\]

For \( 1 \leq i \leq m-2 \),

\[
f'[P_2] = f(v_i) + f(u_{i-1}) + f(v_i, u_i) + f(u_i, u_{i+1})
\]

\[
= 5m-1+i+6m+(m-i-1) + m+1+i+2m+m-i
\]

\[
= 16m-1 \quad \text{(B)}
\]
\[ f'[P_3] = f(v_{m-1}) + f(v_m) + f(v_{m-1}um-1) + f(um-1um) + f(umvm) \\
= 6m - 2 + 1 + 2m + 2m + 1 + 6m - 1 \\
= 16m - 1 \quad \text{(C)} \]

From (A), (B) and (C), we conclude that G admits \( \psi \) - magic graphoidal total labeling. Hence, \( P_m \cong K_{1,3} \) is magic graphoidal.

For example, consider the graph \( P_4 \cong K_{1,3} \) shown in figure 7.

![Figure 7. \( P_4 \cong K_{1,3} \)](image)

Clearly, \( \psi = \{(v_{11}, v_1, v_{12}), (v_{21}, v_2, v_{22}), (v_{31}, v_3, v_{32}), (v_{41}, v_4, v_{42}), (v_1, u_1, u_2), (v_1, u_2, u_3), (v_3, u_3, u_4, v_4)\} \)

is magic graphoidal. Here the constant \( K = 63 \).

**REFERENCES**


