Magic Graphoidal on Class of Trees

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Abstract: B.D.Acharya and E. Sampathkumar [1] defined Graphoidal cover as partition of edge set of a graph G into internally disjoint paths (not necessarily open). The minimum cardinality of such cover is known as graphoidal covering number of G.

Let G = {V, E} be a graph and let ψ be a graphoidal cover of G. Define f:

\[ V \cup E \to \{1, 2, \ldots, p+q\} \]  such that for every path \( P = (v_0,v_1,v_2, \ldots v_n) \) in \( \psi \) with \( f^*(P) = f(v_0) + f(v_n) + \sum_{i=1}^{n} f(v_i-1) = k \), a constant, where \( f^* \) is the induced labeling on \( \psi \). Then, we say that G admits \( \psi \) - magic graphoidal total labeling of G.

A graph G is called magic graphoidal if there exists a minimum graphoidal cover \( \psi \) of G such that G admits \( \psi \) - magic graphoidal total labeling.

In this paper, we proved that [P_n;S_1], [P_n;S_2], T(n), P_m \( \ominus \) K_{1,3}, P_m \( \ominus \) 2K_1 and K_{1,n} 2 \( \ominus \) 1 are magic graphoidal

1. INTRODUCTION

By a graph, we mean a finite simple and undirected graph. The vertex set and edge set of a graph G are denoted by V(G) and E(G) respectively. Terms and notations not used here are as in [3].

1.1. Definition: Let \( S_1 = (v_0, v_1) \) be a star and let \( [P_n; S_1] \) be the graph obtained from n copies of \( S_1 \) and the path \( P_n = (u_1, u_2, \ldots, u_n) \) by joining \( u_j \) with the vertex \( v_0 \) of the \( j^{th} \) copy of \( S_1 \) by means of an edge, for \( 1 \leq j \leq n \).

1.2. Definition: Let \( S_2 = (v_0,v_1,v_2) \) be a star and let \( [P_n; S_2] \) be the graph obtained from n copies of \( S_2 \) and the path \( P_n = (u_1, u_2, \ldots, u_n) \) by joining \( u_j \) with the vertex \( v_0 \) of the \( j^{th} \) copy of \( S_2 \) by means of an edge, for \( 1 \leq j \leq n \).

1.3. Definition: Let T be any Tree. Denote the tree, obtained from T by considering two copies of T by adding an edge between them, by \( T(2) \) and in general the graph obtained from \( T(n-1) \) and T by adding an edge between them is denoted by \( T(n) \).

1.4. Result [11]: For a Tree T, \( \gamma(T) = n-1 \) where n is the number of pendent vertices of G.

2. PRELIMINARIES

Let G = {V, E} be a graph with p vertices and q edges. A graphoidal cover \( \psi \) of G is a collection of (open) paths such that

(i) every edge is in exactly one path of \( \psi \)

(ii) every vertex is an interval vertex of atmost one path in \( \psi \).

We define \( \gamma(G) = \min_{\psi \in \zeta} |\psi| \),

where \( \zeta \) is the collection of graphoidal covers \( \psi \) of G and \( \gamma \) is graphoidal covering number of G.
Let $\psi$ be a graphoidal cover of $G$. Then we say that $G$ admits $\psi$ - magic graphoidal total labeling of $G$ if there exists a bijection $f : V \cup E \rightarrow \{1, 2, \ldots, p + q\}$ such that for every path $P = (v_0v_1v_2 \ldots v_n)$ in $\psi$, then $\hat{f}(P) = f(v_0) + f(v_n) + \sum_{i=1}^{n} f(v_{i-1}v_i) = k$, a constant, where $\hat{f}$ is the induced labeling of $\psi$. A graph $G$ is called magic graphoidal if there exists a minimum graphoidal cover $\psi$ of $G$ such that $G$ admits $\psi$ - magic graphoidal total labeling.

3. Magical Graphoidal on Trees

3.1. Theorem : $[P_n; S_1], (n - even)$ is magic graphoidal.

Proof: Let $G = [P_n; S_1]$

Let $V(G) = \{u_i, v_i, w_i : 1 \leq i \leq n\}$ and $E(G) = \{(u_i, v_i) \cup (v_i, w_i) : 1 \leq i \leq n \} \cup \{(u_i, u_{i+1}) : 1 \leq i \leq n-1\}$

Define $f : V \cup E \rightarrow \{1, 2, \ldots, p+q\}$ by

$f(w_i) = 1$

$f(w_i, v_i) = 2$

$f(v_i, u_i) = 3$

$f(u_{i+1}) = 6 + i \quad 1 \leq i \leq n - 2$

$f(w_i + 1) = 6n - i \quad 1 \leq i \leq n - 1$

$f(v_i + 1, w_i + 1) = 4n + 1 + i \quad 1 \leq i \leq n - 1$

$f(u_i + 1, v_i + 1) = 3n + 3 - 2i \quad 1 \leq i \leq (n-2) - 1$

$f\left(\begin{array}{c}
u_n \\
v_{n+1}
\end{array}\right) = 3n + 4 - 2i \quad 1 \leq i \leq \frac{n}{2}$

$f(u_i, u_{i+1}) = \frac{3n}{2} + 4 + i \quad 1 \leq i \leq \frac{n}{2} - 1$

$f\left(\begin{array}{c}
u_n \\
u_{n+1}
\end{array}\right) = n + 4 + i \quad 1 \leq i \leq \frac{n}{2}$

Let $\psi = \{P_1=(w_1, v_1, u_1, u_2, v_2, w_2), P_2=(u_i, u_{i+1}, v_{i+1}, w_{i+1}) : 2 \leq i \leq n - 1\}$

$f^\prime[P_1] = f(w_1) + f(w_2) + f(v_1, v_1) + f(v_1, u_1) + f(u_1, u_2) + f(u_2, v_2) + f(v_2, w_2)$

$= 1 + 6n - 1 + 2 + 3 + \frac{3n}{2} + 4 + 1 + 3n + 3 - 2 + 4n + 2$

$= 13n + \frac{3n}{2} + 13 \quad \text{(A)}$

For $2 \leq i \leq (n/2) - 1$,

$f^\prime[P_2] = f(u_i) + f(w_{i+1}) + f(u_{i+1}, u_{i+1}) + f(u_{i+1}, v_{i+1}) + f(v_{i+1}, w_{i+1})$

$= 6 + i - 1 + 6n - i + (3n/2) + 4 + i + 3n + 3 - 2i + 4n + 1 + i$

$= 13n + (3n/2) + 13 \quad \text{(B)}$

For $(n/2) \leq i \leq n - 1$, 


\[ f^*[P_2] = f(u_i) + f(w_{i+1}) + f(u_{i}u_{i+1}) + f(u_{i+1}v_{i+1}) + f(v_{i+1}w_{i+1}) \]
\[ = 6 + i - 1 + 6n - i + n + 4 + i - (n/2) + 1 + 3n + 4 - 2(i + 1 - (n/2)) + 4n + 1 + i \]
\[ = 13n + (3n/2) + 13 \quad \text{(C)} \]

From (A), (B) and (C), we conclude that \( G \) admits \( \psi \) - magic graphoidal total labeling. Hence, \([P_n ; S_1]\) (n - even) is magic graphoidal.

For example, consider the graph \([P_6 ; S_1]\) shown in figure 1.

**Figure 1** \([P_6 ; S_1]\)

Clearly, \( \psi = \{(w_1, v_1, u_1, u_2, v_2, w_2), (u_2, u_3, v_3, w_3), (u_3, u_4, v_4, w_4), (u_4, u_5, v_5, w_5), (u_5, u_6, v_6, w_6)\} \) is a minimum graphoidal cover and \([P_6 ; S_1]\) is magic graphoidal. Here the constant \( K = 100 \)

**3.2. Theorem:** \([P_n ; S_1]\), (n - odd) is magic graphoidal.

**Proof:** Let \( G = [P_n ; S_1] \)

Let \( V(G) = \{u_i, v_i, w_i: 1 \leq i \leq n\} \) and \( E(G) = \{(u_iu_j) \cup (v_iw_i): 1 \leq i \leq n\} \cup [(u_iu_{i+1}): 1 \leq i \leq n-1] \)

Define \( f: V \cup E \rightarrow \{1, 2, ..., p+q\} \) by

\[
\begin{align*}
f(w_1) &= 1 \\
f(w_1v_1) &= 2 \\
f(v_1u_1) &= 3 \\
f(u_{i+1}) &= 5 + 2i \quad 1 \leq i \leq (n-1)/2 \\
f(w_{i+1}) &= 6n - i \quad 1 \leq i \leq n - 1 \\
f(v_{i+1}w_{i+1}) &= 4n + 1 + i \quad 1 \leq i \leq n - 1 \\
f(u_1u_2) &= n + 5 \\
f(u_{i+1}u_{i+2}) &= \left\lfloor \frac{3n+1}{2} \right\rfloor - i \quad 1 \leq i \leq \frac{n-1}{2} \\
f(u_{n+1}u_{n+2}) &= \left\lfloor \frac{3n+9}{2} \right\rfloor + i \quad 1 \leq i \leq \frac{n-3}{2} \\
f(u_{i+2}v_{i+2}) &= 3n + 2 - i \quad 1 \leq i \leq n - 2 \\
f(u_3v_2) &= \left\lfloor \frac{7n+3}{2} \right\rfloor \\
\end{align*}
\]
Let $\psi = \{P_1 = (w_1,v_1,u_1,u_2,v_2,w_2), P_2 = (u_i,u_{i+1},v_{i+1},w_{i+1}) : 2 \leq i \leq n - 1\}$

\[
\ell_P^f = f(w_1) + f(w_2) + f(v_1w_1) + f(u_1u_2) + f(u_2v_2) + f(v_2w_2)
\]

\[
= 1 + 6n - 1 + 2 + 3 + n + 5 + \frac{7n+3}{2} + 4n + 1 + 1
\]

\[
= 14n + \frac{n+1}{2} + 13 \quad \text{(A)}
\]

For $2 \leq i < \frac{n+1}{2}$

\[
\ell_{P_2}^f = f(u_i) + f(w_{i+1}) + f(u_iw_{i+1}) + f(v_{i+1}w_{i+1})
\]

\[
= 5 + 2(i-1) + 6n - i + \frac{3n+11}{2} - 1 + 3n + 2 - (i-1) + 4n + 1 + 1
\]

\[
= 14n + \frac{n+1}{2} + 13 \quad \text{(B)}
\]

For $\frac{n+1}{2} \leq i \leq n - 1$.

\[
\ell_{P_2}^f = f(u_i) + f(w_{i+1}) + f(u_iw_{i+1}) + f(v_{i+1}w_{i+1})
\]

\[
= 6 + 2 \left(i - \frac{n+1}{2}\right) + 6n - i + \frac{3n+9}{2} + (n-i) + 3n + 2 - (i-1) + 4n + 1 + 1
\]

\[
= 14n + \frac{n+1}{2} + 13 \quad \text{(C)}
\]

From (A), (B) and (C), we conclude that $\psi$ is minimum magic graphoidal cover. Hence, $[P_n ; S_1]$, $(n - \text{even})$ is magic graphoidal.

For example, consider the graph $[P_7 ; S_1]$ shown in figure 2.

Clearly, $\psi = \{(w_1,v_1,u_1,u_2,v_2,w_2), (u_2,u_3,v_3,w_3), (u_3,u_4,v_4,w_4), (u_4,u_5,v_5,w_5), (u_5,u_6,v_6,w_6), (u_6,u_7,v_7,w_7)\}$ is a minimum graphoidal cover and $[P_7 ; S_1]$ is magic graphoidal. Here the constant $K = 115$.

### 3.3. Theorem:

$[P_n ; S_2] \text{ is magic graphoidal.}$

**Proof:** Let $G = [P_n ; S_2]$

$V(G) = \{(u_i, v_i) : 1 \leq i \leq n\}$
\[E(G) = \{ (u_i u_{i+1}) : 1 \leq i \leq n-1 \} \cup \{(u_i v_i) : 1 \leq i \leq n \} \cup \{(v_i v_{i+1}) : 1 \leq i \leq n \}\]

Define \( f : V \cup E \rightarrow \{1, 2, \ldots, p+q\} \) by

\[
\begin{align*}
&f(v_1) = 1 \\
&f(u_1 v_1) = 2 f(u_1) = 3 \\
&f(u_{i+1}) = 3 + i \quad 1 \leq i \leq n - 2 \\
&f(u_n) = p + q \\
&f(v_i) = n + 1 + i \quad 1 \leq i \leq n \\
&f(v_{i+1}) = 8n - 1 - i \quad 1 \leq i \leq n - 1 \\
&f(v_{i+2}) = 7n - i \quad 1 \leq i \leq n \\
&f(u_i v_{i+1}) = 4n + i \quad 1 \leq i \leq n - 1 \\
&f(u_i u_{i+1}) = 4n + 1 - i \quad 1 \leq i \leq n - 1 \\
&f(v_i v_{i+1}) = 3n + 2 - i \quad 1 \leq i \leq n
\end{align*}
\]

Let \( \psi = \{(v_1, v_2, v_3) : 1 \leq i \leq n\}, P_2 = (v_1, u_1, u_2, v_2), P_3 = [(u_i u_{i+1}, v_{i+1}) : 2 \leq i \leq n-1]\) \( P_3 = [(u_i u_{i+1}, v_{i+1}) : 2 \leq i \leq n-1]\)

For \(1 \leq i \leq n\),

\[
\begin{align*}
&f^* [P_1] = f(v_1) + f(v_2) + f(v_1 v_1) + f(v_1 v_2) \\
&= 8 + 1 + 7n - i + 3n - 3 + 2 - i + 5n - 1 + i \\
&= 16n + 2 \quad \text{(A)}
\end{align*}
\]

\[
\begin{align*}
&f^* [P_2] = f(v_1) + f(v_2) + f(v_1 u_1) + f(u_1 u_2) + f(u_2 v_2) \\
&= 1 + 8n - 2 + 2 + 4n + 4n + 1 \\
&= 16n + 2 \quad \text{(B)}
\end{align*}
\]

For \(2 \leq i \leq n-1\),

\[
\begin{align*}
&f^* [P_3] = f(u_i) + f(v_{i+1}) + f(u_i u_{i+1}) + f(u_{i+1} v_{i+1}) \\
&= 3 + i + 1 + 8n - 1 - i + 4n + 1 - i + 4n + 1 \\
&= 16n + 2 \quad \text{(C)}
\end{align*}
\]

From (A), (B) and (C), we conclude that \( G \) admits \( \psi \) - magic graphoidal total labeling. Hence, \([P_n : S_2]\) is magic graphoidal.

For example, consider the graphs \([P_3 ; S_2]\) and \([P_4 ; S_2]\) shown in figure 3.1 and 3.2.

![Figure 3.1 [P_3 ; S_2]](image)

Clearly, \( \psi = \{(v_{11}, v_1, v_{12}), (v_{21}, v_2, v_{22}), (v_{31}, v_3, v_{32}), (v_1, u_1, u_2, v_2), (u_2, u_3, v_3)\} \) is a minimum graphoidal cover and \([P_3 ; S_2]\) is magic graphoidal. Here the constant \( K = 50 \)
Clearly, $\psi = \{(v_{11},v_{1},v_{12}), (v_{21},v_{2},v_{22}), (v_{31},v_{3},v_{32}), (v_{41},v_{4},v_{42}), (v_{1},u_{1},u_{2},v_{2}), \ldots, (A), (B) \text{ and } (C)\}$ we conclude that $G$ admits $\psi$ - magic graphoidal total labeling. Hence, $T(n)$ is magic graphoidal.

3.4. Theorem: For a Tree, $T(n)$ is magic graphoidal.

Proof:

Let $T(n)$ be a graph such that

$V[T(n)] = \{u_{i1}, u_{i2}, u_{i3}, u_{i4}, u_{i5} : 1 \leq i \leq n \}$ and

$E[T(n)] = \{(u_{i1}u_{i2}), (u_{i2}u_{i3}), (u_{i3}u_{i4}), (u_{i4}u_{i5}) : 1 \leq i \leq n \} \cup \{(u_{in}u_{i+1}n) : 1 \leq i \leq n-1\}$

Define $f: V \cup E \rightarrow \{1, 2, \ldots, p+q\}$ by

- $f(u_{i1}) = i \quad 1 \leq i \leq n$
- $f(u_{i3}) = n + 1$
- $f(u_{i4}) = n + 2$
- $f(u_{i2}) = 2n + 3 - i \quad 1 \leq i \leq n$
- $f(u_{i+1,2}u_{i+3,5}) = 5n - 3 - 2i \quad 1 \leq i \leq n - 3$
- $f(u_{i+1,4}u_{i+1,5}) = 4n - 1 + i \quad n - 2 \leq i \leq n - 1$
- $f(u_{i+1,2}n_{i+1,i,3}) = 5n - 2 + i \quad 1 \leq i \leq n$
- $f(u_{i+1,3}u_{i+1,i,4}) = 6n - 2 + i \quad 1 \leq i \leq n$
- $f(u_{i+1,3}u_{i+1,i,5}) = 7n - 2 + i \quad 1 \leq i \leq n - 1$
- $f(u_{i+1,3}) = 8n - 2 + 2(i-1) \quad 1 \leq i \leq n$
- $f(u_{i+1,3}) = 8n - 1 + 2(i-1) \quad 1 \leq i \leq n - 1$

Let $\psi = \{P_1 = (u_{i1},u_{i2},u_{i3},u_{i4}), P_2 = (u_{i3},u_{i5},u_{i25},u_{i23}), P_3 = \{(u_{i5},u_{i+1,5},u_{i+1,3}) : 2 \leq i \leq n - 1\}\}$

- $f'[P_1] = f(u_{i1}) + f(u_{i4}) + f(u_{i1}u_{i2}) + f(u_{i2}u_{i3}) + f(u_{i4}u_{i4})$
  \[= i + 8n - 2 + 2(i-1) + 2n + 3 - i + 5n - 2 + (n + 1 - i) + 6n - 2 + (n + 1 - i)\]
  \[= 23n - 3 \quad \text{(A)}\]

- $f'[P_2] = f(u_{i3}) + f(u_{i+1,3}) + f(u_{i3}u_{i+1,3}) + f(u_{i+1,5}u_{i+1,3})$
  \[= 2n + 2i - 1 + 8n - 1 + 2(i - 1) + 5n - 3 - 2(i - 2) + 7n - 2 + n - i\]
  \[= 23n - 3 \quad \text{(B)}\]

- $f'[P_3] = f(u_{i5}) + f(u_{i3}u_{i5}) + f(u_{i1,5}u_{25}) + f(u_{i3}u_{i25})$
  \[= n + 1 + 8n - 1 + 2(i - 1) + 5n - 3 - 2(i - 2) + 7n - 2 + n - i\]
  \[= 23n - 3 \quad \text{(C)}\]

From (A), (B) and (C), we conclude that $G$ admits $\psi$ - magic graphoidal total labeling. Hence, $T(n)$ is magic graphoidal.
For example, consider the graph $T_3$ shown in figure 4.

\[
\begin{align*}
\psi &= \{(u_{11}, u_{12}, u_{13}, u_{14}), (u_{21}, u_{22}, u_{23}, u_{24}), (u_{31}, u_{32}, u_{33}, u_{34}), (u_{13}, u_{15}, u_{25}, u_{23}), (u_{25}, u_{35}, u_{33})\} \\
\text{Clearly, } \psi &= 2n + 3 + 3n + 4 + i - 2 + 4n + 2 + n + 1 - i \\
&= 10n + 8 \quad \text{(B)}
\end{align*}
\]

For $1 \leq i \leq n$, 

\[
\text{Proof: Let } G = K_{1,n} \otimes 2K_1.
\]

\[
\begin{align*}
V(G) &= \{u, [u_1 : 1 \leq i \leq n], [(u_{i1}, u_{i2}) : 1 \leq i \leq n]\} \quad \text{and} \\
E(G) &= \{ ((u_{i1}), 1 \leq i \leq n) \cup (u_{i1}u_{i2}) : 1 \leq i \leq n \}
\end{align*}
\]

Define $f : V \cup E \rightarrow \{1, 2, ..., p+q\}$ by

\[
\begin{align*}
f(u) &= 2n + 3 \\
f(u_{i1}) &= i \\
f(u_{i2}) &= n + 2 \\
f(u_{i1}u_{i2}) &= n + 2i + 1 \\
f(u_{i1}u_{i1}) &= 2n + 3 + i \quad 1 \leq i \leq n \\
f(u_{i1}u_{i2}) &= 3n + 4 \\
f(u_{i2}u_{i1}) &= 3n + 4 + i \quad 1 \leq i \leq n - 2 \\
f(u_{i1}u_{i1}) &= 4n + 2 + i \quad 1 \leq i \leq n - 1 \\
f(u_{n+1,i}u_{n+1,i+2}) &= 5n + 1 + i \quad 1 \leq i \leq n
\end{align*}
\]

Let $\psi = \{P_1 = (u_1, u_2), P_2 = [(u_{i1}, u_{i2}) : 3 \leq i \leq n], P_3 = [(u_{i1}, u_{i1}) : 1 \leq i \leq n]\}$

\[
\begin{align*}
f'[P_1] &= f(u_{i1}) + f(u_{i2}) + f(u_{i1}u_{i2}) \\
&= n + 1 + n + 2 + 3n + 4 + 5n + 1 \\
&= 10n + 8 \quad \text{(A)}
\end{align*}
\]

For $3 \leq i \leq n$, 

\[
\begin{align*}
f'[P_2] &= f(u) + f(u_{i1}) + f(u_{i2}) \\
&= 2n + 3 + 3n + 4 + i - 2 + 4n + 2 + n + 1 - i \\
&= 10n + 8 \quad \text{(B)}
\end{align*}
\]

For $1 \leq i \leq n$, 

\[
\begin{align*}
f'[P_3] &= f(u_{i1}) + f(u_{i1}u_{i2}) + f(u_{i1}u_{i1}) \\
&= 2n + 3 + 3n + 4 + i - 2 + 4n + 2 + n + 1 - i \\
&= 10n + 8 \quad \text{(B)}
\end{align*}
\]
\[ f[P_3] = f(u_{i1}) + f(u_{i2}) + f(u_{i1}u_i) + f(u_{i1}u_{i2}) \]
\[ = i + n + 2 + n + 1 - i + 2n + 3 + i + 5n + 1 + n + 1 - i \]
\[ = 10n + 8 \quad \text{--------- (C)} \]

From (A), (B) and (C), we conclude that G admits \( \psi \) - magic graphoidal total labeling. Hence, Double Crowned Star \( K_{1,n} \odot 2K_1 \) is magic graphoidal.

For example, consider the graph \( K_{1,5} \odot 2K_1 \) shown in Figure 5.

Clearly, \( \psi = \{(u_{11},u_{1},u_{12}), (u_{21},u_{2},u_{22}), (u_{11},u_{3},u_{32}), (u_{41},u_{4},u_{42}), (u_{1},u,v,u_{2}), (u,u_3), (u,u_4), (u,u_5)\} \) is a minimum graphoidal cover and \( K_{1,5} \odot 2K_1 \) is magic graphoidal. Here the constant \( K = 58 \).

3.6. Theorem: \( P_m \odot 2K_1 \) is magical graphoidal.

Proof: Let \( G = P_m \odot 2K_1 \)

\[ V(G) = \{ (u_i : 1 \leq i \leq m), (u_{ij} : 1 \leq i \leq m, 1 \leq j \leq 2) \} \text{ and} \]
\[ E(G) = \{ [(u_i,u_{i+1}) : 1 \leq i \leq m-1] \cup [(u_i,u_{ij}) : 1 \leq i \leq m, 1 \leq j \leq 2] \} \]

Let \( \psi = \{ P_1 = [(u_{i1},u_{i},u_{i2}) : 1 \leq i \leq m], P_2 = [(u_i,u_{i+1}) : 1 \leq i \leq m-1] \} \)

Define \( f: V \to \{0, 1, 2, ..., 6m-1\} \) by
\[ f(u_{i1}) = i \quad 1 \leq i \leq m \]
\[ f(u_{i2}) = 2m+1-i \quad 1 \leq i \leq m \]
\[ f(u_{2i}) = \begin{cases} 
2m+i & 1 \leq i \leq \frac{m}{2} \quad \text{if } m \text{ is even} \\
2m+i & 1 \leq i \leq \frac{m-1}{2} \quad \text{if } m \text{ is odd}
\end{cases} \]
\[
\begin{align*}
f(u_1) &= \begin{cases} 
\frac{5m}{2} + i & 1 \leq i \leq m \text{ if } m \text{ is even} \\
\frac{5m - 1}{2} + i & 1 \leq i \leq m \text{ if } m \text{ is odd}
\end{cases} \\
f(u_{2i-1}) &= \begin{cases} 
\frac{7m}{2} + i & 1 \leq i \leq \frac{m}{2} \text{ if } m \text{ is even} \\
\frac{7m - 1}{2} + i & 1 \leq i \leq \frac{m+1}{2} \text{ if } m \text{ is odd}
\end{cases} \\
f(u_{m+1-i, 2}) &= 4m + i \quad 1 \leq i \leq m - 1 \\
f(u_{m+1-i, 2}) &= 5m - 1 + i \quad 1 \leq i \leq m 
\end{align*}
\]

**Case (i):** when \( m \) is odd

For \( 1 \leq i \leq m \),
\[
\begin{align*}
f^*[P_1] &= f(u_1) + f(u_2) + f(u_{i1}u_i) + f(u_{i2}) \\
&= \frac{5m - 1}{2} + i + 5m - 1 + m + 1 - i + 2m + 1 - i \\
&= \frac{21m + 1}{2} \quad \text{(A)}
\end{align*}
\]

For \( 1 \leq i \leq m - 1, \ i \equiv 1 \text{ mod } 2 \)
\[
\begin{align*}
f^*[P_2] &= f(u_i) + f(u_{i+1}) + f(u_{i1}u_{i+1}) \\
&= \frac{7m - 1}{2} + \frac{i + 1}{2} + 2m + \frac{i + 1}{2} + 4m + m - i \\
&= \frac{21m + 1}{2} \quad \text{(B)}
\end{align*}
\]

For \( 1 \leq i \leq m - 1, \ i \equiv 0 \text{ mod } 2 \)
\[
\begin{align*}
f^*[P_2] &= f(u_i) + f(u_{i+1}) + f(u_{i1}u_{i+1}) \\
&= 2m + \frac{i}{2} + \frac{7m - 1}{2} + \frac{i + 2}{2} + 4m + m - i \\
&= \frac{21m + 1}{2} \quad \text{(C)}
\end{align*}
\]

From (A), (B) and (C), we conclude that \( G \) admits \( \psi \) - magic graphoidal total labeling. Hence, \( P_m \bigcirc 2K_1 \) (m-odd) is magic graphoidal.

For example, consider the graph \( P_5 \bigcirc 2K_1 \) shown in figure 6.1.

**Case (ii):** when \( m \) is even

For \( 1 \leq i \leq m \),
\[
f^*[P_1] = f(u_1) + f(u_2) + f(u_{i1}u_i) + f(u_{i2})
\]
\[
\begin{align*}
5m \\
2 = & \quad i + 5m - 1 + m + 1 - i + i + 2m + 1 - i \\
= & \quad 21m + 2 \\
\frac{2}{2} \quad \text{------- (A)}
\end{align*}
\]

For \(1 \leq i \leq m - 1, \ i \equiv 1 \mod 2\)

\[
\begin{align*}
\hat{f}[P_2] & = f(u_i) + f(u_{i+1}) + f(u_{i+1}) \\
& = \frac{7m}{2} + \frac{i+1}{2} + 2m + \frac{i+1}{2} + 4m + m - i \\
& = \frac{21m + 2}{2} \quad \text{------- (B)}
\end{align*}
\]

For \(1 \leq i \leq m - 1, \ i \equiv 0 \mod 2\)

\[
\begin{align*}
\hat{f}[P_2] & = f(u_i) + f(u_{i+1}) + f(u_{i+1}) \\
& = 2m + \frac{i}{2} + \frac{7m}{2} - \frac{i+2}{2} + 4m + m - i \\
& = \frac{21m + 2}{2} \quad \text{------- (C)}
\end{align*}
\]

From (A), (B) and (C), we conclude that \(G\) admits \(\psi\) - magic graphoidal total labeling. Hence, \(P_m \odot 2K_1\) (m-even) is magic graphoidal.

For example, consider the graph \(P_4 \odot 2K_1\) shown in figure 6.2.

![Graph 6.1 P_5 \odot 2K_1](image)

Figure 6.1 \(P_5 \odot 2K_1\)

Clearly, \(\psi = \{(u_{11}, u_1, u_{12}), (u_{21}, u_2, u_{22}), (u_{31}, u_3, u_{32}), (u_{41}, u_4, u_{42}), (u_{51}, u_5, u_{52}), (u_1, u_2), (u_2, u_3), (u_3, u_4), (u_4, u_5)\}\) is a minimum graphoidal cover and \(P_4 \odot 2K_1\) is magic graphoidal. Here the constant \(K = 53\).
Clearly, \( \psi = \{(u_1, u_1, u_2), (u_1, u_2, u_2), (u_3, u_3, u_2), (u_3, u_2, u_3), (u_1, u_2), (u_1, u_2), (u_2, u_2), (u_3, u_3)\} \) is a minimum graphoidal cover and \( P_4 \odot 2K_1 \) is magic graphoidal. Here the constant \( K = 43 \).

3.7. Theorem: \( P_m \odot K_{1,3} \) is magic graphoidal.

Proof: Let \( G = P_m \odot K_{1,3} \)

\[
V(G) = \{(u_i, v_j) : 1 \leq i \leq m, 1 \leq j \leq 3\}
\]

\[
E(G) = \{(u_i, u_{i+1}) : 1 \leq i \leq m-1\} \cup \{(v_i, v_j) : 1 \leq i \leq m, 1 \leq j \leq 3\}
\]

with \( u_i = u_{3i}, 1 \leq i \leq m \)

Let \( \psi = \{(v_1, v_1, v_2) : 1 \leq i \leq m\}, P_1 = (v_i, u_i, u_{i+1}) : 1 \leq i \leq m-2\), \( P_3 = (v_{m-1}, u_{m-1}, u_m, v_m)\)

Define \( f : V \cup E \rightarrow \{1, 2, ..., 8m-1\} \) by

\[
\begin{align*}
f(v_m) &= 1 \\
f(v_i, v_j) &= 1 + i \\
f(u_i, v_j) &= m + 1 + i \\
f(u_i, u_{i+1}) &= 2m + i \\
f(v_{m+1-i}, v_{m+1-i,2}) &= 3m - 1 + i \\
f(v_{i+1}) &= 4m - 1 + i \\
f(v_i) &= 5m - 1 + i \\
f(u_m) &= 6m - 1 \\
f(u_{m-i}) &= 6m + i \\
f(v_{m+1-i,2}) &= 7m - 2 + i
\end{align*}
\]

For \( 1 \leq i \leq m, \)

\[
f'[P_1] = f(v_1) + f(v_2) + f(v_1, v_2) + f(v_1, v_2) = 4m - 1 + i + 7m - 2 + m - 1 + i + 1 + 3m - 1 + m + 1 - i
\]

\[
= 16m - 1 \quad \text{-------- (A)}
\]

For \( 1 \leq i \leq m - 2, \)

\[
f'[P_2] = f(v_i) + f(u_{i+1}) + f(v_i, u_i) + f(u_i, u_{i+1}) = 5m - 1 + i + 6m + (m - i - 1) + m + 1 + i + 2m + m - i
\]

\[
= 16m - 1 \quad \text{-------- (B)}
\]
\[ f[P_3] = f(v_{m-1}) + f(v_m) + f(u_{m-1}u_m) + f(u_m v_m) \]
\[ = 6m - 2 + 1 + 2m + 2m + 1 + 6m - 1 \]
\[ = 16m - 1 \quad \text{(C)} \]

From (A), (B) and (C), we conclude that G admits \( \psi \) - magic graphoidal total labeling. Hence, \( P_m \bowtie K_{1,3} \) is magic graphoidal.

For example, consider the graph \( P_4 \bowtie K_{1,3} \) shown in figure 7.

![Figure 7. P_4 \bowtie K_{1,3}](image)

Clearly, \( \psi = \{v_{11}, v_1, v_{12}, (v_{21}, v_2, v_{22}), (v_{31}, v_3, v_{32}), (v_{41}, v_4, v_{42}), (v_1, u_1, u_2), (v_1, u_2, u_3), (v_3, u_3, u_4, v_4)\} \) is magic graphoidal. Here the constant \( K = 63 \).

REFERENCES