Magic Graphoidal on Class of Trees
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Abstract: B.D.Acharya and E. Sampathkumar [1] defined Graphoidal cover as partition of edge set of a graph G into internally disjoint paths (not necessarily open). The minimum cardinality of such cover is known as graphoidal covering number of G.

Let G = \{V, E\} be a graph and let \( \psi \) be a graphoidal cover of G. Define \( f_\psi \) such that for every path \( P = (v_0, v_1, v_2, \ldots, v_n) \) in \( \psi \) with \( f_\psi(P) = f(v_0) + f(v_n) + \sum_{i=1}^{n} f(v_i - v_{i-1}) \), a constant, where \( f_\psi \) is the induced labeling on \( \psi \). Then, we say that G admits \( \psi \) - magic graphoidal total labeling of G.

A graph G is called magic graphoidal if there exists a minimum graphoidal cover \( \psi \) of G such that G admits \( \psi \) - magic graphoidal total labeling.

In this paper, we proved that \([P_n:S_1], [P_n:S_2], T(n), P_m \circlearrowleft K_{1,3}, P_m \circlearrowleft 2K_1\) and \(K_{1,n} \circlearrowleft 2\) are magic graphoidal.

1. INTRODUCTION

By a graph, we mean a finite simple and undirected graph. The vertex set and edge set of a graph G are denoted by V(G) and E(G) respectively. Terms and notations not used here are as in [3].

1.1. Definition : Let \( S_1 = (v_0, v_1) \) be a star and let \([P_n ; S_1] \) be the graph obtained from n copies of \( S_1 \) and the path \( P_n = (u_1, u_2, \ldots, u_n) \) by joining \( u_j \) with the vertex \( v_0 \) of the \( j^{th} \) copy of \( S_1 \) by means of an edge, for \( 1 \leq j \leq n \).

1.2. Definition : Let \( S_2 = (v_0, v_1, v_2) \) be a star and let \([P_n ; S_2] \) be the graph obtained from n copies of \( S_2 \) and the path \( P_n = (u_1, u_2, \ldots, u_n) \) by joining \( u_j \) with the vertex \( v_0 \) of the \( j^{th} \) copy of \( S_2 \) by means of an edge, for \( 1 \leq j \leq n \).

1.3. Definition : Let T be any Tree. Denote the tree, obtained from T by considering two copies of T by adding an edge between them, by \( T(2) \) and in general the graph obtained from \( T(n-1) \) and T by adding an edge between them is denoted by \( T(n) \).

1.4. Result [11] : For a Tree T, \( \gamma(T) = n-1 \) where n is the number of pendent vertices of G.

2. PRELIMINARIES

Let G = \{V, E\} be a graph with p vertices and q edges. A graphoidal cover \( \psi \) of G is a collection of (open) paths such that

\( (i) \) \hspace{1cm} every edge is in exactly one path of \( \psi \)
\( (ii) \) \hspace{1cm} every vertex is an interval vertex of atmost one path in \( \psi \).

We define \( \gamma(G) = \min_{\psi \epsilon \zeta} |\psi| \),

where \( \zeta \) is the collection of graphoidal covers \( \psi \) of G and \( \gamma \) is graphoidal covering number of G.
Let $\psi$ be a graphoidal cover of $G$. Then we say that $G$ admits $\psi$-magic graphoidal total labeling of $G$ if there exists a bijection $f: V \cup E \rightarrow \{1, 2, \ldots, p + q\}$ such that for every path $P=(v_0v_1v_2 \ldots v_n)$ in $\psi$, then, $\hat{f^*}(P) = f(v_0) + f(v_n) + \sum_{i=1}^{n} f(v_{i-1}v_{i}) = k$, a constant, where $\hat{f^*}$ is the induced labeling of $\psi$. A graph $G$ is called magic graphoidal if there exists a minimum graphoidal cover $\psi$ of $G$ such that $G$ admits $\psi$-magic graphoidal total labeling.

3. Magical Graphoidal on Trees

3.1. Theorem: $[P_0; S_1]$, (n - even) is magic graphoidal.

Proof: Let $G = [P_n; S_1]$ 

Let $V(G) = \{u_i, v_i, w_i: 1 \leq i \leq n\}$ and $E(G) = \{(u_i, v_i) \cup (v_i, w_i): 1 \leq i \leq n\} \cup [(u_i, u_{i+1}): 1 \leq i \leq n-1]$

Define $f: V \cup E \rightarrow \{1, 2, \ldots, p+q\}$ by

$f(w_1) = 1$
$f(w_1v_1) = 2$
$f(v_1u_1) = 3$
$f(u_{i+1}) = 6 + i \quad 1 \leq i \leq n - 2$
$f(w_{i+1}) = 6n - i \quad 1 \leq i \leq n - 1$
$f(v_{i+1}w_{i+1}) = 4n + 1 + i \quad 1 \leq i \leq n - 1$
$f(u_{i+1}v_{i+1}) = 3n + 3 - 2i \quad 1 \leq i \leq (n/2) - 1$

$f\left(\frac{u_{n+1}}{2}, v_{n+1} \frac{v_{n+1}}{2}\right) = 3n + 4 - 2i \quad 1 \leq i \leq \frac{n}{2}$

$f(u_iu_{i+1}) = \frac{3n}{2} + 4 + i \quad 1 \leq i \leq \frac{n}{2} - 1$

$f\left(\frac{u_{n-1+i}}{2}, u_{n-1+i} \frac{u_{n-1+i}}{2}\right) = n + 4 + i \quad 1 \leq i \leq \frac{n}{2}$

Let $\psi = \{P_1 = (w_1v_1u_1u_2v_2w_2), P_2 = (u_iu_{i+1}, v_{i+1}, w_{i+1}): 2 \leq i \leq n - 1\}$

$f^*[P_1] = f(w_1) + f(w_2) + f(w_1v_1) + f(v_1u_1) + f(u_1u_2) + f(u_2v_2) + f(v_2w_2)$

$= 1 + 6n - 1 + 2 + 3 + \frac{3n}{2} + 4 + 1 + 3n + 3 - 2 + 4n + 2$

$= 13n + \frac{3n}{2} + 13 \quad \text{(A)}$

For $2 \leq i \leq (n/2) - 1,$

$f^*[P_2] = f(u_i) + f(w_{i+1}) + f(u_{i+1}u_{i+1}) + f(u_{i+1}v_{i+1}) + f(v_{i+1}w_{i+1})$

$= 6 + i - 1 + 6n - i + (3n/2) + 4 + i + 3n + 3 - 2i + 4n + 1 + i$

$= 13n + (3n/2) + 13 \quad \text{(B)}$

For $(n/2) \leq i \leq n - 1,$
\[ f[P_2] = f(u_i) + f(w_{i+1}) + f(u_iu_{i+1}) + f(u_{i+1}v_{i+1}) + f(v_{i+1}w_{i+1}) \]

\[ = 6 + i - 1 + 6n - i + n + 4 + i - (n/2) + 1 + 3n + 4 - 2(i + 1 - (n/2)) + 4n + 1 + i \]

\[ = 13n + (3n^2)/2 + 13 \quad \text{(C)} \]

From (A), (B) and (C), we conclude that G admits \( \psi \) - magic graphoidal total labeling. Hence, \([P_n; S_1]\), (n - even) is magic graphoidal.

For example, consider the graph \([P_6; S_1]\) shown in figure 1.

![Figure 1](image-url)

Clearly, \( \psi = \{(w_1,v_1,u_1,u_2,v_2,w_2), (u_2,u_3,v_3,w_3), (u_3,u_4,v_4,w_4), (u_4,u_5,v_5,w_5), (u_5,u_6,v_6,w_6)\} \) is a minimum graphoidal cover and \([P_6; S_1]\) is magic graphoidal. Here the constant \( K = 100 \)

**Theorem:** \([P_n; S_1]\), (n - odd) is magic graphoidal.

**Proof:** Let \( G = [P_n; S_1]\)

Let \( V(G) = \{u_i, v_i, w_i: 1 \leq i \leq n\} \) and

\[ E(G) = \{[u_iw_i] \cup [v_iw_i]: 1 \leq i \leq n \} \cup ([u_iu_{i+1}]: 1 \leq i \leq n-1) \}

Define \( f: V \cup E \rightarrow \{1, 2, \ldots, p+q\} \) by

\[
\begin{align*}
    f(w_1) &= 1 \\
    f(w_1v_1) &= 2 \\
    f(v_1u_1) &= 3 \\
    f(u_{i+1}) &= 5 + 2i \quad 1 \leq i \leq (n-1)/2 \\
    f(w_{i+1}) &= 6n - i \quad 1 \leq i \leq n - 1 \\
    f(v_{i+1}w_{i+1}) &= 4n + 1 + i \quad 1 \leq i \leq n - 1 \\
    f(u_1u_2) &= n + 5 \\
    f(u_{i+1}u_{i+2}) &= \begin{cases} 
        \frac{3n + 1}{2} - i & 1 \leq i \leq \frac{n-1}{2} \\
        \frac{3n + 9}{2} + i & 1 \leq i \leq \frac{n-3}{2} \\
        3n + 2 - i & 1 \leq i \leq n - 2 \\
        \frac{7n+3}{2} & 
    \end{cases} \\
    f(u_1v_2) &= 7n + 3/2
\end{align*}
\]
Let \( \psi = \{P_1 = (w_1,v_1,u_1,u_2,v_2,w_2), P_2 = (u_i,u_{i+1},v_{i+1},w_{i+1}) : 2 \leq i \leq n - 1 \}\)

\[
f'[P_1] = f(w_1) + f(w_2) + f(w_1 v_1) + f(u_1 u_2) + f(u_2 v_2) + f(v_2 w_2)
  = 1 + 6n - 1 + 2 + 3 + n + 5 + \frac{7n + 3}{2} + 4n + 1 + 1
  = 14n + \frac{n+1}{2} + 13 \quad \text{(A)}
\]

For \( 2 \leq i < \frac{n+1}{2} \)

\[
f'[P_2] = f(u_i) + f(w_{i+1}) + f(u_i v_{i+1}) + f(v_{i+1} w_{i+1})
  = 5 + 2(i - 1) + 6n - i + \frac{3n + 11}{2} - (i - 1) + 3n + 2 + (i - 1) + 4n + 1 + i
  = 14n + \frac{n+1}{2} + 13 \quad \text{(B)}
\]

For \( \frac{n+1}{2} \leq i \leq n - 1 \),

\[
f'[P_2] = f(u_i) + f(w_{i+1}) + f(u_i v_{i+1}) + f(v_{i+1} w_{i+1})
  = 6 + 2 \left( i - \frac{n+1}{2} \right) + 6n - i + \frac{3n + 9}{2} + (n - i) + 3n + 2 - (i - 1) + 4n + 1 + i
  = 14n + \frac{n+1}{2} + 13 \quad \text{(C)}
\]

From (A), (B) and (C), we conclude that \( \psi \) is minimum magic graphoidal cover. Hence, \([P_n ; S_i], (n - \text{even}) \) is magic graphoidal.

For example, consider the graph \([P_7 ; S_1]\) shown in figure 2.

Clearly, \( \psi = \{(w_1,v_1,u_1,u_2,v_2,w_2), (u_2,u_3,v_3,w_3), (u_3,u_4,v_4,w_4), (u_4,u_5,v_5,w_5), (u_5,u_6,v_6,w_6), (u_6,u_7,v_7,w_7)\} \) is a minimum graphoidal cover and \([P_7 ; S_1]\) is magic graphoidal. Here the constant \( K = 115 \).

3.3. Theorem : \([P_n ; S_2]\) is magic graphoidal.

Proof: Let \( G = [P_n ; S_2] \)

\[ V(G) = \{(u_i,v_i) : 1 \leq i \leq n\} \]
E(G) = \{(u_iu_{i+1}) : 1 \leq i \leq n-1\} \cup \{(u_iu_i) : 1 \leq i \leq n\} \\
\cup \{(v_iv_{i1}) \cup (v_iv_{i2}) : 1 \leq i \leq n\}

Define \( f : V \cup E \rightarrow \{1, 2, \ldots, p+q\} \) by

\[
\begin{align*}
f(v_i) &= 1 \\
f(u_iv_i) &= 2 \text{ if } & 1 \leq i \leq n-1 \\
f(u_i) &= 3 \\
f(u_{i+1}) &= 3 + i & 1 \leq i \leq n-2 \\
f(u_i) &= p + q \\
f(v_i) &= n + 1 + i & 1 \leq i \leq n \\
f(u_{i+1}) &= 8n - 1 - i & 1 \leq i \leq n-1 \\
f(v_i) &= 7n - i & 1 \leq i \leq n \\
f(v_{i2}) &= 5n - 1 + i & 1 \leq i \leq n \\
f(u_{i+1}v_{i+1}) &= 4n + i & 1 \leq i \leq n-1 \\
f(u_iu_{i+1}) &= 4n + 1 - i & 1 \leq i \leq n-1 \\
f(v_i) &= 3n + 2 - i & 1 \leq i \leq n \\
\end{align*}
\]

Let \( \psi = \{(v_i, v_{i1}, v_{i2}) : 1 \leq i \leq n\}, P_2=(v_1, u_1, u_2, v_2), P_3= \{(u_i, u_{i+1}, v_{i+1}) : 2 \leq i \leq n-1\}\)

For \( 1 \leq i \leq n\),

\[
\begin{align*}
\hat{f}^1[P_1] &= f(v_i) + f(v_{i2}) + f(v_{i1}v_i) + f(v_{i2}) \\
&= n + 1 + i + 7n - i + 3n + 2 - i + 5n - 1 + i \\
&= 16n + 2 \quad \text{(A)}
\end{align*}
\]

\[
\begin{align*}
\hat{f}^2[P_2] &= f(v_i) + f(v_{i2}) + f(v_{i1}v_i) + f(u_1u_2) + f(u_2v_2) \\
&= 1 + 8n - 2 + 2 + 4n + 4n + 1 \\
&= 16n + 2 \quad \text{(B)}
\end{align*}
\]

For \( 2 \leq i \leq n-1\),

\[
\begin{align*}
\hat{f}^3[P_3] &= f(u_i) + f(v_{i1}) + f(u_{i+1}) + f(u_{i+1}v_{i+1}) \\
&= 3 + i - 1 + 8n - 1 - i + 4n + 1 - i + 4n + i \\
&= 16n + 2 \quad \text{(C)}
\end{align*}
\]

From (A), (B) and (C), we conclude that \( G \) admits \( \psi \) - magic graphoidal total labeling. Hence, \( [P_n : S_2] \) is magic graphoidal.

For example, consider the graphs \( [P_3 : S_2] \) and \( [P_4 : S_2] \) shown in figure 3.1 and 3.2.

![Figure 3.1 [P_3 : S_2]](image)

Clearly, \( \psi = \{(v_{i1}, v_{i1}, v_{i2}), (v_{21}, v_{21}, v_{22}), (v_{31}, v_{31}, v_{32}), (v_1, u_1, u_2, v_2), (u_2, u_3, v_3)\} \) is a minimum graphoidal cover and \( [P_3 : S_2] \) is magic graphoidal. Here the constant \( K = 50 \)
Clearly, $\psi = \{(v_{11}, v_{1}, v_{12}), (v_{21}, v_2, v_{22}), (v_{31}, v_3, v_{32}), (v_{41}, v_4, v_{42}), (v_1, u_1, u_2, v_2), ... (A), (B) and (C), we conclude that G admits $\psi$ - magic graphoidal total labeling. Hence, $T(n)$ is magic graphoidal.

3.4. Theorem: For a Tree, $T(n)$ is magic graphoidal.

Proof:

Let $T(n)$ be a graph such that
$V[T(n)] = \{u_{i1}, u_{i2}, u_{i3}, u_{i4}, u_{i5} : 1 \leq i \leq n \}$ and
$E[T(n)] = \{(u_{i1}u_{i2}), (u_{i2}u_{i3}), (u_{i3}u_{i4}), (u_{i4}u_{i5}) : 1 \leq i \leq n \} \cup \{(u_{in}u_{i+1}n) : 1 \leq i \leq n-1\}$

Define $f: V \cup E \rightarrow \{1, 2, ..., p+q\}$ by

$f(u_{i1}) = i \quad 1 \leq i \leq n$

$f(u_{i3}) = n+1$

$f(u_{i2}) = n+2$

$f(u_{i1}u_{i2}) = 2n+3 - i \quad 1 \leq i \leq n$

$f(u_{i1}u_{i2}u_{i3}) = 5n - 3 - 2i \quad 1 \leq i \leq n - 3$

$f(u_{i1}u_{i1}u_{i3}) = 4n - 1 + i \quad n - 2 \leq i \leq n - 1$

$f(u_{i1}u_{i2}u_{i3}u_{i4}) = 5n - 2 + i \quad 1 \leq i \leq n$

$f(u_{i1}u_{i2}u_{i3}u_{i4}u_{i5}) = 8n - 2 + 2\cdot(i-1) \quad 1 \leq i \leq n$

$f(u_{i1}u_{i2}u_{i3}) = 8n - 1 + 2(i-1) \quad 1 \leq i \leq n - 1$

Let $\psi = \{P_1 = (u_{i1}, u_{i2}, u_{i3}, u_{i4}), P_2 = (u_{i3}, u_{i4}, u_{i5}, u_{i23}), P_3 = [(u_{i5}, u_{i15}, u_{i13}) : 2 \leq i \leq n - 1]\}$

$f[P_1] = f(u_{i1}) + f(u_{i4}) + f(u_{i1}u_{i2}) + f(u_{i2}u_{i3}) + f(u_{i4}u_{i4})$

$= i + 8n - 2 + 2(i-1) + 2n + 3 - i + 5n - 2 + (n + 1 - i) + 6n - 2 + (n + 1 - i)$

$= 23n - 3 \quad \text{(A)}$

$f[P_2] = f(u_{i5}) + f(u_{i1}u_{i3}) + f(u_{i5} u_{i+1}u_{i3}) + f(u_{i1}u_{i5} u_{i1}u_{i3})$

$= 2n + 2 + i - 1 + 8n - 1 + 2(i - 1) + 5n - 3 - 2(i - 2) + 7n - 2 + n - i$

$= 23n - 3 \quad \text{(B)}$

$f[P_3] = f(u_{i5}) + f(u_{i23}) + f(u_{i3}u_{i15}) + f(u_{i15}u_{i23}) + f(u_{i25}u_{23})$

$= n + 1 + 8n - 1 + 2(i - 1) + 5n - 3 - 2(i - 2) + 7n - 2 + n - i$

$= 23n - 3 \quad \text{(C)}$

From (A), (B) and (C), we conclude that $G$ admits $\psi$ - magic graphoidal total labeling. Hence, $T(n)$ is magic graphoidal.
For example, consider the graph $T(3)$ shown in figure 4.

\[ \psi = \{(u_{11}, u_{12}, u_{13}, u_{14}), \ldots, + f(u_{i}) + f(u_{ui}) \}
\]

\[ = 2n + 3 + 3n + 4 + i - 2 + 4n + 2 + n + 1 - i 
\]

\[ = 10n + 8 \quad \text{---------- (B)} 
\]

For $1 \leq i \leq n$,

**Figure 4 $T(3)$**

Clearly, $\psi = \{(u_{11}, u_{12}, u_{13}, u_{14}), (u_{21}, u_{22}, u_{23}, u_{24}), (u_{31}, u_{32}, u_{33}, u_{34}), (u_{13}, u_{15}, u_{25}, u_{23}), (u_{25}, u_{35}, u_{33})\}$ is a minimum graphoidal cover and $T(3)$ is magic graphoidal. Here the constant $K = 66$

**3.5. Theorem:** The graph Double Crowned Star $K_{1,n} \oplus K_1$ is magic graphoidal.

**Proof:** Let $G = K_{1,n} \oplus 2K_1$

\[ V(G) = \{u, [u_i : 1 \leq i \leq n], [(u_{1i}, u_{i2}) : 1 \leq i \leq n] \} \quad \text{and} \]

\[ E(G) = \{[(u_{1i}) : 1 \leq i \leq n] \cup [(u_{ui}) : 1 \leq i \leq n] \} \]

Define $f : V \cup E \rightarrow \{1, 2, \ldots, p+q\}$ by

\[ f(u) = 2n + 3 \]

\[ f(u_{1}) = i \quad 1 \leq i \leq n \]

\[ f(u_{i2}) = n + 2 \]

\[ f(u_{n+1,i}) = 3n + 4 \]

\[ f(u_{n+1,i}) = 4n + 2 + i \quad 1 \leq i \leq n - 2 \]

\[ f(u_{n+1,i}) = 5n + 1 + i \quad 1 \leq i \leq n \]

Let $\psi = \{P_1 = (u_1, u_2), P_2 = [(u, u_i) : 3 \leq i \leq n], P_3 = [(u_{1i}, u_{1i+1}) : 1 \leq i \leq n] \}$

\[ f'[P_1] = f(u_1) + f(u_2) + f(u_{1i}) + f(u_{1i}) 
\]

\[ = n + 1 + n + 2 + 3n + 4 + 5n + 1 
\]

\[ = 10n + 8 \quad \text{---------- (A)} 
\]

For $3 \leq i \leq n$,

\[ f'[P_2] = f(u) + f(u_i) + f(u_{ui}) 
\]

\[ = 2n + 3 + 3n + 4 + i - 2 + 4n + 2 + n + 1 - i 
\]

\[ = 10n + 8 \quad \text{---------- (B)} 
\]

For $1 \leq i \leq n$,
\[ f[P_3] = f(u_{i1}) + f(u_{i2}) + f(u_{i1}u_i) + f(u_{i2}) \]

\[ = i + n + 2 + n + 1 - i + 2n + 3 + i + 5n + 1 + n + 1 - i \]

\[ = 10n + 8 \quad \text{(C)} \]

From (A), (B) and (C), we conclude that G admits \( \psi \) - magic graphoidal total labeling. Hence, Double Crowned Star \( K_{1,n} \odot 2K_1 \) is magic graphoidal.
For example, consider the graph \( K_{1,5} \odot 2K_1 \) shown in figure 5.

![Figure 5. \( K_{1,5} \odot 2K_1 \)](image)

Clearly, \( \psi = \{(u_{i1},u_{i1}u_{i2}), (u_{i2},u_{i2}u_{i3}), (u_{i3},u_{i3}u_{i4}), (u_{i4},u_{i4}u_{i5}), (u_{i5},u_{i5}u_{i6})\} \) is a minimum graphoidal cover and \( K_{1,5} \odot 2K_1 \) is magic graphoidal. Here the constant \( K = 58 \).

### 3.6. Theorem

\( P_m \odot 2K_1 \) is magical graphoidal.

**Proof:** Let \( G = P_m \odot 2K_1 \)

\[ V(G) = \{(u_i : 1 \leq i \leq m), (u_{ij} : 1 \leq i \leq m, 1 \leq j \leq 2)\} \] and

\[ E(G) = \{[(u_i, u_{i+1}) : 1 \leq i \leq m-1] \cup [(u_i, u_{ij}) : 1 \leq i \leq m, 1 \leq j \leq 2]\} \]

Let \( \psi = \{P_1 = [(u_{i1},u_{i1}u_{i2}) : 1 \leq i \leq m], P_2 = [(u_{i1},u_{i1+1}) : 1 \leq i \leq m-1]\} \)

Define \( f: V \rightarrow \{0, 1, 2, \ldots, 6m-1\} \) by

\[ f(u_i) = i \quad 1 \leq i \leq m \]

\[ f(u_{i1}) = 2m+1-i \quad 1 \leq i \leq m \]

\[ f(u_{i1}u_{i2}) = \begin{cases} 2m+i & 1 \leq i \leq \frac{m}{2} \\ 2m+i & 1 \leq i \leq \frac{m-1}{2} \end{cases} \text{ if } m \text{ is even} \]

\[ f(u_{i2}) = \begin{cases} 2m+i & 1 \leq i \leq \frac{m}{2} \\ 2m+i & 1 \leq i \leq \frac{m-1}{2} \end{cases} \text{ if } m \text{ is odd} \]
\[
f(u_m+1-i) = \begin{cases} 
\frac{5m+i}{2} & 1 \leq i \leq m \text{ if } m \text{ is even} \\
\frac{5m-1+i}{2} & 1 \leq i \leq m \text{ if } m \text{ is odd} 
\end{cases}
\]

\[
f(u_{m+1-i}, 2) = \begin{cases} 
\frac{7m+i}{2} & 1 \leq i \leq \frac{m}{2} \text{ if } m \text{ is even} \\
\frac{7m-1+i}{2} & 1 \leq i \leq \frac{m+1}{2} \text{ if } m \text{ is odd} 
\end{cases}
\]

Case (i) : when m is odd

For \(1 \leq i \leq m\),
\[
f^*[P_1] = f(u_{i_1}) + f(u_{i_2}) + f(u_{i_1}u_{i_2}) + f(u_{i_2}u_{i_1}) = \frac{5m-1+i}{2} + 5m - 1 + m + 1 - i + 2m + 1 - i = \frac{21m+1}{2} \quad (A)
\]

For \(1 \leq i \leq m-1\), \(i \equiv 1 \mod 2\)
\[
f^*[P_2] = f(u_i) + f(u_{i+1}) + f(u_{i}u_{i+1}) = \frac{7m-1+i+1}{2} + 2m + \frac{i+1}{2} + 4m + m - i = \frac{21m+1}{2} \quad (B)
\]

For \(1 \leq i \leq m-1\), \(i \equiv 0 \mod 2\)
\[
f^*[P_2] = f(u_i) + f(u_{i+1}) + f(u_{i}u_{i+1}) = \frac{2m + i}{2} + \frac{7m-1+i+2}{2} + 4m + m - i = \frac{21m+1}{2} \quad (C)
\]

From (A), (B) and (C), we conclude that G admits \(\psi\) - magic graphoidal total labeling. Hence, 
\(P_m \bowtie 2K_1\) (m-odd) is magic graphoidal.

For example, consider the graph \(P_5 \bowtie 2K_1\) shown in figure 6.1.

Case (ii) : when m is even

For \(1 \leq i \leq m\),
\[
f^*[P_1] = f(u_{i_1}) + f(u_{i_2}) + f(u_{i_1}u_{i}) + f(u_{i_2}u_{i_1})
\]
5m
2 = i + 5m –1 + m + 1 – i + i + 2m + 1 - i

21m + 2
2 = \text{------------ (A)}

For 1 \leq i \leq m -1, \; i \equiv 1 \mod 2

f'(P_2) = f(u_i) + f(u_{i+1}) + f(u_{ui+1})
= \frac{7m}{2} \cdot \frac{i + 1}{2} + 2m \cdot \frac{i + 1}{2} + 4m + m - i
= \frac{21m + 2}{2} \text{------------ (B)}

For 1 \leq i \leq m -1, \; i \equiv 0 \mod 2

f'[P_2] = f(u_i) + f(u_{i+1}) + f(u_{ui+1})
= \frac{2m}{2} + \frac{7m}{2} + \frac{i + 2}{2} + 4m + m - i
= \frac{21m + 2}{2} \text{------------ (C)}

From (A), (B) and (C), we conclude that G admits \( \psi \) - magic graphoidal total labeling. Hence, \( P_m \odot 2K_1 \) (m-even) is magic graphoidal.

For example, consider the graph \( P_4 \odot 2K_1 \) shown in Figure 6.2.

Figure 6.1 \( P_5 \odot 2K_1 \)

Clearly, \( \psi = \{(u_{i1},u_{i1},u_{i2}), (u_{21},u_{21},u_{22}), (u_{31},u_{31},u_{32}), (u_{41},u_{41},u_{42}), (u_{51},u_{51},u_{52}), (u_{1},u_{2}), (u_{2},u_{3}), (u_{3},u_{4}), (u_{4},u_{5})\} \) is a minimum graphoidal cover and \( P_4 \odot 2K_1 \) is magic graphoidal. Here the constant \( K = 53. \)
Clearly, $\psi = \{(u_{11},u_{11},ui_2),(u_{21},u_{22},ui_3),(u_{31},u_{32},ui_4),(u_{41},u_{42},ui_3),(u_{51},u_{52},ui_4)\}$ is a minimum graphoidal cover and $P_4 \cong 2K_1$ is magic graphoidal. Here the constant $K = 43$.

3.7. Theorem: $P_m \cong K_{1,3}$ is magic graphoidal.

Proof: Let $G = P_m \cong K_{1,3}$

$V(G) = \{(u_i,v_i) : 1 \leq i \leq m\}, [(v_j) : 1 \leq i \leq m, 1 \leq i \leq 3]\}

E(G) = \{(u_i,u_{i+1}) : 1 \leq i \leq m, 1 \leq j \leq m\}

with $u_i = u_{3i}, 1 \leq i \leq m$

Let $\psi = \{(v_1,v_2,v_1) : 1 \leq i \leq m\}, P_2 = [(v_i,v_{i+1}) : 1 \leq i \leq m-2]$,

$P_3 = (v_{m-1},u_{m-1},u_m,v_m)\}$

Define $f: V \cup E \rightarrow \{1, 2, ..., 8m-1\}$ by

$f(v_m) = 1$

$f(v_{i+1}) = i+1 \leq i \leq m$

$f(u_i,v_i) = m+1+i \leq i \leq m-1$

$f(u_{i+1},u_{i+2}) = 2m+i \leq i \leq m-1$

$f(u_{i+1},v_{i+1+2}) = 3m+1+i \leq i \leq m$

$f(v_{i+1}) = 4m-1+i \leq i \leq m$

$f(v_{m-1}) = 5m-1+i \leq i \leq m-1$

$f(u_{m+1}) = 6m-1$

$f(u_{m-1}) = 6m+i \leq i \leq m-2$

$f(v_{m+1}) = 7m-2+i \leq i \leq m$

For $1 \leq i \leq m$,

$f^r(P_1) = f(v_{i+1}) + f(v_{i+2}) + f(v_{i}v_{i+2}) + f(v_{i+1}v_{i+2})$

$= 4m-1+i+7m-2+m+1-i+i+1+3m-1+m+1-i$

$= 16m-1 \quad \text{--- (A)}$

For $1 \leq i \leq m-2$,

$f^r(P_2) = f(v_{i}) + f(u_{i+1}) + f(v_{i}u_{i+1}) + f(u_{i+1}u_{i+1})$

$= 5m-1+i+6m+(m-i-1)+m+1+i+2m+m-i$

$= 16m-1 \quad \text{--- (B)}$
\[ f[P_3] = f(v_{m-1}) + f(v_m) + f(v_{m-1}u_{m-1}) + f(u_{m-1}u_m) + f(u_mv_m) \]
\[ = 6m - 2 + 1 + 2m + 2m + 1 + 6m - 1 \]
\[ = 16m - 1 \quad \text{------ (C)} \]

From (A), (B) and (C), we conclude that G admits \( \psi \) - magic graphoidal total labeling. Hence, \( P_m \circ K_{1,3} \) is magic graphoidal.

For example, consider the graph \( P_4 \circ K_{1,3} \) shown in figure 7.

![Figure 7. \( P_4 \circ K_{1,3} \)](image)

Clearly, \( \psi = \{ (v_{11}, v_1, v_{12}), (v_{21}, v_2, v_{22}), (v_{31}, v_3, v_{32}), (v_{41}, v_4, v_{42}), (v_1, u_1, u_2), (v_1, u_2, u_3), (v_3, u_3, u_4, v_4) \} \) is magic graphoidal. Here the constant \( K = 63 \).

**REFERENCES**


