Magic Graphoidal on Class of Trees

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Abstract: B.D.Acharya and E. Sampathkumar [1] defined Graphoidal cover as partition of edge set of a graph G into internally disjoint paths (not necessarily open). The minimum cardinality of such cover is known as graphoidal covering number of G.

Let G = {V, E} be a graph and let ψ be a graphoidal cover of G. Define f:

\[ V \cup E \rightarrow \{1, 2, \ldots, p+q\} \text{ such that for every path } P = (v_0, v_1, v_2, \ldots, v_n) \text{ in } \psi \text{ with } f^*(P) = f(v_0) + f(v_n) + \sum_{i=1}^{n} f(v_{i-1}, v_i) = k, \text{ a constant, where } f^* \text{ is the induced labeling on } \psi. \text{ Then, we say that } G \text{ admits } \psi - \text{ magic graphoidal total labeling of } G. \]

A graph G is called magic graphoidal if there exists a minimum graphoidal cover ψ of G such that G admits ψ - magic graphoidal total labeling.

In this paper, we proved that [P_n; S_1], [P_n; S_2], T(n), P_m \bigodot K_{1,3}, P_m \bigodot 2K_1 \text{ and } K_{1,n} \bigodot 1 \text{ are magic graphoidal}

1. INTRODUCTION

By a graph, we mean a finite simple and undirected graph. The vertex set and edge set of a graph G are denoted by V(G) and E(G) respectively. Terms and notations not used here are as in [3].

1.1. Definition: Let S_1 = (v_0, v_1) be a star and let [P_n ; S_1] be the graph obtained from n copies of S_1 and the path P_n = (u_1, u_2, \ldots, u_n) by joining u_j with the vertex v_0 of the j^{th} copy of S_1 by means of an edge, for 1 \leq j \leq n.

1.2. Definition: Let S_2 = (v_0, v_1, v_2) be a star and let [P_n ; S_2] be the graph obtained from n copies of S_2 and the path P_n = (u_1, u_2, \ldots, u_n) by joining u_j with the vertex v_0 of the j^{th} copy of S_2 by means of an edge, for 1 \leq j \leq n.

1.3. Definition: Let T be any Tree. Denote the tree, obtained from T by considering two copies of T by adding an edge between them, by T_{(2)} and in general the graph obtained from T_{(n-1)} and T by adding an edge between them is denoted by T_{(n)}.

1.4. Result [11]: For a Tree T, \( \gamma(T) = n-1 \) where n is the number of pendent vertices of G.

2. PRELIMINARIES

Let G = {V, E} be a graph with p vertices and q edges. A graphoidal cover \( \psi \) of G is a collection of (open) paths such that

(i) every edge is in exactly one path of \( \psi \)

(ii) every vertex is an interval vertex of atmost one path in \( \psi \).

We define \( \gamma(G) = \min_{\psi \in \zeta} |\psi| \),

where \( \zeta \) is the collection of graphoidal covers \( \psi \) of G and \( \gamma \) is graphoidal covering number of G.
Let $\psi$ be a graphoidal cover of $G$. Then we say that $G$ admits $\psi$ - magic graphoidal total labeling of $G$ if there exists a bijection $f: V \cup E \rightarrow \{1, 2, \ldots, p+q\}$ such that for every path $P = (v_0, v_1, v_2, \ldots, v_n)$ in $\psi$, then $f^*(P) = f(v_0) + f(v_n) + \sum_{i=1}^{n} f(v_{i+1}) = k$, a constant, where $f^*$ is the induced labeling of $\psi$. A graph $G$ is called magic graphoidal if there exists a minimum graphoidal cover $\psi$ of $G$ such that $G$ admits $\psi$ - magic graphoidal total labeling.

3. Magical Graphoidal on Trees

3.1. Theorem: $[P_n ; S_1], (n - \text{even})$ is magic graphoidal.

Proof: Let $G = [P_n ; S_1]$

Let $V(G) = \{u_i, v_i, w_i : 1 \leq i \leq n\}$ and $E(G) = \{(u_i, v_i) \cup (v_i, w_i) : 1 \leq i \leq n\} \cup \{(u_i, u_{i+1}) : 1 \leq i \leq n - 1\}$

Define $f: V \cup E \rightarrow \{1, 2, \ldots, p+q\}$ by

$f(w_1) = 1$

$f(w_1v_1) = 2$

$f(v_1u_1) = 3$

$f(u_{i+1}) = 6 + i$ for $1 \leq i \leq n - 2$

$f(w_{i+1}) = 6n - i$ for $1 \leq i \leq n - 1$

$f(v_{i+1}w_{i+1}) = 4n + 1 + i$ for $1 \leq i \leq n - 1$

$f(u_{i+1}v_{i+1}) = 3n + 3 - 2i$ for $1 \leq i \leq (n/2) - 1$

$f\left(\begin{array}{c}
\frac{u_i}{2} \\
\frac{v_i}{2}
\end{array}\right) = 3n + 4 - 2i$ for $1 \leq i \leq \frac{n}{2}$

$f(u_{i+1}u_{i+1}) = \frac{3n}{2} + 4 + i$ for $1 \leq i \leq \frac{n}{2} - 1$

$f\left(\begin{array}{c}
\frac{u_i}{2} \\
\frac{u_{i+1}}{2}
\end{array}\right) = n + 4 + i$ for $1 \leq i \leq \frac{n}{2}$

Let $\psi = \{P_1 = (w_1, v_1, u_1, u_2, v_2, w_2), P_2 = (u_i, u_{i+1}, v_i, v_{i+1}, w_{i+1}) : 2 \leq i \leq n - 1\}$

$f^*[P_1] = f(w_1) + f(w_2) + f(v_1v_1) + f(v_1u_1) + f(u_1u_2) + f(u_2v_2) + f(v_2w_2)$

$= 1 + 6n - 1 + 2 + 3 + \frac{3n}{2} + 4 + 1 + 3n + 3 - 2 + 4n + 2$

$= 13n + \frac{3n}{2} + 13 \quad \text{---------- (A)}$

For $2 \leq i \leq (n/2) - 1$,

$f^*[P_2] = f(u_i) + f(w_{i+1}) + f(u_{i+1}u_{i+1}) + f(u_{i+1}v_{i+1}) + f(v_{i+1}w_{i+1})$

$= 6 + i - 1 + 6n - i + (3n/2) + 4 + i + 3n + 3 - 2i + 4n + 1 + i$

$= 13n + (3n/2) + 13 \quad \text{---------- (B)}$

For $(n/2) \leq i \leq n - 1$,
\[ f[P_2] = f(u_i) + f(w_{i+1}) + f(u_{i+1}v_i) + f(v_{i+1}w_{i+1}) \]
\[ = 6 + i - 1 + 6n - i + n + 4 + i - (n/2) + 1 + 3n + 4 - 2(i + 1 - (n/2)) + 4n + 1 + i \]
\[ = 13n + (3n/2) + 13 \quad \text{(C)} \]

From (A), (B) and (C), we conclude that \( G \) admits \( \psi \) - magic graphoidal total labeling. Hence, \([P_n ; S_1], \text{ (n} \text{ even)}\) is magic graphoidal.

For example, consider the graph \([P_6 ; S_1]\) shown in figure 1.

![Figure 1 [P6 ; S1]](image-url)

Clearly, \( \psi = \{(w_1, v_1, u_1, u_2, v_2, w_2), (u_2, u_3, v_3, w_3), (u_3, u_4, v_4, w_4), (u_4, u_5, v_5, w_5), (u_5, u_6, v_6, w_6)\} \) is a minimum graphoidal cover and \([P_6 ; S_1]\) is magic graphoidal. Here the constant \( K = 100 \)

**3.2. Theorem :** \([P_n ; S_1], \text{ (n} \text{- odd)}\) is magic graphoidal.

**Proof:** Let \( G = [P_n ; S_1] \)

Let \( V(G) = \{ u_i, v_i, w_i : 1 \leq i \leq n \} \) and

\[ E(G) = \{ (u_i, v_i) \cup (v_i, w_i) : 1 \leq i \leq n \} \cup \{(u_1u_{i+1}) : 1 \leq i \leq n-1\} \]

Define \( f : V \cup E \rightarrow \{1, 2, \ldots, p+q\} \) by

\[ f(w_1) = 1 \]
\[ f(w_1v_1) = 2 \]
\[ f(v_1u_1) = 3 \]
\[ f(u_{i+1}) = 5 + 2i \quad 1 \leq i \leq (n-1)/2 \]
\[ f(v_{i+1}) = 6n - i \quad 1 \leq i \leq n - 1 \]
\[ f(v_{i+1}w_{i+1}) = 4n + 1 + i \quad 1 \leq i \leq n - 1 \]
\[ f(u_1u_2) = n + 5 \]
\[ f(u_{i+1}u_{i+2}) = \frac{3n + 1}{2} \quad 1 \leq i \leq \frac{n-1}{2} \]
\[ f(u_{i+1}u_{i+3}) = \frac{3n + 9}{2} + i \quad 1 \leq i \leq \frac{n-3}{2} \]
\[ f(u_{i+2}v_{i+2}) = 3n + 2 - i \quad 1 \leq i \leq n - 2 \]
\[ f(u_2v_2) = \frac{7n + 3}{2} \]
Let $\psi = \{P_1 = (w_1,v_1,u_1,u_2,v_2,w_2), P_2 = (u_i,u_{i+1},v_{i+1},w_{i+1}) : 2 \leq i \leq n - 1\}$

\[f^* [P_1] = f(w_1) + f(w_2) + f(v_1u_1) + f(u_1u_2) + f(u_2v_2) + f(v_2w_2)\]

\[= 1 + 6n - 1 + 2 + 3 + n + 5 + \frac{7n + 3}{2} + 4n + 1 + 1\]

\[= 14n + \frac{n + 1}{2} + 13 \quad \text{(A)}\]

For $2 \leq i < \frac{n + 1}{2}$

\[f^* [P_2] = f(u_i) + f(w_{i+1}) + f(u_{i+1}v_{i+1}) + f(v_{i+1}w_{i+1})\]

\[= 5 + 2(i - 1) + 6n - i + \frac{3n + 9}{2} - (i - 1) + 3n + 2 - (i - 1) + 4n + 1 + i\]

\[= 14n + \frac{n + 1}{2} + 13 \quad \text{(B)}\]

For $\frac{n + 1}{2} \leq i \leq n - 1$,

\[f^* [P_2] = f(u_i) + f(w_{i+1}) + f(u_{i+1}v_{i+1}) + f(v_{i+1}w_{i+1})\]

\[= 6 + 2 \left( i - \frac{n + 1}{2} \right) + 6n - i + \frac{3n + 9}{2} + (n - i) + 3n + 2 - (i - 1) + 4n + 1 + i\]

\[= 14n + \frac{n + 1}{2} + 13 \quad \text{(C)}\]

From (A), (B) and (C), we conclude that $\psi$ is minimum magic graphoidal cover
Hence, $[P_n ; S_1]$, (n - even) is magic graphoidal.

For example, consider the graph $[P_7 ; S_1]$ shown in figure 2.

Clearly, $\psi = \{(w_1,v_1,u_1,u_2,v_2,w_2), (u_2,u_3,v_3,w_3), (u_3,u_4,v_4,w_4), (u_4,u_5,v_5,w_5), (u_5,u_6,v_6,w_6), (u_6,u_7,v_7,w_7)\}$ is a minimum graphoidal cover and $[P_7 ; S_1]$ is magic graphoidal. Here the constant $K = 115$.

3.3. Theorem : $[P_n ; S_2]$ is magic graphoidal.

Proof: Let $G = [P_n ; S_2]
V(G) = \{(u_i, v_i) : 1 \leq i \leq n\}$
E(G) = \{ \[ (u_iu_{i+1}) : 1 \leq i \leq n-1 \] \cup \[ (u_iv_i) : 1 \leq i \leq n \] \cup \[(v_iv_{i1}) \cup (v_iv_{i2}) : 1 \leq i \leq n \] \}

Define \( f : V \cup E \rightarrow \{1, 2, ..., p+q\} \) by

\[
\begin{align*}
    f(v_1) &= 1 \\
    f(u_1v_1) &= 2f \\
    (u_i) &= 3 \\
    f(u_{i+1}) &= 3 + i & 1 \leq i \leq n - 2 \\
    f(v_i) &= p + q \\
    f(v_i+1) &= n + 1 + i & 1 \leq i \leq n \\
    f(v_{i+1}) &= 8n - 1 - i & 1 \leq i \leq n - 1 \\
    f(v_{i2}) &= 7n - i & 1 \leq i \leq n \\
    f(v_{i2}) &= 5n - 1 + i & 1 \leq i \leq n \\
    f(u_{i+1}v_{i+1}) &= 4n + i & 1 \leq i \leq n - 1 \\
    f(u_iu_{i+1}) &= 4n + 1 - i & 1 \leq i \leq n - 1 \\
    f(v_{i1}) &= 3n + 2 - i & 1 \leq i \leq n
\end{align*}
\]

Let \( \psi = \{(v_{i1}, v_{i2}, v_{i3}) : 1 \leq i \leq n\}, \) \( P_2=(v_1u_1u_2v_2), \) \( P_3=\{(u_iu_{i+1}, v_{i+1}) : 2 \leq i \leq n-1\} \)

For \( 1 \leq i \leq n, \)

\[
\begin{align*}
    f^\star[P_1] &= f(v_{i1}) + f(v_{i2}) + f(v_{i1}v_{i1}) + f(v_{i1}v_{i2}) \\
    &= n + 1 + i + 7n - i + 3n + 2 - i + 5n - 1 + i \\
    &= 16n + 2 \quad \text{(A)}
\end{align*}
\]

\[
\begin{align*}
    f^\star[P_2] &= f(v_{i1}) + f(v_{i2}) + f(v_{i1}u_{i1}) + f(u_{i2}u_{i2}) + f(u_{i2}v_{i2}) \\
    &= 1 + 8n - 2 + 2 + 4n + 4n + 1 \\
    &= 16n + 2 \quad \text{(B)}
\end{align*}
\]

For \( 2 \leq i \leq n - 1, \)

\[
\begin{align*}
    f^\star[P_3] &= f(u_i) + f(v_{i+1}) + f(u_iu_{i+1}) + f(u_{i+1}v_{i+1}) \\
    &= 3 + i - 1 + 8n - 1 - i + 4n + 1 - i + 4n + i \\
    &= 16n + 2 \quad \text{(C)}
\end{align*}
\]

From (A), (B) and (C), we conclude that \( G \) admits \( \psi \) - magic graphoidal total labeling. Hence, \([P_n : S_2]\) is magic graphoidal.

For example, consider the graphs \([P_3 : S_2]\) and \([P_4 : S_2]\) shown in figure 3.1 and 3.2.

![Figure 3.1 [P_3 ; S_2]](image)

Clearly, \( \psi = \{(v_{11}, v_{12}, v_{13}), (v_{21}, v_{22}, v_{23}), (v_{31}, v_{32}, v_{33}), (v_{14}, u_1, u_2, v_2), (u_3, v_3)\} \) is a minimum graphoidal cover and \([P_3 : S_2]\) is magic graphoidal. Here the constant \( K = 50 \)
Clearly, $\psi = \{(v_{11}, v_1, v_{12}), (v_{21}, v_2, v_{22}), (v_{31}, v_3, v_{32}), (v_{41}, v_4, v_{42}), (v_1, u_{11}, u_1, v_2), (u_2, u_3, v_3), (u_3, u_4, v_4)\}$ is a minimum graphoidal cover and $[P_4; S_2]$ is magic graphoidal. Hence, $T(n)$ is magic graphoidal.

**3.4. Theorem:** For a Tree, $T_{(a)}$ is magic graphoidal.

**Proof:**

Let $T_{(a)}$ be a graph such that

$\begin{align*}
V[T_{(a)}] & = \{u_{11}, u_{12}, u_{13}, u_{44}, u_{55} : 1 \leq i \leq n \} \quad \text{and} \\
E[T_{(a)}] & = \{(u_{11}u_{12}), (u_{12}u_{33}), (u_{33}u_{44}), (u_{44}u_{55}) : 1 \leq i \leq n \} \cup \{(u_{in}u_{i+1n}) : 1 \leq i \leq n-1\}
\end{align*}$

Define $f: V \cup E \to \{1, 2, \ldots, p+q\}$ by

$\begin{align*}
f(u_{11}) & = i \\
f(u_{12}) & = n+1 \\
f(u_{13}) & = n+2 \\
f(u_{11}u_{12}) & = 2n+3-i \\
f(u_{12}u_{33}) & = 5n-3-2i \\
f(u_{33}u_{44}) & = 7n-2+i \\
f(u_{44}u_{55}) & = 8n-2+2(i-1) \\
f(u_{i1}) & = 8n-1+2(i-1) \\
\end{align*}$

Let $\psi = \{P_1 = (u_{11}, u_{12}, u_{13}, u_{44}), P_2 = (u_{13}, u_{25}, u_{25}, u_{32}), P_3 = [(u_{55}, u_{5+15}, u_{5+13}) : 2 \leq i \leq n-1]\}$

$\begin{align*}
f'[P_1] & = f(u_{11}) + f(u_{12}) + f(u_{11}u_{12}) + f(u_{25}u_{32}) + f(u_{44}u_{55}) \\
& = i+8n-2+2(i-1)+2n+3-i+5n-2+(n+1-i)+6n-2+(n+1-i) \\
& = 23n-3 \quad \text{(A)}
\end{align*}$

$\begin{align*}
f'[P_2] & = f(u_{13}) + f(u_{1+13}) + f(u_{25}u_{32}) + f(u_{13}u_{44}) \\
& = 2n+2+i-1+8n-1+2(i-1)+5n-3-2(i-2)+7n-2+n-i \\
& = 23n-3 \quad \text{(B)}
\end{align*}$

$\begin{align*}
f'[P_3] & = f(u_{11}) + f(u_{25}) + f(u_{32}u_{13}) + f(u_{55}u_{25}) + f(u_{25}u_{25}) \\
& = n+1+8n-1+2(i-1)+5n-3-2(i-2)+7n-2+n-i \\
& = 23n-3 \quad \text{(C)}
\end{align*}$

From (A), (B) and (C), we conclude that $G$ admits $\psi$ - magic graphoidal total labeling. Hence, $T_{(a)}$ is magic graphoidal.
For example, consider the graph $T(3)$ shown in figure 4.

![Figure 4 $T(3)$](image)

Clearly, $\psi = \{(u_{11}, u_{12}, u_{13}, u_{14}), \ldots \}$ is a minimum grphoidal cover and $T(3)$ is magic grphoidal. Here the constant $K = 66$

### 3.5. Theorem:

The graph Double Crowned Star $K_{1,n} \circ K_1$ is magic grphoidal.

**Proof:** Let $G = K_{1,n} \circ 2K_1$

$V(G) = \{u, [u_i : 1 \leq i \leq n], [(u_{i1}, u_{i2}) : 1 \leq i \leq n]\}$ and $E(G) = \{[(u_{i1}) : 1 \leq i \leq n] \cup [(u_{i1}, u_{i2}) : 1 \leq i \leq n]\}$

Define $f : V \cup E \to \{1, 2, \ldots, p+q\}$ by

- $f(u) = 2n + 3$
- $f(u_{i1}) = i \hspace{1cm} 1 \leq i \leq n$
- $f(u_{i2}) = n + 2$
- $f(u_{n+1-i,2}) = n + 2 + i \hspace{1cm} 1 \leq i \leq n$
- $f(u_{u_i, u_{i1}}) = 2n + 3 + i \hspace{1cm} 1 \leq i \leq n$
- $f(u_{u_i}) = 3n + 4$
- $f(u_{u_2, u_1}) = 3n + 4 + i \hspace{1cm} 1 \leq i \leq n - 2$
- $f(u_{u_{n+1-i}}, u_{n+1, i}) = 4n + 2 + i \hspace{1cm} 1 \leq i \leq n - 1$
- $f(u_{u_{n+1-i}, u_{n+1, i}}, u_{u_{n+1}}, u_{n+1, i}) = 5n + 1 + i \hspace{1cm} 1 \leq i \leq n$

Let $\psi = \{P_1 = (u_1, u_2), P_2 = [(u_i, u_i) : 3 \leq i \leq n], P_3 = [(u_{i1}, u_{i2}) : 1 \leq i \leq n]\}$

$\hat{f} [P_1] = f(u_1) + f(u_2) + f(u_{i1}) + f(u_{u_2})$

$= n + 1 + n + 2 + 3n + 4 + 5n + 1$

$= 10n + 8 \hspace{1cm} \text{(A)}$

For $3 \leq i \leq n$,

$\hat{f} [P_2] = f(u) + f(u_i) + f(u_{i1})$

$= 2n + 3 + 3n + 4 + i - 2 + 4n + 2 + n + 1 - i$

$= 10n + 8 \hspace{1cm} \text{(B)}$

For $1 \leq i \leq n$, 


From (A), (B) and (C), we conclude that \( G \) admits \( \psi \) - magic graphoidal total labeling. Hence, Double Crowned Star \( K_{1,n} \otimes 2K_1 \) is magic graphoidal.

For example, consider the graph \( K_{1,5} \otimes 2K_1 \) shown in figure 5.

\[ 3.6. \textbf{Theorem} : P_n \otimes 2K_1 \text{ is magical graphoidal.} \]

\textbf{Proof :} Let \( G = P_n \otimes 2K_1 \)

\[ V(G) = \{ (u_i : 1 \leq i \leq m), (u_j : 1 \leq i \leq m, 1 \leq j \leq 2) \} \text{ and} \]
\[ E(G) = \{ [(u_i, u_{i+1}) : 1 \leq i \leq m-1] \cup [(u_i, u_{i+2}) : 1 \leq i \leq m, 1 \leq j \leq 2] \} \]

Let \( \psi = \{ P_1 = [(u_i, u_{i+1}) : 1 \leq i \leq m], P_2 = [(u_i, u_{i+2}) : 1 \leq i \leq m-1] \} \)

Define \( f : V \rightarrow \{0, 1, 2, \ldots, 6m-1\} \) by

\[ f(u_i) = i \quad 1 \leq i \leq m \]
\[ f(u_{i+1}) = 2m+1-i \quad 1 \leq i \leq m \]

\[ f(u_{i+2}) = \begin{cases} 2m+i & 1 \leq i \leq \frac{m}{2} \quad \text{if } m \text{ is even} \\ 2m+i & 1 \leq i \leq \frac{m-1}{2} \quad \text{if } m \text{ is odd} \end{cases} \]
\[ f(u_i) = \begin{cases} 
\frac{5m}{2} + i & \text{if } m \text{ is even and } 1 \leq i \leq m \\
\frac{5m-1}{2} + i & \text{if } m \text{ is odd and } 1 \leq i \leq m 
\end{cases} \]

\[ f(u_{3i+1}) = \begin{cases} 
\frac{7m}{2} + i & \text{if } m \text{ is even and } 1 \leq i \leq \frac{m}{2} \\
\frac{7m-1}{2} + i & \text{if } m \text{ is odd and } 1 \leq i \leq \frac{m+1}{2} 
\end{cases} \]

\[ f(u_{m+1-i, 2}) = 4m + i \quad 1 \leq i \leq m-1 \]

\[ f(u_{m+1-i, u_{m-i}}) = 5m - 1 + i \quad 1 \leq i \leq m \]

Case (i): when m is odd

For \(1 \leq i \leq m\),

\[ f^*[P_1] = f(u_{i1}) + f(u_{i2}) + f(u_{i1}u_i) + f(u_{i2}i) \]

\[ = \frac{5m-1}{2} + i + 5m - 1 + m + 1 - i + 2m + 1 - i \]

\[ = \frac{21m + 1}{2} \quad \text{(A)} \]

For \(1 \leq i \leq m - 1, \ i \equiv 1 \mod 2\)

\[ f^*[P_2] = f(u_i) + f(u_{i+1}) + f(u_{i1}u_i) \]

\[ = \frac{7m-1}{2} + i + 1 + 2m + \frac{i+1}{2} + 4m + m - i \]

\[ = \frac{21m + 1}{2} \quad \text{(B)} \]

For \(1 \leq i \leq m - 1, \ i \equiv 0 \mod 2\)

\[ f^*[P_2] = f(u_i) + f(u_{i+1}) + f(u_{i2}i) \]

\[ = 2m + \frac{i}{2} + \frac{7m-1}{2} + \frac{i+2}{2} + 4m + m - i \]

\[ = \frac{21m + 1}{2} \quad \text{(C)} \]

From (A), (B) and (C), we conclude that \(G\) admits \(\psi\) - magic graphoidal total labeling. Hence, \(P_m \bigcirc 2K_1\) \((m\text{-odd})\) is magic graphoidal.

For example, consider the graph \(P_5 \bigcirc 2K_1\) shown in figure 6.1.

Case (ii): when m is even

For \(1 \leq i \leq m\),

\[ f^*[P_1] = f(u_{i1}) + f(u_{i2}) + f(u_{i1}u_i) + f(u_{i2}i) \]
\[
\begin{align*}
5m & \quad = i + 5m - 1 + m + 1 - i + i + 2m + 1 - i \\
\frac{21m + 2}{2} & \quad = (A)
\end{align*}
\]

For \(1 \leq i \leq m - 1\), \( i \equiv 1 \mod 2 \)
\[
f'[P_2] = f(u_i) + f(u_{i+1}) + f(u_{i+1})
\]
\[
= \frac{7m}{2} + \frac{i + 1}{2} + \frac{2m}{2} + \frac{i + 1}{2} + 4m + m - i
\]
\[
= \frac{21m + 2}{2} \quad (B)
\]

For \(1 \leq i \leq m - 1\), \( i \equiv 0 \mod 2 \)
\[
f'[P_2] = f(u_i) + f(u_{i+1}) + f(u_{i+1})
\]
\[
= \frac{2m}{2} + \frac{i}{2} + \frac{7m}{2} + \frac{i + 2}{2} + 4m + m - i
\]
\[
= \frac{21m + 2}{2} \quad (C)
\]

From (A), (B) and (C), we conclude that \( G \) admits \( \psi \) - magic graphoidal total labeling. Hence, \( P_m \bowtie 2K_1 \) (m-even) is magic graphoidal.

For example, consider the graph \( P_4 \bowtie 2K_1 \) shown in figure 6.2.

\[
\text{Figure 6.1 } P_5 \bowtie 2K_1
\]

Clearly, \( \psi = \{(u_{11},u_{11},u_{12}), (u_{21},u_{22},u_{22}), (u_{31},u_{31},u_{32}), (u_{41},u_{42},u_{42}), (u_{51},u_{52},u_{52}), (u_1,u_2), (u_2,u_3), (u_3,u_4), (u_4,u_5)\} \) is a minimum graphoidal cover and \( P_4 \bowtie 2K_1 \) is magic graphoidal. Here the constant \( K = 53 \).
Clearly, $\psi = (u_{i1}, u_{i1}, u_{i2}), (u_{i1}, u_{i1}, u_{i2}), (u_{i1}, u_{i2}), (u_{i1}, u_{i2}), (u_{i1}, u_{i2}), (u_{i1}, u_{i2}), (u_{i1}, u_{i2}), (u_{i1}, u_{i2})$ is a minimum graphoidal cover and $P_4 \bigcirc 2K_1$ is magic graphoidal. Here the constant $K = 43$.

**3.7. Theorem:** $P_m \bigcirc K_{1,3}$ is magic graphoidal.

**Proof:** Let $G = P_m \bigcirc K_{1,3}$

- $V(G) = \{(v_i, v_j), 1 \leq i \leq m, 1 \leq j \leq 3\}$
- $E(G) = \{(u_{i1}, u_{i1}), 1 \leq i \leq m\}$

Let $\psi = \{(v_{i1}, v_{i1}, v_{i2}), 1 \leq i \leq m\}$, $P_2 = (v_i, u_{i1}, u_{i1})$, $1 \leq i \leq m-2$, $P_3 = (v_{i1}, u_{i1}, u_{i2}, v_{i3})$

Define $f: V \cup E \rightarrow \{1, 2, ..., 8m-1\}$ by

- $f(v_{i1}) = i+1$
- $f(v_{i1}) = m+1+i$
- $f(u_{i1}) = 2m+i$
- $f(v_{i1}) = 3m-1+i$
- $f(v_{i1}) = 4m-1+i$
- $f(v_{i1}) = 5m-1+i$
- $f(u_{i1}) = 6m-1$
- $f(u_{i1}) = 6m+i$
- $f(u_{i1}) = 7m-2+i$

For $1 \leq i \leq m$,

- $f'[P_1] = f(v_{i1}) + f(v_{i2}) + f(v_{i1}) + f(v_{i2})$
- $= 4m-1+i + 7m-2+m+1-i+i+1+3m-1+m+1-i$
- $= 16m-1$ \[\text{(A)}\]

For $1 \leq i \leq m-2$,

- $f'[P_2] = f(v_{i1}) + f(u_{i1}) + f(v_{i2}) + f(u_{i1})$
- $= 5m-1+i+6m+(m-i-1)+m+1+i+2m+m-i$
- $= 16m-1$ \[\text{(B)}\]
\[ f'[P_3] = f(v_{m-1}) + f(v_m) + f(v_{m-1}u_{m-1}) + f(u_{m-1}u_m) + f(u_mv_m) \]
\[ = 6m - 2 + 1 + 2m + 2m + 1 + 6m - 1 \]
\[ = 16m - 1 \quad \text{(C)} \]

From (A), (B) and (C), we conclude that G admits $\psi$ - magic graphoidal total labeling. Hence, $P_m \boxf K_{1,3}$ is magic graphoidal.

For example, consider the graph $P_4 \boxf K_{1,3}$ shown in figure 7.

![Figure 7. $P_4 \boxf K_{1,3}$](image)

Clearly, $\psi = \{(v_{11},v_{1},v_{12}), (v_{21},v_{2},v_{22}), (v_{31},v_{3},v_{32}), (v_{41},v_{4},v_{42}), (v_{1},u_{1},u_{2}), (v_{1},u_{2},u_{3}), (v_{3},u_{3},u_{4},u_{4})\}$ is magic graphoidal. Here the constant $K = 63$.

**REFERENCES**


