Magic Graphoidal on Class of Trees
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Abstract: B.D.Acharya and E. Sampathkumar [1] defined Graphoidal cover as partition of edge set of a graph G into internally disjoint paths (not necessarily open). The minimum cardinality of such cover is known as graphoidal covering number of G.

Let G = {V, E} be a graph and let ψ be a graphoidal cover of G. Define f:

V ∪ E → {1, 2, ..., p + q} such that for every path \( P = (v_0,v_1,v_2, \ldots, v_n) \) in ψ with \( f^*(P) = f(v_0) + f(v_n) + \sum_{i=1}^{n} f(v_{i-1}v_i) \) = k, a constant, where \( f^* \) is the induced labeling on ψ. Then, we say G admits ψ - magic graphoidal total labeling of G.

A graph G is called magic graphoidal if there exists a minimum graphoidal cover ψ of G such that G admits ψ - magic graphoidal total labeling.

In this paper, we proved that \([P_n;S_1], [P_n;S_2], T(n), P_m \接手 K_{1,3}, P_m \接手 2K_1 \) and \( K_{1,n} \接手 2K_1 \) are magic graphoidal

1. INTRODUCTION

By a graph, we mean a finite simple and undirected graph. The vertex set and edge set of a graph G are denoted by V(G) and E(G) respectively. Terms and notations not used here are as in [3].

1.1. Definition : Let \( S_1 = (v_0, v_1) \) be a star and let \([P_n; S_1] \) be the graph obtained from n copies of \( S_1 \) and the path \( P_n = (u_1, u_2, \ldots, u_n) \) by joining \( u_j \) with the vertex \( v_0 \) of the \( j^{th} \) copy of \( S_1 \) by means of an edge, for \( 1 \leq j \leq n \).

1.2. Definition : Let \( S_2 = (v_0, v_1, v_2) \) be a star and let \([P_n; S_2] \) be the graph obtained from n copies of \( S_2 \) and the path \( P_n = (u_1, u_2, \ldots, u_n) \) by joining \( u_j \) with the vertex \( v_0 \) of the \( j^{th} \) copy of \( S_2 \) by means of an edge, for \( 1 \leq j \leq n \).

1.3. Definition : Let T be any Tree. Denote the tree, obtained from T by considering two copies of T by adding an edge between them, by \( T(2) \) and in general the graph obtained from \( T(n-1) \) and T by adding an edge between them is denoted by \( T(n) \).

1.4. Result [11] : For a Tree T, \( \gamma(T) = n-1 \) where n is the number of pendant vertices of G.

2. PRELIMINARIES

Let G = {V, E} be a graph with p vertices and q edges. A graphoidal cover ψ of G is a collection of (open) paths such that

(i) every edge is in exactly one path of ψ

(ii) every vertex is an interval vertex of atmost one path in ψ.

We define \( \gamma(G) = \min_{\psi \in \zeta} |\psi| \),

where \( \zeta \) is the collection of graphoidal covers ψ of G and \( \gamma \) is graphoidal covering number of G.
Let \( \psi \) be a graphoidal cover of \( G \). Then we say that \( G \) admits \( \psi \) - magic graphoidal total labeling of \( G \) if there exists a bijection \( f: V \cup E \rightarrow \{1, 2, \ldots, p+q\} \) such that for every path \( P = (v_0v_1v_2 \ldots v_n) \) in \( \psi \), then, \( f^*(P) = f(v_0) + f(v_n) + \sum_{i=1}^{n} f(v_{i-1}v_i) = k \), a constant, where \( f^* \) is the induced labeling of \( \psi \). A graph \( G \) is called magic graphoidal if there exists a minimum graphoidal cover \( \psi \) of \( G \) such that \( G \) admits \( \psi \) - magic graphoidal total labeling.

3. Magical Graphoidal on Trees

3.1. Theorem : \([P_n; S_1], (n \text{ - even}) \) is magic graphoidal.

Proof: Let \( G = [P_n; S_1] \)

Let \( V(G) = \{ u_i, v_i, w_i : 1 \leq i \leq n \} \) and \( E(G) = \{ \{u_i, v_i\} \cup (v_i, w_i) : 1 \leq i \leq n \} \cup \{(u_i, u_{i+1}) : 1 \leq i \leq n-1\} \)

Define \( f : V \cup E \rightarrow \{1, 2, \ldots, p+q\} \) by

\[
\begin{align*}
f(w_1v_1) & = 1 \\
f(w_1v_1) & = 2 \\
f(v_1u_1) & = 3 \\
f(u_{i+1}) & = 6 + i \quad \text{for} \quad 1 \leq i \leq n-2 \\
f(w_{i+1}) & = 6n - i \quad \text{for} \quad 1 \leq i \leq n-1 \\
f(v_{i+1}v_{i+1}) & = 4n + 1 + i \quad \text{for} \quad 1 \leq i \leq n-1 \\
f(u_{i+1}v_{i+1}) & = 3n + 3 - 2i \quad \text{for} \quad 1 \leq i \leq (n/2) - 1 \\
\end{align*}
\]

\[
\begin{align*}
f \left( \begin{array}{c}
u_n \\
u_{n-i+1}
\end{array} \right) & = 3n + 4 - 2i \quad \text{for} \quad 1 \leq i \leq \frac{n}{2} \\
f(u_iu_{i+1}) & = \frac{3n}{2} + 4 + i \quad \text{for} \quad 1 \leq i \leq \frac{n}{2} - 1 \\
f \left( \begin{array}{c}
u_n \\
u_{n-i+1}
\end{array} \right) & = n + 4 + i \quad \text{for} \quad 1 \leq i \leq \frac{n}{2} \\
\end{align*}
\]

Let \( \psi = \{ P_1 = (w_1, v_1, u_1, u_2, v_2, w_2), P_2 = (u_i, u_{i+1}, v_{i+1}, w_{i+1}) : 2 \leq i \leq n - 1 \} \)

\[
\begin{align*}
f^*[P_1] & = f(w_1) + f(w_2) + f(v_1v_1) + f(u_1u_1) + f(u_1u_2) + f(v_2v_2) + f(v_2w_2) \\
& = 1 + 6n - 1 + 2 + 3 + \frac{3n}{2} + 4 + 1 + 3n + 3 - 2 + 4n + 2 \\
& = 13n + \frac{3n}{2} + 13 \quad \text{------- (A)}
\end{align*}
\]

For \( 2 \leq i \leq (n/2) - 1, \)

\[
\begin{align*}
f^*[P_2] & = f(u_i) + f(w_{i+1}) + f(u_iu_{i+1}) + f(u_{i+1}v_{i+1}) + f(v_{i+1}w_{i+1}) \\
& = 6 + i - 1 + 6n - i + (3n/2) + 4 + i + 3n + 3 - 2i + 4n + 1 + i \\
& = 13n + (3n/2) + 13 \quad \text{------- (B)}
\end{align*}
\]

For \( (n/2) \leq i \leq n - 1, \)
\[ f(P_2) = f(u_i) + f(w_{i+1}) + f(u_i u_{i+1}) + f(v_{i+1} w_{i+1}) \]

\[ = 6i - 1 + 6n - i + n + 4i - (n/2) + 1 + 3n + 4 - 2(i + 1 - (n/2)) + 4n + 1 + i \]

\[ = 13n + (3n/2) + 13 \quad \text{(C)} \]

From (A), (B) and (C), we conclude that \( G \) admits \( \psi \) - magic graphoidal total labeling. Hence, \([P_n ; S_1], \) (n - even) is magic graphoidal.

For example, consider the graph \([P_6 ; S_1]\) shown in figure 1.

![Figure 1 [P6 ; S1]](image-url)

Clearly, \( \psi = \{(w_1, v_1, u_1, u_2, v_2, w_2), (u_2, u_3, v_3, w_3), (u_3, u_4, v_4, w_4), (u_4, u_5, v_5, w_5), (u_5, u_6, v_6, w_6)\} \) is a minimum graphoidal cover and \([P_6 ; S_1]\) is magic graphoidal. Here the constant \( K = 100 \)

### 3.2. Theorem:
\([P_n ; S_1], \) (n - odd) is magic graphoidal.

**Proof:** Let \( G = [P_n ; S_1] \)

Let \( V(G) = \{u_i, v_i, w_i: \ 1 \leq i \leq n\} \) and

\[ E(G) = \{(u_i v_i) \cup (v_i w_i): 1 \leq i \leq n \} \cup \{(u_i u_{i+1}): 1 \leq i \leq n-1\} \]

Define \( f: V \cup E \rightarrow \{1, 2, ..., p+q\} \) by

\[
\begin{align*}
f(w_1) &= 1 \\
f(w_1 v_1) &= 2 \\
f(v_1 u_1) &= 3 \\
f(u_{i+1}) &= 5 + 2i \quad 1 \leq i \leq (n-1)/2 \\
f(w_{i+1}) &= 6n - i \quad 1 \leq i \leq n - 1 \\
f(v_{i+1} w_{i+1}) &= 4n + 1 + i \quad 1 \leq i \leq n - 1 \\
f(u_1 u_2) &= n + 5 \\
f(u_{i+1} u_{i+2}) &= \frac{3n + 1}{2} - i \quad 1 \leq i \leq \frac{n-1}{2} \\
f(u_{n+1}, u_{n+2}) &= \frac{3n + 9}{2} + i \quad 1 \leq i \leq \frac{n-3}{2} \\
f(u_{i+2} v_{i+2}) &= 3n + 2 - i \quad 1 \leq i \leq n - 2 \\
f(u_2 v_2) &= \frac{7n + 3}{2}
\end{align*}
\]
Let \( \psi = \{P_1 = (w_1,v_1,u_1,u_2,v_2,w_2), \ P_2 = (u_i,u_{i+1},v_{i+1},w_{i+1}) : 2 \leq i \leq n - 1\} \)

\[
f^* [P_1] = f(w_1) + f(w_2) + f(v_1u_1) + f(u_1u_2) + f(u_2v_2) + f(v_2w_2)
= 1 + 6n - 1 + 2 + 3 + n + 5 + \frac{7n + 3}{2} + 4n + 1 + 1
= 14n + \frac{n + 1}{2} + 13 \quad \text{(A)}
\]

For \( 2 \leq i < \frac{n+1}{2} \),

\[
f^* [P_2] = f(u_i) + f(w_{i+1}) + f(u_{i+1}v_{i+1}) + f(v_{i+1}w_{i+1})
= 5 + 2(i - 1) + 6n - i + \frac{3n + 11}{2} - (i - 1) + 3n + 2 - (i - 1) + 4n + 1 + i
= 14n + \frac{n + 1}{2} + 13 \quad \text{(B)}
\]

For \( \frac{n+1}{2} \leq i \leq n - 1 \),

\[
f^* [P_2] = f(u_i) + f(w_{i+1}) + f(u_{i+1}v_{i+1}) + f(v_{i+1}w_{i+1})
= 6 + 2\left(i - \frac{n+1}{2}\right) + 6n - i + \frac{3n + 9}{2} + (n - i) + 3n + 2 - (i - 1) + 4n + 1 + i
= 14n + \frac{n + 1}{2} + 13 \quad \text{(C)}
\]

From (A), (B) and (C), we conclude that \( \psi \) is minimum magic graphoidal cover
Hence, \([P_n ; S_1]\), \((n \text{ even})\) is magic graphoidal.

For example, consider the graph \([P_7 ; S_1]\) shown in figure 2.

Clearly, \( \psi = \{(w_1,v_1,u_1,u_2,v_2,w_2), (u_2,u_3,v_3,w_3), (u_3,u_4,v_4,w_4), (u_4,u_5,v_5,w_5), (u_5,u_6,v_6,w_6), (u_6,u_7,v_7,w_7)\}\) is a minimum graphoidal cover and \([P_7 ; S_1]\) is magic graphoidal. Here the constant \(K = 115\).

3.3. Theorem : \([P_n ; S_2]\) is magic graphoidal.

Proof: Let \( G = [P_n ; S_2]\)

\[V(G) = \{(u_i,v_i) : 1 \leq i \leq n\}\]
E(G) = \{ (uiui+1) : 1 \leq i \leq n-1 \} \cup \{(uivi) : 1 \leq i \leq n \} \\ \cup \{(vi vi_1) \cup (vi vi_2) : 1 \leq i \leq n \}

Define f : V \cup E \rightarrow \{1, 2, \ldots, p+q\} by

f(v_1) = 1
f(u_1v_1) = 2 f
(u_1) = 3
f(u_i+1) = 3 + i \quad 1 \leq i \leq n-2
f(u_n) = p + q
f(v_i) = n + 1 + i \quad 1 \leq i \leq n
f(v_i+1) = 8n - 1 - i \quad 1 \leq i \leq n-1
f(v_{i2}) = 7n - i \quad 1 \leq i \leq n
f(v_{i2}) = 5n - 1 + i \quad 1 \leq i \leq n
f(u_{i1}v_{i+1}) = 4n + i \quad 1 \leq i \leq n-1
f(u_{i+1}u_{i+1}) = 4n + 1 - i \quad 1 \leq i \leq n-1
f(v_{i1}) = 3n + 2 - i \quad 1 \leq i \leq n

Let \( \psi = \{(v_{i1},v_{i2},v_{i2}): 1 \leq i \leq n\}, P_2=(v_1,u_1,u_2,v_2), P_3=\{(u_{i1},u_{i+1},v_{i+1}): 2 \leq i \leq n-1\}\)

For \( 1 \leq i \leq n \),
\[
\hat{f}^f[P_1] = f(v_{i1}) + f(v_{i2}) + f(v_{1}v_{i}) + f(v_{i}v_{i2})
\]
\[
= n + 1 + i + 7n - i + 3n + 2 - i + 5n - 1 + i
\]
\[
= 16n + 2 \quad \text{--------- (A)}
\]
\[
\hat{f}^f[P_2] = f(v_{i1}) + f(v_{i2}) + f(v_{1}u_{i1}) + f(u_{1}u_{2}) + f(u_{2}v_{2})
\]
\[
= 1 + 8n - 2 + 2 + 4n + 4n + 1
\]
\[
= 16n + 2 \quad \text{--------- (B)}
\]

For \( 2 \leq i \leq n-1 \),
\[
\hat{f}^f[P_3] = f(u_{i1}) + f(v_{i+1}) + f(u_{i1}u_{i+1}) + f(u_{i+1}v_{i+1})
\]
\[
= 3 + i - 1 + 8n - 1 - i + 4n + 1 - i + 4n + i
\]
\[
= 16n + 2 \quad \text{--------- (C)}
\]

From (A), (B) and (C), we conclude that G admits \( \psi \) - magic graphoidal total labeling. Hence, \([P_n : S_2]\) is magic graphoidal.

For example, consider the graphs \([P_3 : S_2]\) and \([P_4 : S_2]\) shown in figure 3.1 and 3.2.

![Figure 3.1 \([P_3 : S_2]\)](image)

Clearly, \( \psi = \{(v_{11},v_{11},v_{12}), (v_{21},v_{22},v_{22}), (v_{31},v_{33},v_{32}), (v_{11},u_{11},u_{22}), (u_{22},u_{33},v_{33})\} \) is a minimum graphoidal cover and \([P_3 : S_2]\) is magic graphoidal. Here the constant \( K = 50 \)
Clearly, $\psi = \{(v_{11}, v_1, v_{12}), (v_{21}, v_2, v_{22}), (v_{31}, v_3, v_{32}), (v_{41}, v_4, v_{42}), (v_{1}, u_1, u_{2}), (u_2, u_3, v_3), (u_3, u_4, v_4)\}$ is a minimum graphoidal cover and $[P_4; S_2]$ is magic graphoidal. Here the constant $K = 66$

3.4. Theorem: For a Tree, $T_{(n)}$ is magic graphoidal.

Proof:

Let $T_{(n)}$ be a graph such that 
$V[T_{(n)}] = \{u_{i1}, u_{i2}, u_{i3}, u_{i4}, u_{i5} : 1 \leq i \leq n\}$ and 
$E[T_{(n)}] = \{(u_{i1}u_{i2}), (u_{i2}u_{i3}), (u_{i3}u_{i4}), (u_{i4}u_{i5}) : 1 \leq i \leq n\} \cup [(u_{i5}u_{i+1}n) : 1 \leq i \leq n-1]\}$

Define $f: V \cup E \rightarrow \{1, 2, ..., p+q\}$ by

- $f(u_{i1}) = i \quad 1 \leq i \leq n$
- $f(u_{i3}) = n + 1$
- $f(u_{i3}u_{i5}) = n + 2$
- $f(u_{i1}u_{i2}) = 2n + 3 - i \quad 1 \leq i \leq n$
- $f(u_{i1}u_{i2}u_{i3}u_{i4}) = 5n - 3 - 2i \quad 1 \leq i \leq n - 3$
- $f(u_{i5}u_{i+1}i, u_{i+1}u_{i+2}) = 4n - 1 + i \quad n - 2 \leq i \leq n - 1$
- $f(u_{i3}u_{i+1}i, u_{i+1}u_{i+2}) = 5n - 2 + i \quad 1 \leq i \leq n$
- $f(u_{i1}u_{i3}u_{i+1}u_{i+2}) = 6n - 2 + i \quad 1 \leq i \leq n$
- $f(u_{i5}u_{i+1}i, u_{i+1}u_{i+2}) = 7n - 2 + i \quad 1 \leq i \leq n - 1$
- $f(u_{i4}) = 8n - 2 + 2(i-1) \quad 1 \leq i \leq n$
- $f(u_{i1}u_{i+1}i) = 8n - 1 + 2(i-1) \quad 1 \leq i \leq n - 1$

Let $\psi = \{P_1 = (u_{i1}u_{i+2}u_{i+3}u_{i+4}), P_2 = (u_{i3}u_{i5}u_{i+2}u_{i+3}), P_3 = [(u_{i5}u_{i+1}i, u_{i+1}i'] : 2 \leq i \leq n - 1]\}$

$f'[P_1] = f(u_{i1}) + f(u_{i4}) + f(u_{i1}u_{i2}) + f(u_{i2}u_{i3}) + f(u_{i3}u_{i4})$

$= i + 8n - 2 + 2(i-1) + 2n + 3 - i + 5n - 2 + (n + 1 - i) + 6n - 2 + (n + 1 - i)\$

$= 23n - 3 \quad \text{(A)}$

$f'[P_2] = f(u_{i3}) + f(u_{i3}u_{i4}) + f(u_{i+1}u_{i+2}u_{i+3}) + f(u_{i+1}u_{i+2}u_{i+3})$

$= 2n + 2 + i - 1 + 8n - 1 + 2(i - 1) + 5n - 3 - 2(i - 2) + 7n - 2 + n - i\$

$= 23n - 3 \quad \text{(B)}$

$f'[P_3] = f(u_{i5}) + f(u_{i2}u_{i3}u_{i4}) + f(u_{i5}u_{i+2}u_{i+3}) + f(u_{i5}u_{i+2}u_{i+3})$

$= n + 1 + 8n - 1 + 2(i - 1) + 5n - 3 - 2(i - 2) + 7n - 2 + n - i\$

$= 23n - 3 \quad \text{(C)}$

From (A), (B) and (C), we conclude that $G$ admits $\psi$ - magic graphoidal total labeling. Hence, $T_{(n)}$ is magic graphoidal.
For example, consider the graph \( T(3) \) shown in figure 4.

![Figure 4 T(3)](image)

Clearly, \( \psi = \{(u_{11}, u_{12}, u_{13}, u_{14}), \ldots + f(u_i) + f(u_{ui}) \} \)

= 2n + 3 + 3n + 4 + i – 2 + 4n + 2 + n + 1 – i

= 10n + 8  ---------- (B)

For 1 \( \leq i \leq n \),

**3.5. Theorem:** The graph Double Crowned Star \( K_{1,n} \circ K_1 \) is magic graphoidal.

**Proof:** Let \( G = K_{1,n} \circ 2K_1 \)

\[ V(G) = \{u, [u_i : 1 \leq i \leq n], [(u_{1i}, u_{i2}) : 1 \leq i \leq n] \} \] and

\[ E(G) = \{ [(u_{1i}) : 1 \leq i \leq n] \cup [(u_{1i}) \cup (u_{i2}) : 1 \leq i \leq n] \} \]

Define \( f : V \cup E \to \{1, 2, \ldots, p+q\} \) by

\[
\begin{align*}
f(u) & = 2n + 3 \\
f(u_{1i}) & = i \quad 1 \leq i \leq n \\
f(u_{i2}) & = n + 2 \\
f(u_{n+1-i,2}) & = n + 2 + i \quad 1 \leq i \leq n \\
f(u_{ui}) & = 2n + 3 + i \quad 1 \leq i \leq n \\
f(u_{u1}) & = 3n + 4 \\
f(u_{u2}) & = 3n + 4 + i \quad 1 \leq i \leq n - 2 \\
f(u_{un+1-i}) & = 4n + 2 + i \quad 1 \leq i \leq n - 1 \\
f(u_{un+1-i, u_{n+1-i,2}}) & = 5n + 1 + i \quad 1 \leq i \leq n 
\end{align*}
\]

Let \( \psi = \{P_1 = (u_1, u_2, u_3), P_2 = [(u_i) : 3 \leq i \leq n], P_3 = [(u_i, u_i, u_{i2}) : 1 \leq i \leq n] \} \)

\[
f^f[P_1] = f(u_1) + f(u_2) + f(u_{1i}) + f(u_{u1})
\]

= \( n + 1 + n + 2 + 3n + 4 + 5n + 1 \)

= \( 10n + 8 \)  ---------- (A)

For \( 3 \leq i \leq n \),

\[
f^f[P_2] = f(u) + f(u_i) + f(u_{u1})
\]

= \( 2n + 3 + 3n + 4 + i – 2 + 4n + 2 + n + 1 – i \)

= \( 10n + 8 \)  ---------- (B)

For \( 1 \leq i \leq n \),
\[ f'[P_3] = f(u_{i1}) + f(u_{i2}) + f(u_{i1}u_i) + f(u_{i1}u_{i2}) \]
\[ = i + n + 2 + n + 1 - i + 2n + 3 + i + 5n + 1 + n + 1 - i \]
\[ = 10n + 8 \quad \text{(C)} \]

From (A), (B) and (C), we conclude that \( G \) admits \( \psi \) - magic graphoidal total labeling. Hence, Double Crowned Star \( K_{1,n} \odot 2K_1 \) is magic graphoidal.

For example, consider the graph \( K_{1,5} \odot 2K_1 \) shown in figure 5.

Clearly, \( \psi = \{(u_{i1},u_{i1}u_{i2}), (u_{i2},u_{i3},u_{i2}), (u_{i4},u_{i5},u_{i2}), (u_{i5},u_{i6},u_{i2}), (u_{i3}), (u_{i4}), (u_{i5})\} \) is a minimum graphoidal cover and \( K_{1,5} \odot 2K_1 \) is magic graphoidal. Here the constant \( K = 58 \).

3.6. **Theorem** : \( P_m \odot 2K_1 \) is magical graphoidal.

**Proof** : Let \( G = P_m \odot 2K_1 \)

- \( V(G) = \{(u_i : 1 \leq i \leq m), (u_{ij} : 1 \leq i \leq m, 1 \leq j \leq 2)\} \) and
- \( E(G) = \{[(u_i, u_{i+1}) : 1 \leq i \leq m-1] \cup [(u_i, u_{ij}) : 1 \leq i \leq m, 1 \leq j \leq 2]\} \)

Let \( \psi = \{P_1 = [(u_{i1},u_{i1}u_{i2}) : 1 \leq i \leq m], P_2 = [(u_i, u_{i+1}) : 1 \leq i \leq m-1]\} \)

Define \( f : V \to \{0, 1, 2, \ldots, 6m-1\} \) by

\[
 f(u_i) = \begin{cases} 
 i & 1 \leq i \leq m \\
 2m+1-i & 1 \leq i \leq m 
\end{cases}
\]

\[
 f(u_{i1}u_{i2}) = \begin{cases} 
 2m+i & 1 \leq i \leq \frac{m}{2} \\
 2m+i & 1 \leq i \leq \frac{m-1}{2} 
\end{cases}
\] if \( m \) is even

\[
 f(u_{i2}) = \begin{cases} 
 2m+i & 1 \leq i \leq \frac{m}{2} \\
 2m+i & 1 \leq i \leq \frac{m-1}{2} 
\end{cases}
\] if \( m \) is odd

![Figure 5. K_{1,5} \odot 2K_1](image-url)
\[
\begin{align*}
f(u_{i1}) &= \begin{cases} 
\frac{5m}{2} + i & 1 \leq i \leq m \text{ if } m \text{ is even} \\
\frac{5m-1}{2} + i & 1 \leq i \leq m \text{ if } m \text{ is odd}
\end{cases} \\
f(u_{3i-1}) &= \begin{cases} 
\frac{7m}{2} + i & 1 \leq i \leq \frac{m}{2} \text{ if } m \text{ is even} \\
\frac{7m-1}{2} + i & 1 \leq i \leq \frac{m+1}{2} \text{ if } m \text{ is odd}
\end{cases} \\
f(u_{m+1-i, m-i}) &= 4m + i \quad 1 \leq i \leq m - 1 \\
f(u_{m+1-i, 2}) &= 5m - 1 + i \quad 1 \leq i \leq m
\end{align*}
\]

**Case (i):**  when m is odd

For \(1 \leq i \leq m\),
\[
f^*[P_1] = f(u_{i1}) + f(u_{i2}) + f(u_{i1}u_i) + f(u_{i2}u_i) \\
= \frac{5m-1}{2} + i + 5m - 1 + m + 1 - i + i + 2m + 1 - i \\
= \frac{21m + 1}{2} \quad (A)
\]

For \(1 \leq i \leq m - 1\), \(i \equiv 1 \text{ mod } 2\)
\[
f^*[P_2] = f(u_i) + f(u_{i+1}) + f(u_{i}u_{i+1}) \\
= \frac{7m-1}{2} + i + 1 + i + 2 + 2m + \frac{i+1}{2} + 4m + m - i \\
= \frac{21m + 1}{2} \quad (B)
\]

For \(1 \leq i \leq m - 1\), \(i \equiv 0 \text{ mod } 2\)
\[
f^*[P_2] = f(u_i) + f(u_{i+1}) + f(u_{i}u_{i+1}) \\
= 2m + \frac{i}{2} + \frac{7m-1}{2} + \frac{i+2}{2} + 4m + m - i \\
= \frac{21m + 1}{2} \quad (C)
\]

From (A), (B) and (C), we conclude that \(G\) admits \(\psi\) - magic graphoidal total labeling. Hence, \(P_m \boxtimes 2K_1\) (m-odd) is magic graphoidal.

For example, consider the graph \(P_5 \boxtimes 2K_1\) shown in figure 6.1.

**Case (ii):**  when m is even

For \(1 \leq i \leq m\),
\[
f^*[P_1] = f(u_{i1}) + f(u_{i2}) + f(u_{i1}u_i) + f(u_{i2}u_i)
\]
\[
\frac{5m}{2} = i + 5m - 1 + m + 1 - i + i + 2m + 1 - i
\]

= \[
\frac{21m + 2}{2} \quad \text{------- (A)}
\]

For \(1 \leq i \leq m - 1, \ i \equiv 1 \mod 2\)
\[
f'[P_2] = f(u_i) + f(u_{i+1}) + f(u_iu_{i+1})
\]

= \[
\frac{7m + i + 1}{2} + 2m + \frac{i + 1}{2} + 4m + m - i
\]

= \[
\frac{21m + 2}{2} \quad \text{------- (B)}
\]

For \(1 \leq i \leq m - 1, \ i \equiv 0 \mod 2\)
\[
f'[P_2] = f(u_i) + f(u_{i+1}) + f(u_iu_{i+1})
\]

= \[
2m + \frac{i}{2} + \frac{7m}{2} + \frac{i}{2} + 4m + m - i
\]

= \[
\frac{21m + 2}{2} \quad \text{------- (C)}
\]

From (A), (B) and (C), we conclude that \(G\) admits \(\psi\) - magic graphoidal total labeling. Hence, \(P_m \ominus 2K_1\) (m-even) is magic graphoidal.

For example, consider the graph \(P_4 \ominus 2K_1\) shown in figure 6.2.

![Figure 6.1 P_5 \ominus 2K_1](image)

Clearly, \(\psi = \{(u_{11},u_{1},u_{12}), (u_{21},u_{2},u_{22}), (u_{31},u_{3},u_{32}), (u_{41},u_{4},u_{42}), (u_{51},u_{5},u_{52}), (u_{1},u_{2}), (u_{2},u_{3}), (u_{3},u_{4}), (u_{4},u_{5})\}\) is a minimum graphoidal cover and \(P_4 \ominus 2K_1\) is magic graphoidal. Here the constant \(K = 53\).
Clearly, \( \psi = \{(u_{i1},u_{i2}),(u_{i3},u_{i4}),(u_{i5},u_{i6}),\ldots,(u_{i8},u_{i9})\} \) is a minimum graphoidal cover and \( P_4 \odot 2K_1 \) is magic graphoidal. Here the constant \( K = 43 \).

**3.7. Theorem:** \( P_m \odot K_{1,3} \) is magic graphoidal.

**Proof:** Let \( G = P_m \odot K_{1,3} \)

\[
V(G) = \{(u_i,v_i) : 1 \leq i \leq m, 1 \leq i \leq 3\}
\]

\[
E(G) = \{(u_i,u_{i+1}) : 1 \leq i \leq m-1 \} \cup \{(v_i,v_{i+1}) : 1 \leq i \leq m, 1 \leq j \leq 3\}
\]

with \( u_i = u_{i+1}, 1 \leq i \leq m \)

Let \( \psi = \{(v_{i1},v_{i2}) : 1 \leq i \leq m\} \), \( P_2 = \{(v_i,u_{i+1}) : 1 \leq i \leq m-2\} \),

\[
P_3 = \{(v_{i1},u_{i2},u_m,v_m)\}
\]

Define \( f : V \cup E \to \{1, 2$, ..., $8m-1\} \) by

\[
f(v_m) = 1
\]

\[
f(v_{i1}) = i+1, \quad 1 \leq i \leq m
\]

\[
f(u_{i1}) = m+1+i, \quad 1 \leq i \leq m-1
\]

\[
f(u_{i1}) = 2m+i, \quad 1 \leq i \leq m-1
\]

\[
f(u_{i1}) = 3m-1+i, \quad 1 \leq i \leq m
\]

\[
f(v_{i1}) = 4m-1+i, \quad 1 \leq i \leq m
\]

\[
f(v_{i2}) = 5m-1+i, \quad 1 \leq i \leq m-1
\]

\[
f(u_m) = 6m-1
\]

\[
f(u_{m-1}) = 6m+i, \quad 1 \leq i \leq m-2
\]

\[
f(v_{m-1}) = 7m-2+i, \quad 1 \leq i \leq m
\]

For \( 1 \leq i \leq m \),

\[
f^*[P_1] = f(v_{i1}) + f(v_{i2}) + f(v_{i1}) + f(v_{i2})
\]

\[
= 4m-1+i+7m-2+i+5m-1+i+3m-1+m+1-i
\]

\[
= 16m - 1 \quad \text{--------- (A)}
\]

For \( 1 \leq i \leq m-2 \),

\[
f^*[P_2] = f(v_i) + f(u_{i+1}) + f(u_i) + f(u_{i+1})
\]

\[
= 5m-1+i+6m+(m-i-1)+m+1+i+2m+m-i
\]

\[
= 16m - 1 \quad \text{--------- (B)}
\]
\[ f'[P_3] = f(v_{m-1}) + f(v_m) + f(u_{m-1}u_m) + f(u_mv_m) \]
\[ = 6m - 2 + 1 + 2m + 2m + 1 + 6m - 1 \]
\[ = 16m - 1 \quad \text{(C)} \]

From (A), (B) and (C), we conclude that \( G \) admits \( \psi \) - magic graphoidal total labeling. Hence, \( P_m \bowtie K_{1,3} \) is magic graphoidal.

For example, consider the graph \( P_4 \bowtie K_{1,3} \) shown in figure 7.

\[ \text{Figure 7. } P_4 \bowtie K_{1,3} \]

Clearly, \( \psi = \{(v_{11},v_{1},v_{12}), (v_{21},v_{2},v_{22}), (v_{31},v_{3},v_{32}), (v_{41},v_{4},v_{42}), (v_{1},u_{1},u_{2}), (v_{1},u_{2},u_{3}), (v_{3},u_{3},u_{4},v_{4}) \} \) is magic graphoidal. Here the constant \( K = 63 \).

**REFERENCES**