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Locating Equitable Domination and Independence Subdivision Numbers of Graphs

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Abstract: Let G = (V,E) be a simple, undirected, finite nontrivial graph. A non empty set $D \subseteq V$ of vertices in a graph G is a dominating set if every vertex in V-D is adjacent to some vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set D is a locating equitable dominating set of G if for any two vertices $u,w \in V-D$, $N(u)\cap D \neq N(w)\cap D$, $|N(u)\cap D| = |N(w)\cap D|$. The locating equitable domination number of G is the minimum cardinality of a locating equitable dominating set of G. The locating equitable domination subdivision number of G is the minimum number of edges that must be subdivided(where each edge in G can be subdivided at most once) in order to increase the locating equitable domination number and is denoted by $sd\gamma_{le}(G)$. The independence subdivision number $sd\beta_{le}(G)$ to equal the minimum number of edges that must be subdivided in order to increase the independence number. In this paper, we establish bounds on $sd\gamma_{le}(G)$ and $sd\beta_{le}(G)$ for some families of graphs.

1. Introduction

For notation and graph theory terminology, we in general follow [3]. Specifically, a graph G is a finite nonempty set V(G) of objects called vertices together with a possibly empty set E(G) of 2-element subsets of V(G) called edges. The order of G is $n(G) = |V(G)|$ and the size of G is $m(G) = |E(G)|$. The degree of a vertex $v \in V(G)$ in G is $d_G(v) = |N_G(v)|$. A vertex of degree one is called an end-vertex. The minimum and maximum degree among the vertices of G is denoted by $\delta(G)$ and $\Delta(G)$, respectively. Further for a subset $S \subseteq V(G)$, the degree of v in S, denoted ds(v), is the number of vertices in S adjacent to v; that is,$d_S(v) = |N(v)\cap S|$. In particular, $d_G(v) = d_V(v)$. if the graph G is clear from the context, we simply write V,E,n,m,d(v), δ and Δ rather than V(G),E(G),n(G),m(G),$d_G(v)$, $\delta(G)$ and $\Delta(G)$, respectively.

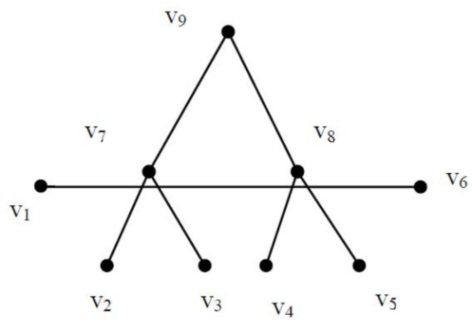
The closed neighborhood of a vertex $u \in V$ is the set $N[u]= \{u\}\cup\{v/uv\in\}$. Given a set $S \subseteq V$ of vertices and a vertex $u \in S$, the private neighbor set of u, with respect to S, is the set $pn[n,S] = N[u] - N[S - \{u\}]$. We say that every vertex $v \in pn[u,S]$ is a private neighbor of u with respect to S. Such a vertex v is adjacent to u but is not adjacent to any other vertex of S, then it is an isolated vertex in the subgraph G[S] induced by S. In this case, $u \in pn[u,S]$, and we say that u is its own private neighbor. We note that if a set s is a $\gamma(G)$-set, then for every vertex $u \in S$, $pn[u,S] \neq \varphi$, i.e., every vertex of S has at least one private neighbor. It can be seen that if S is a $\gamma(G)$-set, and two vertices $u,v \in S$ are adjacent, then each of u and v must have a private neighbor other than itself. We will slao use the following terminology. Let $v \in V$ be a vertex of degree one; v is called a leaf. The only vertex adjacent to a leaf, say u, is called a support vertex, and the edge uv is called a pendant edge.Two edges in a graph G are independent if they are not adjacent in G. The distance $d_G(u,v)$ or d(u,v) between two vertices u and v in a graph G, is the length of a shortest path connecting u and v. The diameter of a connected graph G is defined to be max $\{d_G(u,v) : u,v \in V(G)\}$. A set $D \subseteq V$ of vertices is a dominating set if every vertex in V-D is adjacent to some vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G.

The locating equitable domination subdivision number of a graph G, denoted by $sd\gamma_{le}(G)$, equals the minimum number of edges that must be subdivided in order to create a graph G' for which $\gamma_{le}(G') > \gamma_{le}(G)$. The independence number $\beta(G)$ is the maximum cardinality of an independent set in G.we call an independent set S of cardinality $\beta(G)$ a $\beta(G)$-set. The locating equitable independence subdivision number $sd\beta_{le}(G)$ to equal the minimum number of edges that must be subdivided in order to create a graph G' for which $\beta_{le}(G') > \beta_{le}(G)$.

Results on the locating equitabdomination and independence subdivision numbers are given in sections 2 and 3 respectively.

Example 1.The following is an example of a graph whose $sdg_{le}(G) = 1$.

Let G = (V,E)



V= {v1,v2,v3,v4,v5,v6,v7,v8,v9}, D= { v1,v2,v3,v4,v5,v6,v9} is a γ_{le}- set, $\gamma_{le}(G)= 7$.

After making a single subdivision in G, the number γ_{le} will be change. If we subdivide the edge v_7v_8 then γ_{le} increases. Hence $sd\gamma_{le}(G) = 1$.

2. Results on the Locating Equitable Domination Subdivision numbers

(i) $sd\gamma_{le}(K_n) = 2$

(ii) $sd\gamma_{le}(P_n) = \begin{cases} 1 \text{ if } n \text{ is odd} \\ 2 \text{ if } n \text{ is even} \end{cases}$

(iii) $sd\gamma_{le}(C_n) = \begin{cases} 1 \text{ if } n \text{ is even} \\ 2 \text{ if } n \text{ is odd} \end{cases}$

Proposition 2.1. For any tree T of order $n \geq 3$, $1 \leq sd\gamma_{le}(T) \leq 3$.

Proposition 2.2. For any k-regular graph G where $k \geq 2$, $1 \leq sd\gamma_{le}(G) \leq 3$.

Theorem 2.3. For any connected graph G of order $n \geq 3$, and for any two adjacent vertices u and v, where $\deg(u) \geq 2$ and $\deg(v) \geq 2$, $sd\gamma_{le}(G) \leq \deg(u) + \deg(v) - 1$.

Proof. Let uv be an edge in G, and let G' be the graph which results from subdividing all edges incident to u and v. Thus, deg(u) + deg(v) -1 edges will be subdivided. We assume that both $\deg(u) \geq 2$ and $\deg(v) \geq 2$. We will show that $\gamma_{le}(G') > \gamma_{le}(G)$ by showing that (I) no $\gamma_{le}(G)$-set is also locating equitable dominating set of G' and (II) there is no locating equitable dominating set of G' of cardinality $\gamma(G)$ that contains a subdivision vertex.

(i) Let D be an arbitrary $\gamma_{le}(G)$-set.We will show that D is not a locating equitable dominating set of G'.

Caes (1). $u,v \in D$. In this case, both u and v must have private neighbors other than themselves.But then neither u nor v dominate locating equitably these private neighbors in G'. Case(2). Either $u \notin D$ or $v \notin D$. in this case, D no longer locating equitable dominates $\{u,v\} \cap (V- D)$ in G'.

(ii) Let D be a subset of G' of cardinality $\gamma_{le}(G)$ which contains at least one subdivision vertex. We will show that D is not a locating equitable dominating set of G'.

Assume to the contrary that G' contains a locating equitable dominating set of cardinality $\gamma_{le}(G)$ which contains at least one subdivision vertex.Among all such dominating sets, let D* be one which contains a minimum number of subdivision vertices. Assume, without loss of generality, that D* contains a subdivision vertex adjacent to v, call it v', which subdivides the edge vw ($w \neq u$).

It follows that $v \notin D^*$, since if $v \in D^*$, then $D = D^* - \{v'\} \cup \{w\}$ is a locating equitable dominating set of G' of cardinality $\gamma_{le}(G)$ contains fewer subdivision vertices than D*, contradicting the minimality of D*.

Clearly, v' can only be used to locating equitable dominate vertices v, v' and w. It follows that no other subdivision vertex adjacent to v is in D*, since any such vertices could be exchanced with their neighbors not equal to v, to create a locating equitable dominating set of the same cardinality with fewer subdivision vertices, again contradicting the minimality of D*. It follows, therefore, that $u \in D^*$ since D* is a dominating set and u is the only vertex available to locating equitable dominate the subdivision vertex, say x, between u and v, and $x \notin D^*$. Then no subdivision vertex adjacent to u is in D*, since $x \notin D^*$ and any other such vertex can be exchanged with its neighbor with fewer subdivision vertices than D*, again contradicting the minimality of D*.

At this pont we have established that (i) $v \notin D^*$, (ii) $u, v' \in D^*$, and (iii) every neighbor of v in G other than w is in D*, since the subdivision vertices adjacent to v are not in D* and must be locating equitable dominated. In fact, D* contains only one subdivision vertex, namely v'.But if D* is a locating equitable dominating set of G' of cardinality $\gamma_{le}(G)$, then it follows that $D = D^* - \{u,v'\} \cup \{v\}$ is a dominating set of G of cardinality less than $\gamma_{le}(G)$, a contradiction. (This follows from the observation that v' is only needed to dominate vertices v, v' w in G', and u is only needed to dominate itself and the subdivision vertices adjacent to it in G'). Earlier in the proof, we assumed that D* contains a subdivision vertex adjacent tov, call it v', which subdivides the edge vw ($u \neq w$). It remains to consider the final case that D* contains the subdivision vertex x between vertices u and v.In this case we can assume that D* contains no other subdivision vertex, Otherwise, they could be exchanged, as before, with their neighbors not equal to either u or v, to produce a dominating set of the same cardinality but with fewer subdivision vertices, contradicting the minmimality of D*. but vertex x can only be used to dominate vertices u,x andv, which means that D* cannot contain both u and v (else vertex x is not needed). Therefore there are only three cases to consider.

Case 1. $u \in D^*$ and $v \notin D^*$.

Case2. $u \notin D^*$ and $v \in D^*$.

Case 3. $u \notin D^*$ and $v \notin D^*$.

But in each of the first two cases, it can be seen that the set $D^*-\{x\}$ is a locating equitable dominating set of G of cardinality less than $\gamma_{le}(G)$, a contradiction. In case 3, since $deg(u) \geq 2$ and $deg(v) \geq 2$, then $D^*-\{x\}$ is a locating equitable dominating set of G of cardinality less than $\gamma_{le}(G)$, since every neighbor of u or v in G,other than u and v, is in D*, a contradiction.

Theorem 2.4. For any connected graph G of order $n \geq 3$, and for any two adjacent vertices u and v, where $deg(u) \geq 2$ and $deg(v) \geq 2$, $sd\gamma_{le}(G) \leq deg(u) + deg(v) - |N(u) \cap N(v)| - 1$.

Proof. We may assume that $N(u) \cap N(v) \neq \varphi$ for otherwise our result follows from theorem2.3.let $N(v) = \{v_1, v_2, \ldots v_k\}$ where $u=v_1$ and if $N(u) \cap N[v] \neq \varphi$. Let $N(u) \cap N[v] = \{u_1, u_2, \ldots u_t\}$. Let G' be the graph obtained by subdividing the edge vv_i with subdivision vertex x_i , for $1\leq i\leq k$, and the edge uuj for $1\leq j\leq t$. Let A be the set of the subdivision vertices and D' a $\gamma_{le}(G')$-set. Clearly no vertex of G locating equitable dominates v in G', and so $|D'\cap A|\geq 1$. We Show that $\gamma_{le}(G')> \gamma_{le}(G)$. IT suffices for us to show that $\gamma_{le}(G)\leq |D'|-1$, since then$\gamma_{le}(G')= |D'| \geq \gamma_{le}(G)+1$. One of u or v must be in D' to locating equitable dominate x_1. If both u and v are in D', then D' – A is a LEDS of G, and so $\gamma_{le}(G)\leq |D'-A| \leq |D'|-1$ Assume $u\in D'$ and $v\notin D'$. Then every neighbor of v in G is in D' to locating equitable dominate $\{ x_1,x_2,\ldots.x_k\}$ and some xi is in D' to dominate v. If $|D'\cap A|\geq 2$, then $(D' -A)\cup\{v\}$ is a locating equitable dominating setof G, and so $\gamma_{le}(G)\leq |D'-A|+1\leq |D'|-1$. On the other hand, if $| D'\cap A|=1$, then since D'-{u} locating equitable dominates N(u) – N[v], it follows that $(D'-A-\{u\})\cup\{v\}$ is a LEDS of G and so $\gamma_{le}(G)\leq |D'-A| \leq |D'|-1$.Assume $v\in D'$ and $u\notin D'$. If $|D'\cap A|\geq 2$, then $(D' -A) \cup\{u\}$ is a locating equitable dominating set of G, and so $\gamma_{le}(G)\leq |D'- A|+1\leq |D'|-1$. Therefore ,assume that $| D'\cap A|=1$. The vertex of A in D' is a neighbor of v in G'. If $x_1 \in D'$, then all neighbors of v in G ,except for possible u, are locating equitably dominated by $D'-\{v, x_1\}$. Thus, $(D' - \{v,x_1\}\cup\{v_i\}$ for some vertex $v_i\in N(u)\cap N(v)$ is a locating equitable dominating set of G, and so $\gamma_{le}(G)\leq |D'|-1$.Therefore, we assume that $x_1\notin D'$. But then some $v_i \in N_G(v)\cap N_G(u)$ must be in D' to locating equitably dominate u, whence D' –A is a locating equitable dominating set of G and $\gamma_{le}(G)\leq |D'|-1$.

3. Locating Equitable Independence Subdivision numbers.

The independence subdivision number of any graph is either one or two.We then characterize the class of graphs having independence subdivision number two.

If $sd\beta_{le}(G) = 1$, for some graph G = (V,E), then , by definition, there must exist an edge uv $\in$E,which when subdivided into edges ux and xv results in a graph G' for which $\beta_{le}(G') =\beta_{le}(G) + 1$. This can happen in only one of two ways, either G has a $\beta_{le}(G)$ – set which does not contain either u or v, or uv is a pendant edge and G has a $\beta_{le}(G)$ – set D which contains the support vertex u but not the leaf v, in which case $D\cup\{v\}$ becomes a larger independent set when the edge uv is subdivided into ux and xv.

Proposition 3.1. For every graph G having a $\beta_{le}(G)$ – set D, where the subgraph G[V-D] induced by V-D has at least one edge, $sd\beta_{le}(G) = 1$.

Corollary 3.2. For every graph G having an odd cycle**,** $sd\beta_{le}(G) = 1$.

Proposition 3.3. For every graph G having a $\beta_{le}(G)$ – set D and a pendant edge uv, where D contains the support vertex u (and not the leafv), $sd\beta_{le}(G) = 1$.

Proposition 3.4. For every graph G having a $\beta_{le}(G)$ – set D, and a vertex $u\in D$ which is adjacent to at least two vertices in V-D, $sd\beta_{le}(G) \leq 2$.

Corollary 3.5. For every graph G having an even cycle**,** $sd\beta_{le}(G) \leq 2$.

Proposition 3.6. For any star $K_{1,m}$, $sd\beta_{le} (K_{1,m}) = m$.

Theorem 3.7. For any connected graph G of order $n \geq 3$, either

(i) $G = K_{1,m}$ and $sd\beta_{le}(G) = m$, or

(ii) $1\leq sd\beta_{le}(G) \leq 2$.

Proof. Assume first that G is connected and contains a cycle. By corollary 4, if G contains an odd cycle, then $sd\beta_{le}(G) = 1$.

If G has no odd cycle, then it must have a even cycle. By corollary 7, we can conclude that $sd\beta_{le}(G) \leq 2$.

Assume therefore that $T\neq K_{1,m}$, for $m\geq 3$, and hence that the diameter of T is at least three.

Case 1. If T has a $\beta_{le}(T)$–set D, for which G [V-D] has an edge, then by Proposition 3.1, $sd\beta_{le}(G)=1$.

Case 2. For every $\beta_{1e}(T)$ – set D, V-D is an independent set.Let D be any $\beta_{1e}(T)$ – set.Since T is connected, and has diameter at least three, there must be at least one vertex in D which is adjacent to two or more vertices in V-D. By Proposotion6 it then follows that $sd\beta_{1e}(G) \leq 2$.

It follows from the previous theorem that every connected graph of order $n \geq 3$ can be placed into one of three classes, according to their independence subdivision number:

Class I : Graphs G for which $sd\beta_{1e}(G) = 1$.
Class II : Graphs G for which $sd\beta_{1e}(G) = 2$.
Class III: Graphs $G = K_{1,m}$ for $m \geq 3$.

It follows from Corollary 3.2 that class I contains all graphs which are not bipartite. Class I also contain some bipartite graphs G,i.e., those having a $\beta_{1e}(G)$ – set D, for which the induced subgraph G[V-D] contains at least one edge or those having a $\beta_{1e}(G)$ set which includes at least one support vertex. Class II, which consists of all graphs G for which $sd\beta_{1e}(G) = 2$, contains only bipartite graphs,eg.,C_4, for every $\beta_{1e}(G)$ – set D of which, V-D is an independent set. This class includes, for example, all even cycles C_{2k},all odd paths P_{2k+1}, and all complete bipartite graphs $K_{r,s}$, $2 \leq r \leq s$.

Theorem 3.8. A connected graph G is in Class II if and only if either $G = K_{1,2}$ or G is bipartite with partite sets V_1 and V_2 such that either

(a) $2 \leq |V_1| = |V_2| = \beta_{1e}(G)$, and V_1 and V_2 are the only $\beta_{1e}(G)$,-sets, or
(b) $2 \leq |V_1| < |V_2| = \beta_{1e}(G)$, and V_2 is the unique $\beta_{1e}(G)$-set.

Proof. If $G = K_{1,2}$, then the theorem holds. First assume that $G \neq K_{1,2}$ is bipartite with partite sets V_1 and V_2 such that either (a) or (b) holds. Since G is connected and not a star.it follows from theorem 3.7, that $1 \leq sd\beta_{1e}(G) \leq 2$.

We show that $sd\beta_{1e}(G) \neq 1$. Assume to the contrary that subdividing the edge v_1v_2 yielding v_1vv_2 for some $v_1 \in V_1$ and $v_2 \in V_2$ increases the independence number, and let G' be the graph obtain from G by subdividing edge v_1v_2. If condition (a) holds, then $2 \leq |V_1| = |V_2| = \beta_{1e}(G)$ and V_1 andV_2 are the unique $\beta_{1e}(G)$-sets. If $x_1 \in V_1$ is an endvertex with support $y_2 \in V_2$, then $V_2 - \{y2\} \cup \{x1\}$ is another $\beta_{1e}(G)$-sets. Similarly, V_2 has no endvertices. Thus, $\delta(G) \geq 2$. but since v1 (respectively,v2) has at least two neighbors in V_2 (respectively, V_1), it follows that $\beta_{1e}(G') = \beta_{1e}(G)$, contradicting our assumption. If condition (b) holds, then $2 \leq |V_1| = |V_2| = \beta_{1e}(G)$, and V_2 is the unique $\beta_{1e}(G)$-sets.Hence for every vertex $u \in V_1$, $deg(u) \geq 2$, and the result follows as before. Thus, $G \in$ClassII.

For the converse, assume that connected graph $G \neq K_{1,2} \in$ClassII, i.e. $sd\beta_{1e}(G)=2$.

Let D be a $\beta_{1e}(G)$-sets.Proposition 3 implies that V-D is independent and hence, g is bipartite. Since G is connected and not a star, $2 \leq |V\text{-}D| \leq |D| = \beta_{1e}(G)$. If any vertex,say v,in V-D has exactly one neighbor, say u, in D, then subdividing the edge uv forming uxv increases the independence number since $D \cup \{v\}$ is an independent set, contradicting that $G \in$ClassII. Thus every vertex in V-D has at least two neighbors in D. Note that if $|D| = |V\text{-}D|$, then both d and V-D are $\beta_{1e}(G)$-sets implying that $\delta(G) \geq 2$.Suppose D' is a $\beta_{1e}(G)$-sets that is not a partite set of G, that is, $D' \cap D = A \neq \varphi$ and $D' \cap (V\text{-}D) = B \neq \varphi$. Let C = D-A and F= V-D-B. Note that

$D' = A \cup B$ and $V\text{-}D' = (D\text{-}A) \cup (V\text{-}D\text{-}B) = C \cup F$.If $C \cup F$ contains an edge, then by Proposition 3.1. $sd\beta_{1e}(G) = 1$. But since $G \in$ClassII, $C \cup F$ must be an independent set. But in this case there are no edge between $A \cup F$ and $C \cup B$, implying that G is not a connected graph, a contradiction. Hence, either condition (a) or (b) holds.

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