Locating Equitable Domination and Independence Subdivision Numbers of Graphs

P.Sumathi¹, G.Alarmelumangai²

¹Head & Associate Professor of Mathematics, C.K.N college for Men, Anna Nagar,
Chennai-600102.
²Assistant professor of Mathematics, E.M.G Yadava women’s college,
Madurai-625014

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Abstract: Let $G = (V,E)$ be a simple, undirected, finite nontrivial graph. A non empty set $D \subseteq V$ of vertices in a graph $G$ is a dominating set if every vertex in $V-D$ is adjacent to some vertex in $D$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of $G$. A dominating set $D$ is a locating equitable dominating set of $G$ if for any two vertices $u,w \in V-D$, $N(u) \cap D \neq N(w) \cap D$, $|N(u) \cap D| = |N(w) \cap D|$. The locating equitable domination number of $G$ is the minimum cardinality of a locating equitable dominating set of $G$. The locating equitable domination subdivision number of $G$ is the minimum number of edges that must be subdivided (where each edge in $G$ can be subdivided at most once) in order to increase the locating equitable domination number and is denoted by $sd\gamma(G)$. The independence subdivision number $sd\beta(G)$ equals the minimum number of edges that must be subdivided in order to increase the independence number. In this paper, we establish bounds on $sd\gamma(G)$ and $sd\beta(G)$ for some families of graphs.

1. Introduction

For notation and graph theory terminology, we in general follow [3]. Specifically, a graph $G$ is a finite nonempty set $V(G)$ of objects called vertices together with a possibly empty set $E(G)$ of 2-element subsets of $V(G)$ called edges. The order of $G$ is $n(G) = |V(G)|$ and the size of $G$ is $m(G) = |E(G)|$. The degree of a vertex $v \in V(G)$ in $G$ is $d_G(v) = |N_G(v)|$. A vertex of degree one is called an end-vertex. The minimum and maximum degree among the vertices of $G$ is denoted by $\delta(G)$ and $\Delta(G)$, respectively. Further for a subset $S \subseteq V(G)$, the degree of $v$ in $S$, denoted $d_S(v)$, is the number of vertices in $S$ adjacent to $v$; that is, $d_S(v) = |N(v) \cap S|$. In particular, $d_G(v) = d_S(v)$ if the graph $G$ is clear from the context, we simply write $V,E,n,m,d(v)$, $\delta$ and $\Delta$ rather than $V(G),E(G),n(G),m(G),d_G(v)$, $\delta(G)$ and $\Delta(G)$, respectively.

The closed neighborhood of a vertex $u \in V$ is the set $N[u] = \{u\} \cup \{v \mid uv \in E\}$. Given a set $S \subseteq V$ of vertices and a vertex $u \in S$, the private neighbor set of $u$ with respect to $S$, is the set $p[u,S] = N[u] - N[S - \{u\}]$. We say that every vertex $v \in p[u,S]$ is a private neighbor of $u$ with respect to $S$. Such a vertex $v$ is adjacent to $u$ but is not adjacent to any other vertex of $S$, then it is an isolated vertex in the subgraph $G[S]$ induced by $S$. In this case, $u \in p[u,S]$, and we say that $u$ is its own private neighbor. We note that if a set $S$ is a $\gamma(G)$-set, then for every vertex $u \in S$, $p[u,S] \neq \emptyset$, i.e., every vertex of $S$ has at least one private neighbor. It can be seen that if $S$ is a $\gamma(G)$-set, and two vertices $u,v \in S$ are adjacent, then each of $u$ and $v$ must have a private neighbor other than itself. We will also use the following terminology. Let $v \in V$ be a vertex of degree one; $v$ is called a leaf. The only vertex adjacent to a leaf, say $u$, is called a support vertex, and the edge $uv$ is called a pendant edge. Two edges in a graph $G$ are independent if they are not adjacent in $G$. The distance $d_G(u,v)$ or $d(u,v)$ between two vertices $u$ and $v$ in a graph $G$, is the length of a shortest path connecting $u$ and $v$. The diameter of a connected graph $G$ is defined to be $\max \{d_G(u,v) : u,v \in V(G)\}$. A set $D \subseteq V$ of vertices is a dominating set if every vertex in $V-D$ is adjacent to
D. The domination number \( \gamma(G) \) is the minimum cardinality of a dominating set of \( G \).

The locating equitable domination subdivision number of a graph \( G \), denoted by \( sd\gamma_{le}(G) \), equals the minimum number of edges that must be subdivided in order to create a graph \( G' \) for which \( \gamma_{le}(G') > \gamma_{le}(G) \). The independence number \( \beta(G) \) is the maximum cardinality of an independent set in \( G \). We call an independent set \( S \) of cardinality \( \beta(G) \) a \( \beta(G) \)-set. The locating equitable independence subdivision number \( sd\beta_{le}(G) \) to equal the minimum number of edges that must be subdivided in order to create a graph \( G' \) for which \( \beta_{le}(G') > \beta_{le}(G) \).

Results on the locating equitably dominating and independence subdivision numbers are given in sections 2 and 3 respectively.

Example 1. The following is an example of a graph whose \( sd\gamma_{le}(G) = 1 \).

Let \( G = (V,E) \)

\[
V = \{v_1,v_2,v_3,v_4,v_5,v_6,v_7,v_8,v_9\}, \quad D = \{v_1,v_2,v_3,v_4,v_5,v_6,v_7,v_8,v_9\} \text{ is a } \gamma_{le} \text{- set, } \gamma_{le}(G) = 7.
\]

After making a single subdivision in \( G \), the number \( \gamma_{le} \) will be change. If we subdivide the edge \( v_7v_8 \) then \( \gamma_{le} \) increases. Hence \( sd\gamma_{le}(G) = 1 \).

2. Results on the Locating Equitable Domination Subdivision numbers

(i) \( sd\gamma_{le}(K_n) = 2 \)

(ii) \( sd\gamma_{le}(P_n) = \begin{cases} 
1 & \text{if } n \text{ is odd} \\
2 & \text{if } n \text{ is even}
\end{cases} \)

(iii) \( sd\gamma_{le}(C_n) = \begin{cases} 
1 & \text{if } n \text{ is even} \\
2 & \text{if } n \text{ is odd}
\end{cases} \)

**Proposition 2.1.** For any tree \( T \) of order \( n \geq 3 \), \( 1 \leq sd\gamma_{le}(T) \leq 3 \).

**Proposition 2.2.** For any \( k \)-regular graph \( G \) where \( k \geq 2 \), \( 1 \leq sd\gamma_{le}(G) \leq 3 \).

**Theorem 2.3.** For any connected graph \( G \) of order \( n \geq 3 \), and for any two adjacent vertices \( u \) and \( v \), where \( \deg(u) \geq 2 \) and \( \deg(v) \geq 2 \), \( sd\gamma_{le}(G) \leq \deg(u) + \deg(v) - 1 \).

**Proof.** Let \( uv \) be an edge in \( G \), and let \( G' \) be the graph which results from subdividing all edges incident to \( u \) and \( v \). Thus, \( \deg(u) + \deg(v) - 1 \) edges will be subdivided. We assume that both \( \deg(u) \geq 2 \) and \( \deg(v) \geq 2 \). We will show that \( \gamma_{le}(G') > \gamma_{le}(G) \) by showing that (I) no \( \gamma_{le}(G)-set \) is also locating equitable dominating set of \( G' \) and (II) there is no locating equitable dominating set of \( G' \) of cardinality \( \gamma(G) \) that contains a subdivision vertex.
(i) Let D be an arbitrary $\gamma_e(G)$-set. We will show that D is not a locating equitable dominating set of G'.

Case (1). u,v $\in$ D. In this case, both u and v must have private neighbors other than themselves. But then neither u nor v dominate locating equitably these private neighbors in G'. Case (2). Either u $\notin$ D or v $\notin$ D. In this case, D no longer locating equitable dominates $\{u,v\} \cap (V - D)$ in G'.

(ii) Let D be a subset of G' of cardinality $\gamma_e(G)$ which contains at least one subdivision vertex. We will show that D is not a locating equitable dominating set of G'.

Assume to the contrary that G' contains a locating equitable dominating set of cardinality $\gamma_e(G)$ which contains at least one subdivision vertex. Among all such dominating sets, let D* be one which contains a minimum number of subdivision vertices. Assume, without loss of generality, that D* contains a subdivision vertex adjacent to v, call it v', which subdivides the edge vw (w $\neq$ u).

It follows that v $\notin$ D*, since if v $\in$ D*, then D = D* - \{v'\} $\cup$ \{w\} is a locating equitable dominating set of G' of cardinality $\gamma_e(G)$ contains fewer subdivision vertices than D*, contradicting the minimality of D*.

Clearly, v' can only be used to locating equitable dominate vertices v, v' and w. It follows that no other subdivision vertex adjacent to v is in D*, since any such vertices could be exchanged with their neighbors not equal to v, to create a locating equitable dominating set of the same cardinality with fewer subdivision vertices, again contradicting the minimality of D*. It follows, therefore, that u $\in$ D* since D* is a dominating set and u is the only vertex available to locating equitable dominate the subdivision vertex, say x, between u and v, and x $\notin$ D*. Then no subdivision vertex adjacent to u is in D*, since x $\notin$ D* and any other such vertex can be exchanged with its neighbor with fewer subdivision vertices than D*, again contradicting the minimality of D*.

At this point we have established that (i) v $\notin$ D*, (ii) u, v $\in$ D*, and (iii) every neighbor of v in G other than w is in D*, since the subdivision vertices adjacent to v are not in D* and must be locating equitable dominated. In fact, D* contains only one subdivision vertex, namely v'. But if D* is a locating equitable dominating set of G' of cardinality $\gamma_e(G)$, then it follows that D = D* - \{u,v'\} $\cup$ \{v\} is a dominating set of G of cardinality less than $\gamma_e(G)$, a contradiction. (This follows from the observation that v' is only needed to dominate vertices v, v' w in G', and u is only needed to dominate itself and the subdivision vertices adjacent to it in G'). Earlier in the proof, we assumed that D* contains a subdivision vertex adjacent to v, call it v', which subdivides the edge vw (u$\neq$w). It remains to consider the final case that D* contains the subdivision vertex x between vertices u and v. In this case we can assume that D* contains no other subdivision vertex. Otherwise, they could be exchanged, as before, with their neighbors not equal to either u or v, to produce a dominating set of the same cardinality but with fewer subdivision vertices, contradicting the minimality of D*. But vertex x can only be used to dominate vertices u,v and v, which means that D* cannot contain both u and v (else vertex x is not needed). Therefore there are only three cases to consider.

Case 1. u $\in$ D* and v $\notin$ D*.

Case 2. u $\notin$ D* and v $\in$ D*.

Case 3. u $\notin$ D* and v $\notin$ D*.

But in each of the first two cases, it can be seen that the set D* - \{x\} is a locating equitable dominating set of G of cardinality less than $\gamma_e(G)$, a contradiction. In case 3, since deg(u)$\geq$2 and deg(v)$\geq$2, then D* - \{x\} is a locating equitable dominating set of G of cardinality less than $\gamma_e(G)$, since every neighbor of u or v in G, other than u and v, is in D*, a contradiction.

**Theorem 2.4.** For any connected graph G of order $\geq$3, and for any two adjacent vertices u and v, where deg(u)$\geq$2 and deg(v)$\geq$2, \(sd\gamma_e(G) \leq \deg(u) + \deg(v) - |N(u) \cap N(v)| - 1\).

**Proof.** We may assume that N(u)$\cap$N(v) $\neq$ $\emptyset$ for otherwise our result follows from theorem 2.3. Let N(v) = \{v_1,v_2,\ldots,v_k\} where u = v_1 and if N(u)$\cap$N[v] $\neq$ $\emptyset$. Let N(u)$\cap$N[v] = \{u_1,u_2,\ldots,u_l\}. Let G'}
be the graph obtained by subdividing the edge vv_i with subdivision vertex x_i, for 1 ≤ i ≤ k, and the edge uu_j for 1 ≤ j ≤ t. Let A be the set of the subdivision vertices and D' a γle(G')-set. Clearly no vertex of G locating equitable dominates v in G', and so |D' ∩ A| ≥ 1. We Show that γle(G') > γle(G). It suffices for us to show that γle(G) ≤ |D'| - 1, since then γle(G') = |D'| ≥ γle(G) + 1. One of u or v must be in D' to locating equitable dominate x_1. If both u and v are in D', then D' - A is a LEDS of G, and so γle(G) ≤ |D' - A| ≤ |D'| - 1. Assume u ∈ D' and v ∉ D'. Then every neighbor of v in G is in D' to locating equitable dominate {x_1, x_2, ..., x_k} and some x_i is in D' to dominate v. If |D' ∩ A| ≥ 2, then (D' - A) ∪ {v} is a locating equitable dominating set of G, and so γle(G) ≤ |D' - A| + 1 ≤ |D'| - 1. On the other hand, if |D' ∩ A| = 1, then since D' - {u} locating equitable dominates N(u) - N[v], it follows that (D' - A - {u}) ∪ {v} is a LEDS of G and so γle(G) ≤ |D' - A| ≤ |D'| - 1. Assume v ∈ D' and u ∉ D'. If |D' ∩ A| ≥ 2, then (D' - A) ∪ {u} is a locating equitable dominating set of G, and so γle(G) ≤ |D' - A| + 1 ≤ |D'| - 1. Therefore, assume that |D' ∩ A| = 1. The vertex of A in D' is a neighbor of v in G'. If x_1 ∈ D', then all neighbors of v in G, except for possible u, are locating equivalently dominated by D' - {v, x_1}. Thus, (D' - {v, x_1}) ∪ {v_i} for some vertex v_i ∈ N(u) ∩ N[v] is a locating equitable dominating set of G, and so γle(G) ≤ |D'| - 1. Therefore, we assume that x_1 ∉ D'. But then some v_i ∈ NG(v) ∩ NG(u) must be in D' to locating equivalently dominate u, whence D' - A is a locating equitable dominating set of G and γle(G) ≤ |D'| - 1.

3. Locating Equitable Independence Subdivision numbers.

The independence subdivision number of any graph is either one or two. We then characterize the class of graphs having independence subdivision number two.

If sdβle(G) = 1, for some graph G = (V, E), then, by definition, there must exist an edge uv ∈ E, which when subdivided into edges ux and xv results in a graph G' for which βle(G') = βle(G) + 1. This can happen in only one of two ways, either G has a βle(G) - set which does not contain either u or v, or uv is a pendant edge and G has a βle(G) - set D which contains the support vertex u but not the leaf v, in which case D ∪ {v} becomes a larger independent set when the edge vu is subdivided into ux and xv.

**Proposition 3.1.** For every graph G having a βle(G) - set D, where the subgraph G[V - D] induced by V - D has at least one edge, sdβle(G) = 1.

**Corollary 3.2.** For every graph G having an odd cycle, sdβle(G) = 1.

**Proposition 3.3.** For every graph G having a βle(G) - set D and a pendant edge uv, where D contains the support vertex u (and not the leaf v), sdβle(G) = 1.

**Proposition 3.4.** For every graph G having a βle(G) - set D, and a vertex u ∈ D which is adjacent to at least two vertices in V - D, sdβle(G) ≤ 2.

**Corollary 3.5.** For every graph G having an even cycle, sdβle(G) ≤ 2.

**Proposition 3.6.** For any star K_1.m, sdβle(K_1.m) = m.

**Theorem 3.7.** For any connected graph G of order n ≥ 3, either

(i) G = K_1.m and sdβle(G) = m, or

(ii) 1 ≤ sdβle(G) ≤ 2.

**Proof.** Assume first that G is connected and contains a cycle. By corollary 4, if G contains an odd cycle, then sdβle(G) = 1.

If G has no odd cycle, then it must have a even cycle. By corollary 7, we can conclude that sdβle(G) ≤ 2.

Assume therefore that T=K_1.m, for m≥3, and hence that the diameter of T is at least three.

Case 1. If T has a βle(T) - set D, for which G [V - D] has an edge, then by Proposition 3.1, sdβle(G) = 1.
Case 2. For every $\beta_{le}(T)$ – set D, V-D is an independent set. Let D be any $\beta_{le}(T)$ – set. Since T is connected, and has diameter at least three, there must be at least one vertex in D which is adjacent to two or more vertices in V-D. By Proposition 6 it then follows that $sd\beta_{le}(G) \leq 2$.

It follows from the previous theorem that every connected graph of order $n \geq 3$ can be placed into one of three classes, according to their independence subdivision number:

Class I: Graphs G for which $sd\beta_{le}(G) = 1$.
Class II: Graphs G for which $sd\beta_{le}(G) = 2$.
Class III: Graphs G = $K_{1,m}$ for $m \geq 3$.

It follows from Corollary 3.2 that class I contains all graphs which are not bipartite. Class I also contain some bipartite graphs G, i.e., those having a $\beta_{le}(G)$ – set D, for which the induced subgraph G[V-D] contains at least one edge or those having a $\beta_{le}(G)$ set which includes at least one support vertex. Class II, which consists of all graphs G for which $sd\beta_{le}(G) = 2$, contains only bipartite graphs, e.g., C4, for every $\beta_{le}(G)$ – set D of which, V-D is an independent set. This class includes, for example, all even cycles $C_{2k}$, all odd paths $P_{2k+1}$, and all complete bipartite graphs $K_{r,s}$, $2 \leq r \leq s$.

**Theorem 3.8.** A connected graph G is in Class II if and only if either G = $K_{1,2}$ or G is bipartite with partite sets $V_1$ and $V_2$ such that either

(a) $2 \leq |V_1| = |V_2| = \beta_{le}(G)$, and $V_1$ and $V_2$ are the only $\beta_{le}(G)$-sets, or

(b) $2 \leq |V_1| < |V_2| = \beta_{le}(G)$, and $V_2$ is the unique $\beta_{le}(G)$-set.

**Proof.** If G = $K_{1,2}$, then the theorem holds. First assume that G $\neq K_{1,2}$ is bipartite with partite sets $V_1$ and $V_2$ such that either (a) or (b) holds. Since G is connected and not a star, it follows from Theorem 3.7, that $1 \leq sd\beta_{le}(G) \leq 2$.

We show that $sd\beta_{le}(G) \neq 1$. Assume to the contrary that subdividing the edge $v_1v_2$ yielding $v_1v_2$ for some $v_1 \in V_1$ and $v_2 \in V_2$ increases the independence number, and let G’ be the graph obtained from G by subdividing edge $v_1v_2$. If condition (a) holds, then $2 \leq |V_1| = |V_2| = \beta_{le}(G)$ and $V_1$ and $V_2$ are the unique $\beta_{le}(G)$-sets. If $x_1 \in V_1$ is an endvertex with support $y_2 \in V_2$, then $V_2 - \{y_2\} \cup \{x_1\}$ is another $\beta_{le}(G)$-set. Similarly, $V_2$ has no endvertices. Thus, $\delta(G) \geq 2$. But since $v_1$ (respectively, $v_2$) has at least two neighbors in $V_2$ (respectively, $V_1$), it follows that $\beta_{le}(G’) = \beta_{le}(G)$, contradicting our assumption. If condition (b) holds, then $2 \leq |V_1| = |V_2| = \beta_{le}(G)$, and $V_2$ is the unique $\beta_{le}(G)$-sets. Hence for every vertex $u \in V_1$, $\text{deg}(u) \geq 2$, and the result follows as before. Thus, G is Class II.

For the converse, assume that connected graph G $\neq K_{1,2} \in$ Class II, i.e. $sd\beta_{le}(G) = 2$.

Let D be a $\beta_{le}(G)$-sets. Proposition 3 implies that V-D is independent and hence, G is bipartite. Since G is connected and not a star, $2 \leq |V-D| = |D| = \beta_{le}(G)$. If any vertex, say v, in V-D has exactly one neighbor, say u, in D, then subdividing the edge uv forming uvx increases the independence number, since D $\cup \{v\}$ is an independent set, contradicting that G is Class II. Thus every vertex in V-D has at least two neighbors in D. Note that if $|D| = |V-D|$, then both d and V-D are $\beta_{le}(G)$-sets implying that $\delta(G) \geq 2$. Suppose D’ is a $\beta_{le}(G)$-sets that is not a partite set of G, that is, $D’ \cap D = A \neq \emptyset$ and $D’ \cap (V-D) = B \neq \emptyset$. Let C = D-A and F = V-D-B. Note that $D’ = A \cup B$ and $V-D’ = (D-A) \cup (V-D-B) = C \cup F$. If $C \cup F$ contains an edge, then by Proposition 3.1, $sd\beta_{le}(G) = 1$. But since G $\in$ Class II, $C \cup F$ must be an independent set. But in this case there are no edge between A $\cup F$ and C $\cup B$, implying that G is not a connected graph, a contradiction. Hence, either condition (a) or (b) holds.
References


