

Locating Equitable Domination and Independence Subdivision Numbers of Graphs

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Abstract: Let $G = (V,E)$ be a simple, undirected, finite nontrivial graph. A non empty set $D \subseteq V$ of vertices in a graph G is a dominating set if every vertex in $V-D$ is adjacent to some vertex in D . The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G . A dominating set D is a locating equitable dominating set of G if for any two vertices $u,w \in V-D$, $N(u) \cap D \neq N(w) \cap D$, $|N(u) \cap D| = |N(w) \cap D|$. The locating equitable domination number of G is the minimum cardinality of a locating equitable dominating set of G . The locating equitable domination subdivision number of G is the minimum number of edges that must be subdivided (where each edge in G can be subdivided at most once) in order to increase the locating equitable domination number and is denoted by $sd\gamma_{le}(G)$. The independence subdivision number $sd\beta_{le}(G)$ to equal the minimum number of edges that must be subdivided in order to increase the independence number. In this paper, we establish bounds on $sd\gamma_{le}(G)$ and $sd\beta_{le}(G)$ for some families of graphs.

1. Introduction

For notation and graph theory terminology, we in general follow [3]. Specifically, a graph G is a finite nonempty set $V(G)$ of objects called vertices together with a possibly empty set $E(G)$ of 2-element subsets of $V(G)$ called edges. The order of G is $n(G) = |V(G)|$ and the size of G is $m(G) = |E(G)|$. The degree of a vertex $v \in V(G)$ in G is $d_G(v) = |N_G(v)|$. A vertex of degree one is called an end-vertex. The minimum and maximum degree among the vertices of G is denoted by $\delta(G)$ and $\Delta(G)$, respectively. Further for a subset $S \subseteq V(G)$, the degree of v in S , denoted $d_S(v)$, is the number of vertices in S adjacent to v ; that is, $d_S(v) = |N(v) \cap S|$. In particular, $d_G(v) = d_V(v)$. If the graph G is clear from the context, we simply write $V, E, n, m, d(v)$, δ and Δ rather than $V(G), E(G), n(G), m(G), d_G(v)$, $\delta(G)$ and $\Delta(G)$, respectively.

The closed neighborhood of a vertex $u \in V$ is the set $N[u] = \{u\} \cup \{v | uv \in E\}$. Given a set $S \subseteq V$ of vertices and a vertex $u \in S$, the private neighbor set of u , with respect to S , is the set $pn[u, S] = N[u] - N[S - \{u\}]$. We say that every vertex $v \in pn[u, S]$ is a private neighbor of u with respect to S . Such a vertex v is adjacent to u but is not adjacent to any other vertex of S , then it is an isolated vertex in the subgraph $G[S]$ induced by S . In this case, $u \in pn[u, S]$, and we say that u is its own private neighbor. We note that if a set S is a $\gamma(G)$ -set, then for every vertex $u \in S$, $pn[u, S] \neq \emptyset$, i.e., every vertex of S has at least one private neighbor. It can be seen that if S is a $\gamma(G)$ -set, and two vertices $u, v \in S$ are adjacent, then each of u and v must have a private neighbor other than itself. We will also use the following terminology. Let $v \in V$ be a vertex of degree one; v is called a leaf. The only vertex adjacent to a leaf, say u , is called a support vertex, and the edge uv is called a pendant edge. Two edges in a graph G are independent if they are not adjacent in G . The distance $d_G(u, v)$ or $d(u, v)$ between two vertices u and v in a graph G , is the length of a shortest path connecting u and v . The diameter of a connected graph G is defined to be $\max \{d_G(u, v) : u, v \in V(G)\}$. A set $D \subseteq V$ of vertices is a dominating set if every vertex in $V-D$ is adjacent to

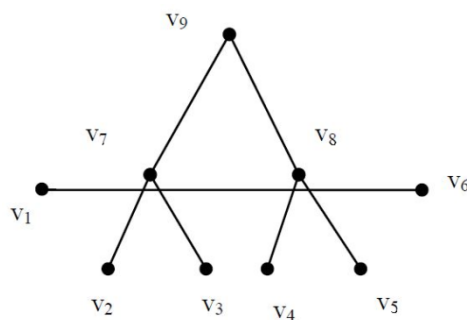
some vertex in D . The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G .

The locating equitable domination subdivision number of a graph G , denoted by $sd\gamma_{le}(G)$, equals the minimum number of edges that must be subdivided in order to create a graph G' for which $\gamma_{le}(G') > \gamma_{le}(G)$. The independence number $\beta(G)$ is the maximum cardinality of an independent set in G . we call an independent set S of cardinality $\beta(G)$ a $\beta(G)$ -set. The locating equitable independence subdivision number $sd\beta_{le}(G)$ to equal the minimum number of edges that must be subdivided in order to create a graph G' for which $\beta_{le}(G') > \beta_{le}(G)$.

Results on the locating equitable domination and independence subdivision numbers are given in sections 2 and 3 respectively.

Example 1. The following is an example of a graph whose $sd\gamma_{le}(G) = 1$.

Let $G = (V, E)$



$$V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\},$$

$$D = \{v_1, v_2, v_3, v_4, v_5, v_6, v_9\} \text{ is a } \gamma_{le}\text{-set, } \gamma_{le}(G) = 7.$$

After making a single subdivision in G , the number γ_{le} will be change. If we subdivide the edge v_7v_8 then γ_{le} increases. Hence $sd\gamma_{le}(G) = 1$.

2. Results on the Locating Equitable Domination Subdivision numbers

(i) $sd\gamma_{le}(K_n) = 2$

(ii) $sd\gamma_{le}(P_n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}$

(iii) $sd\gamma_{le}(C_n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd} \end{cases}$

Proposition 2.1. For any tree T of order $n \geq 3$, $1 \leq sd\gamma_{le}(T) \leq 3$.

Proposition 2.2. For any k -regular graph G where $k \geq 2$, $1 \leq sd\gamma_{le}(G) \leq 3$.

Theorem 2.3. For any connected graph G of order $n \geq 3$, and for any two adjacent vertices u and v , where $\deg(u) \geq 2$ and $\deg(v) \geq 2$, $sd\gamma_{le}(G) \leq \deg(u) + \deg(v) - 1$.

Proof. Let uv be an edge in G , and let G' be the graph which results from subdividing all edges incident to u and v . Thus, $\deg(u) + \deg(v) - 1$ edges will be subdivided. We assume that both $\deg(u) \geq 2$ and $\deg(v) \geq 2$. We will show that $\gamma_{le}(G') > \gamma_{le}(G)$ by showing that (I) no $\gamma_{le}(G)$ -set is also locating equitable dominating set of G' and (II) there is no locating equitable dominating set of G' of cardinality $\gamma(G)$ that contains a subdivision vertex.

(i) Let D be an arbitrary $\gamma_{le}(G)$ -set. We will show that D is not a locating equitable dominating set of G' .

Case (1). $u, v \in D$. In this case, both u and v must have private neighbors other than themselves. But then neither u nor v dominate locating equitably these private neighbors in G' . Case (2). Either $u \notin D$ or $v \notin D$. In this case, D no longer locating equitably dominates $\{u, v\} \cap (V - D)$ in G' .

(ii) Let D be a subset of G' of cardinality $\gamma_{le}(G)$ which contains at least one subdivision vertex. We will show that D is not a locating equitable dominating set of G' .

Assume to the contrary that G' contains a locating equitable dominating set of cardinality $\gamma_{le}(G)$ which contains at least one subdivision vertex. Among all such dominating sets, let D^* be one which contains a minimum number of subdivision vertices. Assume, without loss of generality, that D^* contains a subdivision vertex adjacent to v , call it v' , which subdivides the edge vw ($w \neq u$).

It follows that $v \notin D^*$, since if $v \in D^*$, then $D = D^* - \{v'\} \cup \{w\}$ is a locating equitable dominating set of G' of cardinality $\gamma_{le}(G)$ contains fewer subdivision vertices than D^* , contradicting the minimality of D^* .

Clearly, v' can only be used to locating equitable dominate vertices v, v' and w . It follows that no other subdivision vertex adjacent to v is in D^* , since any such vertices could be exchanged with their neighbors not equal to v , to create a locating equitable dominating set of the same cardinality with fewer subdivision vertices, again contradicting the minimality of D^* . It follows, therefore, that $u \in D^*$ since D^* is a dominating set and u is the only vertex available to locating equitable dominate the subdivision vertex, say x , between u and v , and $x \notin D^*$. Then no subdivision vertex adjacent to u is in D^* , since $x \notin D^*$ and any other such vertex can be exchanged with its neighbor with fewer subdivision vertices than D^* , again contradicting the minimality of D^* .

At this point we have established that (i) $v \notin D^*$, (ii) $u, v' \in D^*$, and (iii) every neighbor of v in G other than w is in D^* , since the subdivision vertices adjacent to v are not in D^* and must be locating equitable dominated. In fact, D^* contains only one subdivision vertex, namely v' . But if D^* is a locating equitable dominating set of G' of cardinality $\gamma_{le}(G)$, then it follows that $D = D^* - \{u, v'\} \cup \{v\}$ is a dominating set of G of cardinality less than $\gamma_{le}(G)$, a contradiction. (This follows from the observation that v' is only needed to dominate vertices v, v', w in G' , and u is only needed to dominate itself and the subdivision vertices adjacent to it in G'). Earlier in the proof, we assumed that D^* contains a subdivision vertex adjacent to v , call it v' , which subdivides the edge vw ($u \neq w$). It remains to consider the final case that D^* contains the subdivision vertex x between vertices u and v . In this case we can assume that D^* contains no other subdivision vertex. Otherwise, they could be exchanged, as before, with their neighbors not equal to either u or v , to produce a dominating set of the same cardinality but with fewer subdivision vertices, contradicting the minimality of D^* . But vertex x can only be used to dominate vertices u, x and v , which means that D^* cannot contain both u and v (else vertex x is not needed). Therefore there are only three cases to consider.

Case 1. $u \in D^*$ and $v \notin D^*$.

Case 2. $u \notin D^*$ and $v \in D^*$.

Case 3. $u \notin D^*$ and $v \notin D^*$.

But in each of the first two cases, it can be seen that the set $D^* - \{x\}$ is a locating equitable dominating set of G of cardinality less than $\gamma_{le}(G)$, a contradiction. In case 3, since $\deg(u) \geq 2$ and $\deg(v) \geq 2$, then $D^* - \{x\}$ is a locating equitable dominating set of G of cardinality less than $\gamma_{le}(G)$, since every neighbor of u or v in G , other than u and v , is in D^* , a contradiction.

Theorem 2.4. For any connected graph G of order $n \geq 3$, and for any two adjacent vertices u and v , where $\deg(u) \geq 2$ and $\deg(v) \geq 2$, $sd\gamma_{le}(G) \leq \deg(u) + \deg(v) - |N(u) \cap N(v)| - 1$.

Proof. We may assume that $N(u) \cap N(v) \neq \emptyset$ for otherwise our result follows from theorem 2.3. Let $N(v) = \{v_1, v_2, \dots, v_k\}$ where $u = v_1$ and if $N(u) \cap N[v] \neq \emptyset$. Let $N(u) \cap N[v] = \{u_1, u_2, \dots, u_t\}$. Let G'

be the graph obtained by subdividing the edge vv_i with subdivision vertex x_i , for $1 \leq i \leq k$, and the edge uv_j for $1 \leq j \leq t$. Let A be the set of the subdivision vertices and D' a $\gamma_{le}(G')$ -set. Clearly no vertex of G locating equitable dominates v in G' , and so $|D' \cap A| \geq 1$. We Show that $\gamma_{le}(G') > \gamma_{le}(G)$. It suffices for us to show that $\gamma_{le}(G) \leq |D'| - 1$, since then $\gamma_{le}(G') = |D'| \geq \gamma_{le}(G) + 1$. One of u or v must be in D' to locating equitable dominate x_1 . If both u and v are in D' , then $D' - A$ is a LEDES of G , and so $\gamma_{le}(G) \leq |D' - A| \leq |D'| - 1$. Assume $u \in D'$ and $v \notin D'$. Then every neighbor of v in G is in D' to locating equitable dominate $\{x_1, x_2, \dots, x_k\}$ and some x_i is in D' to dominate v . If $|D' \cap A| \geq 2$, then $(D' - A) \cup \{v\}$ is a locating equitable dominating set of G , and so $\gamma_{le}(G) \leq |D' - A| + 1 \leq |D'| - 1$. On the other hand, if $|D' \cap A| = 1$, then since $D' - \{u\}$ locating equitable dominates $N(u) - N[v]$, it follows that $(D' - A - \{u\}) \cup \{v\}$ is a LEDES of G and so $\gamma_{le}(G) \leq |D' - A| \leq |D'| - 1$. Assume $v \in D'$ and $u \notin D'$. If $|D' \cap A| \geq 2$, then $(D' - A) \cup \{u\}$ is a locating equitable dominating set of G , and so $\gamma_{le}(G) \leq |D' - A| + 1 \leq |D'| - 1$. Therefore, assume that $|D' \cap A| = 1$. The vertex of A in D' is a neighbor of v in G' . If $x_1 \in D'$, then all neighbors of v in G , except for possible u , are locating equitably dominated by $D' - \{v, x_1\}$. Thus, $(D' - \{v, x_1\}) \cup \{v_1\}$ for some vertex $v_1 \in N(u) \cap N(v)$ is a locating equitable dominating set of G , and so $\gamma_{le}(G) \leq |D'| - 1$. Therefore, we assume that $x_1 \notin D'$. But then some $v_i \in N_G(v) \cap N_G(u)$ must be in D' to locating equitably dominate u , whence $D' - A$ is a locating equitable dominating set of G and $\gamma_{le}(G) \leq |D'| - 1$.

3. Locating Equitable Independence Subdivision numbers.

The independence subdivision number of any graph is either one or two. We then characterize the class of graphs having independence subdivision number two.

If $sd\beta_{le}(G) = 1$, for some graph $G = (V, E)$, then, by definition, there must exist an edge $uv \in E$, which when subdivided into edges ux and xv results in a graph G' for which $\beta_{le}(G') = \beta_{le}(G) + 1$. This can happen in only one of two ways, either G has a $\beta_{le}(G)$ - set which does not contain either u or v , or uv is a pendant edge and G has a $\beta_{le}(G)$ - set D which contains the support vertex u but not the leaf v , in which case $D \cup \{v\}$ becomes a larger independent set when the edge uv is subdivided into ux and xv .

Proposition 3.1. For every graph G having a $\beta_{le}(G)$ - set D , where the subgraph $G[V-D]$ induced by $V-D$ has at least one edge, $sd\beta_{le}(G) = 1$.

Corollary 3.2. For every graph G having an odd cycle, $sd\beta_{le}(G) = 1$.

Proposition 3.3. For every graph G having a $\beta_{le}(G)$ - set D and a pendant edge uv , where D contains the support vertex u (and not the leaf v), $sd\beta_{le}(G) = 1$.

Proposition 3.4. For every graph G having a $\beta_{le}(G)$ - set D , and a vertex $u \in D$ which is adjacent to at least two vertices in $V-D$, $sd\beta_{le}(G) \leq 2$.

Corollary 3.5. For every graph G having an even cycle, $sd\beta_{le}(G) \leq 2$.

Proposition 3.6. For any star $K_{1,m}$, $sd\beta_{le}(K_{1,m}) = m$.

Theorem 3.7. For any connected graph G of order $n \geq 3$, either

- (i) $G = K_{1,m}$ and $sd\beta_{le}(G) = m$, or
- (ii) $1 \leq sd\beta_{le}(G) \leq 2$.

Proof. Assume first that G is connected and contains a cycle. By corollary 4, if G contains an odd cycle, then $sd\beta_{le}(G) = 1$.

If G has no odd cycle, then it must have a even cycle. By corollary 7, we can conclude that $sd\beta_{le}(G) \leq 2$.

Assume therefore that $T \neq K_{1,m}$, for $m \geq 3$, and hence that the diameter of T is at least three.

Case 1. If T has a $\beta_{le}(T)$ -set D , for which $G[V-D]$ has an edge, then by Proposition 3.1, $sd\beta_{le}(G) = 1$.

Case 2. For every $\beta_{1e}(T)$ – set D , $V-D$ is an independent set. Let D be any $\beta_{1e}(T)$ – set. Since T is connected, and has diameter at least three, there must be at least one vertex in D which is adjacent to two or more vertices in $V-D$. By Proposition 6 it then follows that $sd\beta_{1e}(G) \leq 2$.

It follows from the previous theorem that every connected graph of order $n \geq 3$ can be placed into one of three classes, according to their independence subdivision number:

Class I : Graphs G for which $sd\beta_{1e}(G) = 1$.

Class II : Graphs G for which $sd\beta_{1e}(G) = 2$.

Class III: Graphs $G = K_{1,m}$ for $m \geq 3$.

It follows from Corollary 3.2 that class I contains all graphs which are not bipartite. Class I also contain some bipartite graphs G , i.e., those having a $\beta_{1e}(G)$ – set D , for which the induced subgraph $G[V-D]$ contains at least one edge or those having a $\beta_{1e}(G)$ set which includes at least one support vertex. Class II, which consists of all graphs G for which $sd\beta_{1e}(G) = 2$, contains only bipartite graphs, eg., C_4 , for every $\beta_{1e}(G)$ – set D of which, $V-D$ is an independent set. This class includes, for example, all even cycles C_{2k} , all odd paths P_{2k+1} , and all complete bipartite graphs $K_{r,s}$, $2 \leq r \leq s$.

Theorem 3.8. A connected graph G is in Class II if and only if either $G = K_{1,2}$ or G is bipartite with partite sets V_1 and V_2 such that either

(a) $2 \leq |V_1| = |V_2| = \beta_{1e}(G)$, and V_1 and V_2 are the only $\beta_{1e}(G)$,-sets, or

(b) $2 \leq |V_1| < |V_2| = \beta_{1e}(G)$, and V_2 is the unique $\beta_{1e}(G)$ -set.

Proof. If $G = K_{1,2}$, then the theorem holds. First assume that $G \neq K_{1,2}$ is bipartite with partite sets V_1 and V_2 such that either (a) or (b) holds. Since G is connected and not a star, it follows from theorem 3.7, that $1 \leq sd\beta_{1e}(G) \leq 2$.

We show that $sd\beta_{1e}(G) \neq 1$. Assume to the contrary that subdividing the edge v_1v_2 yielding v_1vv_2 for some $v_1 \in V_1$ and $v_2 \in V_2$ increases the independence number, and let G' be the graph obtain from G by subdividing edge v_1v_2 . If condition (a) holds, then $2 \leq |V_1| = |V_2| = \beta_{1e}(G)$ and V_1 and V_2 are the unique $\beta_{1e}(G)$ -sets. If $x_1 \in V_1$ is an endvertex with support $y_2 \in V_2$, then $V_2 - \{y_2\} \cup \{x_1\}$ is another $\beta_{1e}(G)$ -sets. Similarly, V_2 has no endvertices. Thus, $\delta(G) \geq 2$. but since v_1 (respectively, v_2) has at least two neighbors in V_2 (respectively, V_1), it follows that $\beta_{1e}(G') = \beta_{1e}(G)$, contradicting our assumption. If condition (b) holds, then $2 \leq |V_1| = |V_2| = \beta_{1e}(G)$, and V_2 is the unique $\beta_{1e}(G)$ -sets. Hence for every vertex $u \in V_1$, $\deg(u) \geq 2$, and the result follows as before. Thus, $G \in \text{Class II}$.

For the converse, assume that connected graph $G \neq K_{1,2} \in \text{Class II}$, i.e. $sd\beta_{1e}(G) = 2$.

Let D be a $\beta_{1e}(G)$ -sets. Proposition 3 implies that $V-D$ is independent and hence, G is bipartite. Since G is connected and not a star, $2 \leq |V-D| \leq |D| = \beta_{1e}(G)$. If any vertex, say v , in $V-D$ has exactly one neighbor, say u , in D , then subdividing the edge uv forming uxv increases the independence number since $D \cup \{v\}$ is an independent set, contradicting that $G \in \text{Class II}$. Thus every vertex in $V-D$ has at least two neighbors in D . Note that if $|D| = |V-D|$, then both D and $V-D$ are $\beta_{1e}(G)$ -sets implying that $\delta(G) \geq 2$. Suppose D' is a $\beta_{1e}(G)$ -sets that is not a partite set of G , that is, $D' \cap D = A \neq \emptyset$ and $D' \cap (V-D) = B \neq \emptyset$. Let $C = D-A$ and $F = V-D-B$. Note that $D' = A \cup B$ and $V-D' = (D-A) \cup (V-D-B) = C \cup F$. If $C \cup F$ contains an edge, then by Proposition 3.1. $sd\beta_{1e}(G) = 1$. But since $G \in \text{Class II}$, $C \cup F$ must be an independent set. But in this case there are no edge between $A \cup F$ and $C \cup B$, implying that G is not a connected graph, a contradiction. Hence, either condition (a) or (b) holds.

References

- [1] J.A.Bondy and U.S.R.Murty, Graph Theory with Applications (American Elsevier, New York,1977).
- [2] G.Chartant and L.Lesniak,Graphs & Digraphs (Wadsworth and Brooks/cole, Monterey, CA, third edition, 1996).
- [3] F. Harary, Graph theory (Addison-Wesley,Reading,MA,1969).
- [4] T.W.Haynes, S.T.Hedetniemi and P.J. Slater Fundamentals of Domination in Graphs (Marcel dekker, Inc., New York, 1998).
- [5] T.W.Haynes, S.T.Hedetniemi and P.J.Slater, Domination in Graphs Advanced topics (Marcel dekker, Inc., New York, 1998).
- [6] D.B. West, Introduction to Graph Theory (Prentice Hall, New Jersey, 1996).