

Tripled fixed point theorems in partially ordered spaces using a control function

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Abstract: In this paper, We prove some new tripled fixed point theorems using a control function which is also referred to as altering distance function.

1. Introduction

Fixed point theorem is a very active field of research at present. Hundreds of reserachers have contributed in this field since 1922 with Banach's fixed point theorem as a milestone. Generalization of this theorem has been a heavily investigated branch of research. Existence of fixed points in ordered metric spaces was investigated in 2004 by Ran and Reurings [19]. Later, so many results were proved on existence and uniqueness of fixed point in partially ordred metric spaces, see e.g. [2,3,4,8,9,10,11,14,16,18,19,22].

In 2011, Berinde and Bocut [2] reported with the notion of tripled fixed point. Later various results on tripled fixed point have been obtained, see e.g. [1,12,20].

The purpose of this paper is to establish some tripled fixed point results in partially ordered complete metric spaces using a control function which is referred to as altering distance function. This control functoin has been heavily used in metric fixed point theory. Some recent references are Choudhury [5], Sastry and Babu [21], Mihet [15], Naidu [17], Choudhury and Das [6] and Dutta and Choudhury [7].

To begin, we first recall the definitions and notations that will be needed in the sequal.

Definition 1.1 [12]. The triple (X, d, \leq) is called partially ordered metric space if (X, \leq) is a partially ordered set and (X, d) is a metric space.

Definition 1.2 [2]. Let X be a non-empty set and $F : X^3 \rightarrow X$ be a map. An element $(x, y, z) \in X^3$ is called a tripled fixed point of F if

$$F(x, y, z) = x, F(y, x, y) = y, F(z, y, x) = z.$$

Definition 1.3 [13]. A function $\psi : [0, \infty) \rightarrow [0, \infty)$ is called an altering distance function if the following properties are satisfied:

- (i) ψ is monotone increasing and continuous,
- (ii) $\psi(t) = 0$ iff $t = 0$.

Notation.

- Throughout the paper, we assume that $X \neq \emptyset$ and $X^n = \underbrace{X \times X \times \dots \times X}_{n\text{-times}}$.

Let (X, \leq) be a partially order set, we endow the product space X^3 with the partially orde \leq defined by:

For $(x, y, z), (u, v, w) \in X^3$

$$(x, y, z) \leq (u, v, w) \Leftrightarrow x \leq u, y \geq v, z \leq w.$$

2. Main Results

Theorem 2.1. Let (X, d, \leq) be a partially ordered complete metric space, $\xi : [0, \infty) \rightarrow [0, \infty)$ is an altering distance function $\beta \in (0, 1)$. Let $F : X^3 \rightarrow X$ be a mapping such that the following conditions are satisfied:

(i) there exists $(x_0, y_0, z_0) \in X^3$ such that

$$(x_0, y_0, z_0) \leq (F(x_0, y_0, z_0), F(y_0, x_0, y_0), F(z_0, y_0, x_0)),$$

(ii) for $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$,

$$x_1 \leq x_2, y_1 \geq y_2, z_1 \leq z_2$$

$$\Rightarrow F(x_1, y_1, z_1) \leq F(x_2, y_2, z_2), F(z_1, y_1, x_1) \leq F(z_2, y_2, x_2),$$

$$F(y_1, x_1, y_1) \geq F(y_2, x_2, y_2),$$

(iii) If $(x_n, y_n, z_n) \rightarrow (x, y, z)$ is non-decreasing in first and third co-ordinates and non-increasing in second co-ordinates then $x_n \leq x, y_n \geq y, z_n \leq z$ for all n ,

(iv) $\xi(d(F(x_1, y_1, z_1), F(x_2, y_2, z_2))) \leq \beta \xi(\max\{d(x_1, x_2),$

$$\leq \beta \xi(\max\{d(x_1, x_2), d(x_1, F(x_1, y_1, z_1)), d(x_2, F(x_2, y_2, z_2)),$$

$$\left. \frac{d(x_2, F(x_1, y_1, z_1)) + d(x_1, F(x_2, y_2, z_2))}{2} \right\})$$

for all comparable $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$.

Then F has a tripled fixed point.

Proof. Let $F(x_0, y_0, z_0) = x_1, F(y_0, x_0, y_0) = y_1, F(z_0, y_0, x_0) = z_1$, then by hypothesis (i), we have

$$(x_0, y_0, z_0) \leq (x_1, y_1, z_1)$$

$$\Rightarrow x_0 \leq x_1, y_0 \geq y_1 \text{ and } z_0 \leq z_1. \quad (2.1)$$

Now by hypothesis (ii) and using equation (2.1) we have

$$F(x_0, y_0, z_0) \leq F(x_1, y_1, z_1), F(z_0, y_0, x_0) \leq F(z_1, y_1, x_1) \text{ and}$$

$$F(y_0, x_0, y_0) \geq F(y_1, x_1, y_1)$$

Let $F(x_1, y_1, z_1) = x_2, F(y_1, x_1, y_1) = y_2, F(z_1, y_1, x_1) = z_2$.

Then we have

$$x_1 \leq x_2, y_1 \geq y_2 \text{ and } z_1 \leq z_2. \quad (2.2)$$

Again by using hypothesis (ii) and equation (2.2), we have

$$F(x_1, y_1, z_1) \leq F(x_2, y_2, z_2), F(z_1, y_1, x_1) \leq F(z_2, y_2, x_2) \text{ and}$$

$$F(y_1, x_1, y_1) \geq F(y_2, x_2, y_2)$$

Continuing like this we can construct monotone non-decreasing sequences $\{x_n\}, \{z_n\}$ and monotone non-increasing sequence $\{y_n\}$ in X .

That is

$$x_1 \leq x_2 \leq x_3 \leq \dots \leq x_{n-1} \leq x_n \leq \dots,$$

$$y_1 \geq y_2 \geq y_3 \geq \dots \geq y_{n-1} \geq y_n \geq \dots,$$

$$z_1 \leq z_2 \leq z_3 \leq \dots \leq z_{n-1} \leq z_n \leq \dots$$

such that

$$F(x_n, y_n, z_n) = x_{n+1}, F(y_n, x_n, y_n) = y_{n+1}, F(z_n, y_n, x_n) = z_{n+1} \text{ for all } n \geq 0.$$

If there exist an integer ≥ 0 such that

$$x_l = x_{l+1}, y_l = y_{l+1} \text{ and } z_l = z_{l+1}$$

then (x_l, y_l, z_l) will be the coupled fixed point of F .

Hence we assume that either $x_n \neq x_{n+1}$ or $y_n \neq y_{n+1}$ or $z_n \neq z_{n+1}$ for all $n \geq 0$.

First we assume that $x_n \neq x_{n+1}$ for all $n \geq 0$.

Now since $x_n \leq x_{n+1}, y_n \geq y_{n+1}, z_n \leq z_{n+1}$, using hypothesis (iv), we have

$$\begin{aligned} \xi(d(x_{n+1}, x_{n+2})) &= \xi(d(F(x_n, y_n, z_n), F(x_{n+1}, y_{n+1}, z_{n+1}))) \\ &\leq \beta \xi(\max\{d(x_n, x_{n+1}), d(x_n, F(x_n, y_n, z_n)), d(x_{n+1}, F(x_{n+1}, y_{n+1}, z_{n+1}))\}) \end{aligned}$$

$$\begin{aligned} &\left. \frac{d(x_{n+1}, F(x_n, y_n, z_n)) + d(x_n, F(x_{n+1}, y_{n+1}, z_{n+1}))}{2} \right\} \\ &= \beta \xi \left(\max \left\{ d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{d(x_{n+1}, x_{n+1}) + d(x_n, x_{n+2})}{2} \right\} \right) \end{aligned}$$

$$\Rightarrow \xi(d(x_{n+1}, x_{n+2})) \leq \beta \xi(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}). \tag{2.3}$$

$$\left(\because \frac{d(x_n, x_{n+2})}{2} \leq \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\} \right)$$

If we assume $d(x_n, x_{n+1}) \leq d(x_{n+1}, x_{n+2})$ for some n , then by (2.3), we have

$$\xi(d(x_{n+1}, x_{n+2})) \leq \beta \xi(d(x_{n+1}, x_{n+2}))$$

$$\Rightarrow d(x_{n+1}, x_{n+2}) = 0$$

$$\Rightarrow x_{n+1} = x_{n+2}$$

which is a contradiction to our assumption that $x_n \neq x_{n+1}$ for all $n \in N$.

So $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$ for all $n \geq 0$ and $\{d(x_n, x_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers.

$$\Rightarrow \exists \text{ a real number } p \geq 0 \text{ such that } d(x_n, x_{n+1}) \rightarrow p \text{ as } n \rightarrow \infty.$$

Taking the limit $n \rightarrow \infty$ and using continuity of ψ , we have

$$\psi(p) \leq \beta \psi(p). \tag{2.4}$$

$$\Rightarrow p = 0.$$

Hence

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0. \tag{2.5}$$

Similarly if we assume $y_n \neq y_{n+1}$ and $z_n \neq z_{n+1}$, we arrive at contradictions and get

$$\lim_{n \rightarrow \infty} d(y_n, y_{n+1}) = 0. \tag{2.6}$$

and

$$\lim_{n \rightarrow \infty} d(z_n, z_{n+1}) = 0. \tag{2.7}$$

in the respective cases.

Now we will show that $\{x_n\}$ is a Cauchy sequence in X . If possible, let $\{x_n\}$ is not a Cauchy sequence in X . Then there exists an $\varepsilon > 0$ such that

$$d(x_{p(t)}, x_{q(t)}) \geq \varepsilon \text{ for all } t \in N, p(t) > q(t) > t.$$

If $p(t)$ is the smallest such natural number then we have

$$d(x_{q(t)}, x_{p(t)}) \geq \varepsilon \tag{2.8}$$

and

$$d(x_{p(t)-1}, x_{q(t)}) < \varepsilon . \quad (2.9)$$

Then

$$\begin{aligned} d(x_{q(t)}, x_{p(t)}) &\leq d(x_{q(t)}, x_{p(t)-1}) + d(x_{p(t)-1}, x_{p(t)}) \\ &< \varepsilon + d(x_{p(t)-1}, x_{p(t)}) \quad (\text{using (2.9)}) \\ \Rightarrow d(x_{q(t)}, x_{p(t)}) &< \varepsilon + d(x_{p(t)-1}, x_{p(t)}) . \end{aligned} \quad (2.10)$$

Taking limit as $t \rightarrow \infty$ and using (2.5), equations (2.8) and (2.10) jointly give

$$\lim_{t \rightarrow \infty} d(x_{q(t)}, x_{p(t)}) = \varepsilon . \quad (2.11)$$

Now

$$\begin{aligned} d(x_{q(t)}, x_{p(t)}) &\leq d(x_{q(t)}, x_{q(t)+1}) + d(x_{q(t)+1}, x_{p(t)+1}) + d(x_{p(t)+1}, x_{p(t)}) \\ \Rightarrow \lim_{t \rightarrow \infty} d(x_{q(t)+1}, x_{p(t)+1}) &\geq \varepsilon . \end{aligned} \quad (2.12)$$

Also

$$d(x_{q(t)+1}, x_{p(t)+1}) \leq d(x_{q(t)+1}, x_{q(t)}) + d(x_{q(t)}, x_{p(t)}) + d(x_{p(t)}, x_{p(t)+1}) .$$

Again taking limit $t \rightarrow \infty$ and using equations (2.5) and (2.11), we get

$$\lim_{t \rightarrow \infty} d(x_{q(t)+1}, x_{p(t)+1}) \leq \varepsilon . \quad (2.13)$$

Combining (2.12) and (2.13), we have

$$\lim_{t \rightarrow \infty} d(x_{n(t)+1}, x_{p(t)+1}) = \varepsilon . \quad (2.14)$$

Again

$$d(x_{q(t)}, x_{p(t)}) \leq d(x_{q(t)}, x_{p(t)+1}) + d(x_{p(t)+1}, x_{p(t)})$$

and

$$d(x_{q(t)}, x_{p(t)+1}) \leq d(x_{q(t)}, x_{p(t)}) + d(x_{p(t)}, x_{p(t)+1}) .$$

Taking limit as $t \rightarrow \infty$ and using (2.5) and (2.11), we get

$$\lim_{t \rightarrow \infty} d(x_{q(t)}, x_{p(t)+1}) = \varepsilon . \quad (2.15)$$

Again

$$d(x_{p(t)}, x_{q(t)}) \leq d(x_{p(t)}, x_{q(t)+1}) + d(x_{q(t)+1}, x_{q(t)})$$

and

$$d(x_{p(t)}, x_{q(t)+1}) \leq d(x_{p(t)}, x_{q(t)}) + d(x_{q(t)}, x_{q(t)+1}) .$$

Taking limit as $t \rightarrow \infty$ and using (2.5) and (2.11), we get

$$\lim_{t \rightarrow \infty} d(x_{p(t)}, x_{q(t)+1}) = \varepsilon . \quad (2.16)$$

Now $p(t) > q(t) \Rightarrow x_{p(t)} \geq x_{q(t)}, y_{p(t)} \leq y_{q(t)}, z_{p(t)} \geq z_{q(t)}$ for all $t \in N$.

So using hypothesis (iv), we have

$$\begin{aligned} \xi(d(x_{q(t)+1}, x_{p(t)+1})) &= \xi(d(F(x_{q(t)}, y_{q(t)}, z_{q(t)}), F(x_{p(t)}, y_{p(t)}, z_{p(t)}))) \\ &\leq \beta \xi(\max\{d(x_{q(t)}, x_{p(t)}), d(x_{q(t)}, F(x_{q(t)}, y_{q(t)}, z_{q(t)})), d(x_{p(t)}, F(x_{p(t)}, y_{p(t)}, z_{p(t)})), \\ &\quad \frac{d(x_{p(t)}, F(x_{q(t)}, y_{q(t)}, z_{q(t)})) + d(x_{q(t)}, F(x_{p(t)}, y_{p(t)}, z_{p(t)}))}{2}\}) \end{aligned}$$

$$= \beta \xi \left(\max \left\{ d(x_{q(t)}, x_{p(t)}), d(x_{q(t)}, x_{q(t)+1}), d(x_{p(t)}, x_{p(t)+1}), \frac{d(x_{p(t)}, x_{q(t)+1}) + d(x_{q(t)}, x_{p(t)+1})}{2} \right\} \right) .$$
 Taking

limit as $t \rightarrow \infty$ and using (2.5), (2.11) and (2.14)-(2.16), we get $\xi(\varepsilon) \leq \beta \xi(\varepsilon)$, which is not possible by definition of ξ and choice of β . Hence $\{x_n\}$ is a Cauchy sequence in X . Similarly

$\{y_n\}$ and $\{z_n\}$ are also Cauchy sequence in X . But X is given to be complete. So there exist such that

$$x, y, z \in X \quad x_n \rightarrow x, y_n \rightarrow y, z_n \rightarrow z \text{ as } n \rightarrow \infty. \tag{2.17}$$

Then by hypothesis (iii) $x_n \leq x, y_n \geq y$ and $z_n \leq z$ for all n .

So using hypothesis (iv), we get

$$\begin{aligned} & \xi(d(x_{n+1}, F(x, y, x))) = \xi(d(F(x_n, y_n, z_n), F(x, y, z))) \\ & \leq \beta \xi \left(\max \left\{ d(x_n, x), d(x_n, F(x_n, y_n, z_n)), d(x, F(x, y, z)), \right. \right. \\ & \quad \left. \left. \frac{d(x, F(x_n, y_n, z_n)) + d(x_n, F(x, y, z))}{2} \right\} \right) \\ & = \beta \xi \left(\max \left\{ d(x_n, x), d(x_n, x_{n+1}), d(x, F(x, y, z)), \frac{d(x, x_{n+1}) + d(x_n, F(x, y, z))}{2} \right\} \right). \end{aligned}$$

Now taking limit as $n \rightarrow \infty$ and using (2.5) and (2.17), we get

$$\begin{aligned} & \xi(d(x, F(x, y, z))) \leq \beta \xi(d(x, F(x, y, z))) \\ \Rightarrow & d(x, F(x, y, z)) = 0 \\ \Rightarrow & F(x, y, z) = x. \end{aligned} \tag{2.18}$$

Since $y_n \geq y$ and $x_n \leq x$, so again using hypothesis (iv), we have

$$\begin{aligned} & \xi(d(y_{n+1}, F(y, x, y))) = \xi(d(F(y_n, x_n, y_n), F(y, x, y))) \\ & \leq \beta \xi \left(\max \left\{ d(y_n, y), d(y_n, F(y_n, x_n, y_n)), d(y, F(y, x, y)), \right. \right. \\ & \quad \left. \left. \frac{d(z, F(z_n, y_n, x_n)) + d(z_n, F(z, y, x))}{2} \right\} \right) \\ & = \beta \xi \left(\max \left\{ d(z_n, z), d(z_n, z_{n+1}), d(z, F(z, y, x)), \frac{d(z, z_{n+1}) + d(z_n, F(z, y, x))}{2} \right\} \right). \end{aligned}$$

$n \rightarrow \infty$ and using (2.6) and (2.17), we get

$$\begin{aligned} & \xi(d(y, F(y, x, y))) \leq \beta \xi(d(y, F(y, x, y))) \\ \Rightarrow & d(y, F(y, x, y)) = 0 \\ \Rightarrow & F(y, x, y) = y. \end{aligned} \tag{2.19}$$

Finally, since $z_n \leq z, y_n \geq y$ and $x_n \leq x$ for all n , so again using hypothesis (iv), we get

$$\begin{aligned} & \xi(d(z_{n+1}, F(z, y, x))) = \xi(d(F(z_n, y_n, x_n), F(z, y, x))) \\ & \leq \beta \xi \left(\max \left\{ d(z_n, z), d(z_n, F(z_n, y_n, x_n)), d(z, F(z, y, x)), \right. \right. \\ & \quad \left. \left. \frac{d(z, F(z_n, y_n, x_n)) + d(z_n, F(z, y, x))}{2} \right\} \right) \\ & = \beta \xi \left(\max \left\{ d(z_n, z), d(z_n, z_{n+1}), d(z, F(z, y, x)), \frac{d(z, z_{n+1}) + d(z_n, F(z, y, x))}{2} \right\} \right). \end{aligned}$$

Now taking limit as $n \rightarrow \infty$ and using (2.7) and (2.17), we get

$$\begin{aligned} & \xi(d(z, F(z, y, x))) \leq \beta \xi(d(z, F(z, y, x))) \\ \Rightarrow & d(z, F(z, y, x)) = 0 \\ \Rightarrow & F(z, y, x) = z. \end{aligned} \tag{2.20}$$

Thus from (2.18)-(2.20), we conclude that there exist $(x, y, z) \in X^3$ such that $F(x, y, z) = x, F(y, x, y) = y$ and $F(z, y, x) = z$ which implies that (x, y, z) is a tripled fixed point of F .

In the following theorem we will establish the existence of tripled fixed point for a continuous mapping after the removal of hypothesis (iii) from Theorem 2.1.

Theorem 2.2. Let (X, d, \leq) be a partially ordered complete metric space, $\xi : [0, \infty) \rightarrow [0, \infty)$ is an altering distance function and $\beta \in (0, 1)$. Let $F : X^3 \rightarrow X$ be a continuous mapping such that the following conditions are satisfied:

(i) there exist $(x_0, y_0, z_0) \in X^3$ such that

$$(x_0, y_0, z_0) \leq (F(x_0, y_0, z_0), F(y_0, x_0, y_0), F(z_0, y_0, x_0)),$$

(ii) for $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$,

$$(x_1, y_1, z_1) \leq (x_2, y_2, z_2) \Rightarrow F(x_1, y_1, z_1) \leq F(x_2, y_2, z_2),$$

$$F(y_1, x_1, y_1) \geq F(y_2, x_2, y_2) \text{ and } F(z_1, y_1, x_1) \leq F(z_2, y_2, x_2),$$

(iii) $\xi(d(F(x_1, y_1, z_1), F(x_2, y_2, z_2))) \leq \beta \xi(\max\{d(x_1, x_2),$

$$\left. \begin{aligned} & d(x_1, F(x_1, y_1, z_1)), d(x_2, F(x_2, y_2, z_2)), \\ & \frac{d(x_2, F(x_1, y_1, z_1)) + d(x_1, F(x_2, y_2, z_2))}{2} \end{aligned} \right\}$$

for all comparable $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$

Then F has a tripled fixed point.

Proof. From the proof of Theorem 2.1, we find sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ such that

$$\lim_{n \rightarrow \infty} x_n = x, \quad \lim_{n \rightarrow \infty} y_n = y \text{ and } \lim_{n \rightarrow \infty} z_n = z.$$

Then continuity of F implies that

$$x = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} (x_n, y_n, z_n) = F(x, y, z),$$

$$y = \lim_{n \rightarrow \infty} y_{n+1} = \lim_{n \rightarrow \infty} (y_n, x_n, y_n) = F(y, x, y)$$

and

$$z = \lim_{n \rightarrow \infty} z_{n+1} = \lim_{n \rightarrow \infty} (z_n, y_n, x_n) = F(z, y, x)$$

Thus we find $(x, y, z) \in X^3$ such that

$$F(x, y, z) = x, \quad F(y, x, y) = y \text{ and } F(z, y, x) = z$$

$\Rightarrow (x, y, z)$ is a tripled fixed point of F .

Theorem 2.3. Let (X, d, \leq) be a partially ordered complete metric space and $\xi : [0, \infty) \rightarrow [0, \infty)$ is an altering distance function. Let: $F : X^3 \rightarrow X$ be a mapping such that the following conditions are satisfied:

(i) there exist $(x_0, y_0, z_0) \in X^3$ such that

$$(x_0, y_0, z_0) \leq (F(x_0, y_0, z_0), F(y_0, x_0, y_0), F(z_0, y_0, x_0)),$$

(ii) for $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$,

$$(x_1, y_1, z_1) \leq (x_2, y_2, z_2) \Rightarrow F(x_1, y_1, z_1) \leq F(x_2, y_2, z_2),$$

$$F(y_1, x_1, y_1) \geq F(y_2, x_2, y_2), \quad F(z_1, y_1, x_1) \leq F(z_2, y_2, x_2),$$

(iii) If $(x_n, y_n, z_n) \rightarrow (x, y, z)$ is monotone non-decreasing in first and third coordinates and non-increasing in second co-ordinate, then

$$x_n \leq x, \quad y_n \geq y \text{ and } z_n \leq z \text{ for all } n.$$

(iv) $\xi(d(F(x_1, y_1, z_1), F(x_2, y_2, z_2))) \leq \beta \xi(\max\{d(x_1, x_2),$

$$\left. \begin{aligned} & d(x_1, F(x_1, y_1, z_1)), d(x_2, F(x_2, y_2, z_2)), \\ & \frac{d(x_2, F(x_1, y_1, z_1)) + d(x_1, F(x_2, y_2, z_2))}{2} \end{aligned} \right\}$$

$$-\eta(\max\{d(x_1, x_2), d(x_2, F(x_2, y_2, z_2))\})$$

for all comparable $(x_l, y_l, z_l), (x_2, y_2, z_2) \in X^3$,

where $\eta : [0, \infty) \rightarrow [0, \infty)$ is any continuous function with $\eta(t) = 0$ iff $t = 0$. Then F has a tripled fixed point.

Proof. We will construct the same sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ as in Theorem 2.1.

Now if there exist a positive integer l such that $x_l = x_{l+1}, y_l = y_{l+1}$ and $z_l = z_{l+1}$ then (x_l, y_l, z_l) will be a tripled fixed point of F .

Hence we assume that either $x_n \neq x_{n+1}$ or $y_n \neq y_{n+1}$ or $z_n \neq z_{n+1}$ for all n .

First we assume that $x_n \neq x_{n+1}$ for all n .

Now since $x_n \leq x_{n+1}, y_n \geq y_{n+1}$ and $z_n \leq z_{n+1}$, so using hypothesis (iv), we have

$$\begin{aligned} \xi(d(x_{n+1}, x_{n+2})) &= \xi(d(F(x_n, y_n, z_n), F(x_{n+1}, y_{n+1}, z_{n+1}))) \\ &\leq \xi(\max\{d(x_n, x_{n+1}), d(x_n, F(x_n, y_n, z_n)), d(x_{n+1}, F(x_{n+1}, y_{n+1}, z_{n+1})), \\ &\quad \frac{d(x_{n+1}, F(x_n, y_n, z_n)) + d(x_n, F(x_{n+1}, y_{n+1}, z_{n+1}))}{2}\}) \\ &\quad -\eta(\max\{d(x_n, x_{n+1}), d(x_{n+1}, F(x_{n+1}, y_{n+1}, z_{n+1}))\}) \end{aligned}$$

$$= \xi\left(\max\left\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{d(x_{n+1}, x_{n+1}) + d(x_n, x_{n+2})}{2}\right\}\right)$$

$$-\eta(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\})$$

$$\leq \xi(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}) - \eta(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\})$$

$$\Rightarrow \xi(d(x_{n+1}, x_{n+2})) \leq \xi(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\})$$

$$-\eta(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}). \tag{2.21}$$

Suppose $d(x_n, x_{n+1}) \leq d(x_{n+1}, x_{n+2})$ for some n .

Then (2.21) implies

$$\xi(d(x_{n+1}, x_{n+2})) \leq \xi(d(x_{n+1}, x_{n+2})) - \eta(d(x_{n+1}, x_{n+2}))$$

$$\Rightarrow \eta(d(x_{n+1}, x_{n+2})) \leq 0$$

$$\Rightarrow d(x_{n+1}, x_{n+2}) = 0$$

$$\Rightarrow x_{n+1} = x_{n+2} \text{ which is contradiction to our assumption}$$

that $\Rightarrow x_n \neq x_{n+1}$ for all n .

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}) \text{ for all } n \tag{2.22}$$

and $\{d(x_n, x_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers

\Rightarrow there exist a $p \geq 0$ such that

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = p. \tag{2.23}$$

Now using (2.21) and (2.22) we have

$$\xi(d(x_{n+1}, x_{n+2})) \leq \xi(d(x_n, x_{n+1})) - \eta(d(x_n, x_{n+1})).$$

Taking limit as $n \rightarrow \infty$ and using (2.23), we get

$$\xi(p) \leq \xi(p) - \eta(p) \text{ (using continuity of } \xi \text{ and } \eta)$$

which will give a contradiction unless $p = 0$.

So

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0. \quad (2.24)$$

Similarly if we assume $y_n \neq y_{n+1}$ and $z_n \neq z_{n+1}$, then we will arrive at contradiction and get

$$\lim_{n \rightarrow \infty} d(y_n, y_{n+1}) = 0 \quad (2.25)$$

and

$$\lim_{n \rightarrow \infty} d(z_n, z_{n+1}) = 0 \quad (2.26)$$

in the respective cases.

Now we will show that $\{x_n\}$ is a Cauchy sequence in X . If we assume that $\{x_n\}$ is not a Cauchy sequence in X then following the arguments of Theorem 2.1, we have two sequences of positive integers $\{p(t)\}$ and $\{q(t)\}$ such that $p(t) > q(t) > t$ for all t

$$\lim_{t \rightarrow \infty} d(x_{p(t)}, x_{q(t)}) = \varepsilon, \quad (2.27)$$

$$\lim_{t \rightarrow \infty} d(x_{p(t)+1}, x_{q(t)+1}) = \varepsilon, \quad (2.28)$$

$$\lim_{t \rightarrow \infty} d(x_{p(t)}, x_{q(t)+1}) = \varepsilon, \quad (2.29)$$

$$\lim_{t \rightarrow \infty} d(x_{q(t)}, x_{p(t)+1}) = \varepsilon. \quad (2.30)$$

Now $p(t) > q(t) \Rightarrow x_{p(t)} \geq x_{q(t)}, y_{p(t)} \leq y_{q(t)}, z_{p(t)} \geq z_{q(t)}$ for all $t \in N$.

So using hypothesis (vi), we get

$$\begin{aligned} & \xi(d(x_{q(t)+1}, x_{p(t)+1})) = \xi(d(F(x_{q(t)}, y_{q(t)}, z_{q(t)}), F(x_{p(t)}, y_{p(t)}, z_{p(t)}))) \\ & \leq \xi(\max\{d(x_{q(t)}, x_{p(t)}), d(x_{q(t)}, F(x_{q(t)}, y_{q(t)}, z_{q(t)})), d(x_{p(t)}, F(x_{p(t)}, y_{p(t)}, z_{p(t)})), \\ & \quad -\eta(\max\{d(x_{q(t)}, x_{p(t)}), d(x_{p(t)}, F(x_{p(t)}, y_{p(t)}, z_{p(t)})))\}) \\ & = \xi\left(\max\left\{d(x_{q(t)}, x_{p(t)}), d(x_{q(t)}, x_{q(t)+1}), d(x_{p(t)}, x_{p(t)+1}), \frac{d(x_{p(t)}, x_{q(t)+1}) + d(x_{q(t)}, x_{p(t)+1})}{2}\right\}\right) \\ & \quad -\eta(\max\{d(x_{q(t)}, x_{p(t)}), d(x_{p(t)}, x_{p(t)+1})\}). \end{aligned}$$

Taking limit as $t \rightarrow \infty$ and using (2.24) and (2.27)-(2.30), we have $\xi(\varepsilon) \leq \xi(\varepsilon) - \eta(\varepsilon)$, which is not possible by definition of η .

Hence $\{x_n\}$ is a Cauchy sequence in X .

Similarly $\{y_n\}$ and $\{z_n\}$ are also Cauchy sequences in X .

But X is complete so there exists $x, y, z \in X$ such that

$$x_n \rightarrow x, y_n \rightarrow y \text{ and } z_n \rightarrow z \text{ as } n \rightarrow \infty. \quad (2.31)$$

Then by hypothesis (iii) $x_n \leq x, y_n \geq y$ and $z_n \leq z$ for all n .

So we can use condition (iv) and get

$$\begin{aligned} & \xi(d(x_{n+1}, F(x, y, z))) = \xi(d(F(x_n, y_n, z_n), F(x, y, z))) \\ & \leq \xi(\max\{d(x_n, x), d(x_n, F(x_n, y_n, z_n)), d(x, F(x, y, z)), \\ & \quad \frac{d(x_n, F(x, y, z)) + d(x, F(x_n, y_n, z_n))}{2}\}) \\ & \quad -\eta(\max\{d(x_n, x), d(x_n, F(x, y, z))\}). \end{aligned}$$

Taking limit as $n \rightarrow \infty$ and using equations (2.24) and (2.31), we get

$$\begin{aligned} & \xi(d(x, F(x, y, z))) \leq \xi(d(x, F(x, y, z))) - \eta(d(x, F(x, y, z))) \\ & \quad (\because \xi, \eta \text{ are continuous}) \end{aligned}$$

which leads to a contradiction unless $d(x, F(x, y, z)) = 0$.

$$\Rightarrow F(x, y, z) = x. \tag{2.32}$$

Similarly

$$F(y, x, y) = y \text{ and } F(z, y, x) = z. \tag{2.39}$$

Combining equations (2.32) and (2.33) we get (x, y, z) is tripled fixed point of F .

In the following theorem we will show existence of tripled fixed point for a continuous mapping after the removal of hypothesis (iii) of Theorem 2.3.

Theorem 2.4. Let (X, d, \leq) be a partially ordered complete metric space and $\xi : [0, \infty) \rightarrow [0, \infty)$ be an altering distance function. $F : X^3 \rightarrow X$ be a continuous mapping such that the following. Let conditions are satisfied:

- (i) there exist $(x_0, y_0, z_0) \in X^3$ such that $(x_0, y_0, z_0) \leq (F(x_0, y_0, z_0), F(y_0, x_0, y_0), F(z_0, y_0, x_0))$,
- (ii) for $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$, $(x_1, y_1, z_1) \leq (x_2, y_2, z_2) \Rightarrow F(x_1, y_1, z_1) \leq F(x_2, y_2, z_2)$, $F(y_1, x_1, y_1) \geq F(y_2, x_2, y_2)$ and $F(z_1, y_1, x_1) \leq F(z_2, y_2, x_2)$,
- (iii) $\xi(d(F(x_1, y_1, z_1), F(x_2, y_2, z_2))) \leq \beta \xi(\max\{d(x_1, x_2), d(x_1, F(x_1, y_1, z_1)), d(x_2, F(x_2, y_2, z_2)), \frac{d(x_2, F(x_1, y_1, z_1)) + d(x_1, F(x_2, y_2, z_2))}{2}\}) - \eta(\max\{d(x_1, x_2), d(x_2, F(x_2, y_2, z_2))\})$

for all comparable $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$,

where $\eta : [0, \infty) \rightarrow [0, \infty)$ is any continuous function with $\eta(t) = 0$ iff $t = 0$. Then F has a tripled fixed point.

Proof. From the proof of Theorem 2.1, we find sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ such that

$$\lim_{n \rightarrow \infty} x_n \rightarrow x, \lim_{n \rightarrow \infty} y_n \rightarrow y \text{ and } \lim_{n \rightarrow \infty} z_n \rightarrow z.$$

Then continuity of F implies that

$$x = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} F(x_n, y_n, z_n) = F(x, y, z),$$

$$y = \lim_{n \rightarrow \infty} y_{n+1} = \lim_{n \rightarrow \infty} F(y_n, x_n, y_n) = F(y, x, y),$$

and

$$z = \lim_{n \rightarrow \infty} z_{n+1} = \lim_{n \rightarrow \infty} F(z_n, y_n, x_n) = F(z, y, x)$$

that is $\exists (x, y, z) \in X^3$ such that

$$F(x, y, z) = x, F(y, x, y) = y \text{ and } F(z, y, x) = z.$$

$\Rightarrow (x, y, z)$ is tripled fixed point of F .

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