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Tripled fixed point theorems in partially ordered spaces using a control function

SAVITRI and NAWNEET HOODA

Department of Mathematics, DCR University of Science & Technology, Murthal, India

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Abstract: In this paper, We prove some new tripled fixed point theorems using a control function which is also referred to as altering distance function.

1. Introduction

Fixed point thoerem is a very active field of research at present. Hundreds of reserachers have contributed in this field since 1922 with Banach's fixed point theorem as a milestone. Generalization of this theorem has been a heavily investigated branch of research. Existence of fixed points in ordered metric spaces was investigated in 2004 by Ran and Reurings [19]. Later, so many results were proved on existence and uniquenss of fixed point in partially ordred metric spaces, see e.g. [2,3,4,8,9,10,11,14,16,18,19,22].

In 2011, Berinde and Bocut [2] reported with the notion of tripled fixed point. Later various results on tripled fixed point have been obtained, see e.g. [1,12,20].

The purpose of this paper is to establish some tripled fixed point results in partially ordered complete metric spaces using a control function which is referred to as altering distance function. This control functoin has been heavily used in metric fixed point theory. Some recent references are Choudhury [5], Sastry and Babu [21], Mihet [15], Naidu [17], Choudhury and Das [6] and Dutta and Choudhury [7].

To begin, we first recall the definitions and notations that will be needed in the sequal.

Definition 1.1 [12]. The triple $(X, d, \leq)$ is called partially ordered metric space if $(X, \leq)$ is a partially ordered set and (X, d) is a metric space.

Definition 1.2 [2]. Let X be a non-empty set and $F : X^3 \rightarrow X$ be a map. An element $(x, y, z) \in X^3$ is called a tripled fixed point of F if

$F(x, y, z) = x$, $F(y, x, y) = y$, $F(z, y, x) = z$.

Definition 1.3 [13]. A function $\psi : [0, \infty) \rightarrow [0, \infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is monotone increasing and continuous,

(ii) $\psi(t) = 0$ iff $t = 0$.

Notation.

- Througout the paper, we assume that $X \neq \phi$ and $X^n = \underbrace{X \times X \times \ldots \times X}_{n\text{-times}}$.

Let $(X, \leq)$ be a partially order set, we endow the product space X^3 with the partially orde $\leq$ defined by:

For $(x, y, z), (u, v, w) \in X^3$

$(x, y, z) \leq (u, v, w) \Leftrightarrow x \leq u, y \geq v, z \leq w$.

2. Main Results

Theorem 2.1. Let $(X,d,\leq)$ be a partially ordered complete metric space, $\xi:[0,\infty)\to[0,\infty)$ is an altering distance function $\beta\in(0,1)$. Let $F:X^3\to X$ be a mapping such that the following conditions are satisfied:

(i) there exists $(x_0,y_0,z_0)\in X^3$ such that

$$(x_0,y_0,z_0)\leq(F(x_0,y_0,z_0),F(y_0,x_0,y_0),F(z_0,y_0,x_0)),$$

(ii) for $(x_1,y_1,z_1),(x_2,y_2,z_2)\in X^3$,

$$x_1\leq x_2,\ y_1\geq y_2,\ z_1\leq z_2$$

$$\Rightarrow\quad F(x_1,y_1,z_1)\leq F(x_2,y_2,z_2),\ F(z_1,y_1,x_1)\leq F(z_2,y_2,x_2),$$

$$F(y_1,x_1,y_1)\geq F(y_2,x_2,y_2),$$

(iii) If $(x_n,y_n,z_n)\to(x,y,z)$ is non-decreasing in first and third co-ordinates and non-increasing in second co-ordinates then $x_n\leq x$, $y_n\geq y$, $z_n\leq z$ for all n,

(iv) $\xi(d(F(x_1,y_1,z_1),F(x_2,y_2,z_2)))\leq\beta\xi(\max\{d(x_1,x_2),$

$$\leq\beta\xi\left(\max\left\{d(x_1,x_2),d(x_1,F(x_1,y_1,z_1)),d(x_2,F(x_2,y_2,z_2)),\frac{d(x_2,F(x_1,y_1,z_1)+d(x_1,F(x_2,y_2,z_2))}{2}\right\}\right)$$

for all comparable $(x_1,y_1,z_1),(x_2,y_2,z_2)\in X^3$.

Then F has a tripled fixed point.

Proof. Let $F(x_0,y_0,z_0)=x_1$, $F(y_0,x_0,y_0)=y_1$, $F(z_0,y_0,x_0)=z_1$, then by hypothesis (i), we have

$$(x_0,y_0,z_0)\leq(x_1,y_1,z_1)$$

$$\Rightarrow\quad x_0\leq x_1,\ y_0\geq y_1\ \text{and}\ z_0\leq z_1. \tag{2.1}$$

Now by hypothesis (ii) and using equation (2.1) we have

$$F(x_0,y_0,z_0)\leq F(x_1,y_1,z_1),\ F(z_0,y_0,x_0)\leq F(z_1,y_1,x_1).\ \text{and}$$

$$F(y_0,x_0,y_0)\geq F(y_1,x_1,y_1)$$

Let $F(x_1,y_1,z_1)=x_2$, $F(y_1,x_1,y_1)=y_2$, $F(z_1,y_1,x_1)=z_2$.

Then we have

$$x_1\leq x_2,\ y_1\geq y_2\ \text{and}\ z_1\leq z_2. \tag{2.2}$$

Again by using hypothesis (ii) and equation (2.2), we have

$F(x_1,y_1,z_1)\leq F(x_2,y_2,z_2)$, $F(z_1,y_1,x_1)\leq F(z_2,y_2,x_2)$. and

$F(y_1,x_1,y_1)\geq F(y_2,x_2,y_2)$

Continuing like this we can construct monotone non-decreasing sequences $\{x_n\}$, $\{z_n\}$ and monotone non-increasing sequence $\{y_n\}$ in X.

That is

$$x_1\leq x_2\leq x_3\leq\ldots\leq x_{n-1}\leq x_n\leq\ldots,$$

$$y_1\geq y_2\geq y_3\geq\ldots\geq y_{n-1}\geq y_n\geq\ldots,$$

$$z_1\leq z_2\leq z_3\leq\ldots\leq z_{n-1}\leq z_n\leq\ldots$$

such that

$F(x_n,y_n,z_n)=x_{n+1}$, $F(y_n,x_n,y_n)=y_{n+1}$, $F(z_n,y_n,x_n)=z_{n+1}$ for all $n\geq 0$.

If there exist an integer ≥ 0 such that
$x_l = x_{l+1}$, $y_l = y_{l+1}$ and $z_l = z_{l+1}$
then (x_l , y_l , z_l) will be the coupled fixed point of F .

Hence we assume that either $x_n \neq x_{n+1}$ or $y_n \neq y_{n+1}$ or $z_n \neq z_{n+1}$ for all $n \geq 0$.

First we assume that $x_n \neq x_{n+1}$ for all $n \geq 0$.

Now since $x_n \leq x_{n+1}$, $y_n \geq y_{n+1}$, $z_n \leq z_{n+1}$, using hypothesis (iv), we have

$$\xi(d(x_{n+1}, x_{n+2})) = \xi(d(F(x_n, y_n, z_n), F(x_{n+1}, y_{n+1}, z_{n+1})))$$

$$\leq \beta\xi\left(\max\left\{d(x_n, x_{n+1}), d(x_n, F(x_n, y_n, z_n)), d(x_{n+1}, F(x_{n+1}, y_{n+1}, z_{n+1})), \frac{d(x_{n+1}, F(x_n, y_n, z_n)) + d(x_n, F(x_{n+1}, y_{n+1}, z_{n+1}))}{2}\right\}\right)$$

$$= \beta\xi\left(\max\left\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{d(x_{n+1}, x_{n+1}) + d(x_n, x_{n+2})}{2}\right\}\right)$$

$$\Rightarrow \quad \xi(d(x_{n+1}, x_{n+2})) \leq \beta\xi(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}) . \tag{2.3}$$

$$\left(\because \frac{d(x_n, x_{n+2})}{2} \leq \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}\right)$$

If we assume $d(x_n , x_{n+1}) \leq d(x_{n+1} , x_{n+2})$ for some n , then by (2.3), we have

$$\xi(d(x_{n+1}, x_{n+2})) \leq \beta\xi(d(x_{n+1}, x_{n+2}))$$

$$\Rightarrow \quad d(x_{n+1}, x_{n+2}) = 0$$

$$\Rightarrow \quad x_{n+1} = x_{n+2}$$

which is a contradiction to our assumption that $x_n \neq x_{n+1}$ for all $n \in N$.

So $d(x_{n+1} , x_{n+2}) < d(x_n , x_{n+1})$ for all $n \geq 0$ and $\{d(x_n , x_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers.

$\Rightarrow$ $\exists$ a real number $p \geq 0$ such that $d(x_n , x_{n+1}) \to p$ as $n \to \infty$.

Taking the limit $n \to \infty$ and using continuity of ψ , we have

$$\psi(p) \leq \beta\psi(p) . \tag{2.4}$$

$$\Rightarrow \quad p = 0 .$$

Hence

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0 . \tag{2.5}$$

Similarly if we assume $y_n \neq y_{n+1}$ and $z_n \neq z_{n+1}$, we arrive at contradictions and get

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0 . \tag{2.6}$$

and

$$\lim_{n\to\infty} d(z_n, z_{n+1}) = 0 . \tag{2.7}$$

in the respective cases.

Now we will show that $\{x_n\}$ is a Cauchy sequence in X . If possible, let $\{x_n\}$ is not a Cauchy sequence in X .Then there exists an $\varepsilon > 0$ such that

$$d(x_{p(t)}, x_{q(t)}) \geq \varepsilon \text{ for all } t \in N , p(t) > q(t) > t .$$

If $p(t)$ is the smallest such natural number then we have

$$d(x_{q(t)}, x_{p(t)}) \geq \varepsilon \tag{2.8}$$

and

$$d(x_{p(t)-1},x_{q(t)})<\varepsilon\ . \tag{2.9}$$

Then

$$d(x_{q(t)},x_{p(t)})\le d(x_{q(t)},x_{p(t)-1})+d(x_{p(t)-1},x_{p(t)})$$
$$<\varepsilon+d(x_{p(t)-1},x_{p(t)})\quad \text{(using (2.9))}$$
$$\Rightarrow\quad d(x_{q(t)},x_{p(t)})<\varepsilon+d(x_{p(t)-1},x_{p(t)})\ . \tag{2.10}$$

Taking limit as $t\to\infty$ and using (2.5), equations (2.8) and (2.10) jointly give

$$\lim_{t\to\infty} d(x_{q(t)},x_{p(t)})=\varepsilon\ . \tag{2.11}$$

Now

$$d(x_{q(t)},x_{p(t)})\le d(x_{q(t)},x_{q(t)+1})+d(x_{q(t)+1},x_{p(t)+1})+d(x_{p(t)+1},x_{p(t)})$$
$$\Rightarrow\quad \lim_{t\to\infty} d(x_{q(t)+1},x_{p(t)+1})\ge\varepsilon\ . \tag{2.12}$$

Also

$$d(x_{q(t)+1},x_{p(t)+1})\le d(x_{q(t)+1},x_{q(t)})+d(x_{q(t)},x_{p(t)})+d(x_{p(t)},x_{p(t)+1})\ .$$

Again taking limit $t\to\infty$ and using equations (2.5) and (2.11), we get

$$\lim_{t\to\infty} d(x_{q(t)+1},x_{p(t)+1})\le\varepsilon\ . \tag{2.13}$$

Combining (2.12) and (2.13), we have

$$\lim_{t\to\infty} d(x_{n(t)+1},x_{p(t)+1})=\varepsilon\ . \tag{2.14}$$

Again

$$d(x_{q(t)},x_{p(t)})\le d(x_{q(t)},x_{p(t)+1})+d(x_{p(t)+1},x_{p(t)})$$

and

$$d(x_{q(t)},x_{p(t)+1})\le d(x_{q(t)},x_{p(t)})+d(x_{p(t)},x_{p(t)+1})\ .$$

Taking limit as $t\to\infty$ and using (2.5) and (2.11), we get

$$\lim_{t\to\infty} d(x_{q(t)},x_{p(t)+1})=\varepsilon\ . \tag{2.15}$$

Again

$$d(x_{p(t)},x_{q(t)})\le d(x_{p(t)},x_{q(t)+1})+d(x_{q(t)+1},x_{q(t)})$$

and

$$d(x_{p(t)},x_{q(t)+1})\le d(x_{p(t)},x_{q(t)})+d(x_{q(t)},x_{q(t)+1})\ .$$

Taking limit as $t\to\infty$ and using (2.5) and (2.11), we get

$$\lim_{t\to\infty} d(x_{p(t)},x_{q(t)+1})=\varepsilon\ . \tag{2.16}$$

Now $p(t)>q(t)\Rightarrow x_{p(t)}\ge x_{q(t)}$, $y_{p(t)}\le y_{q(t)}$, $z_{p(t)}\ge z_{q(t)}$ for all $t\in N$.

So using hypothesis (iv), we have

$$\xi(d(x_{q(t)+1},x_{p(t)+1}))=\xi(d(F(x_{q(t)},y_{q(t)},z_{q(t)}),F(x_{p(t)},y_{p(t)},z_{p(t)}))$$
$$\le\beta\xi\Big(\max\Big\{d(x_{q(t)},x_{p(t)}),d(x_{q(t)},F(x_{q(t)},y_{q(t)},z_{q(t)})),d(x_{p(t)},F(x_{p(t)},y_{p(t)},z_{p(t)})),$$
$$\frac{d(x_{p(t)},F(x_{q(t)},y_{q(t)},z_{q(t)}))+d(x_{q(t)},F(x_{p(t)},y_{p(t)},z_{p(t)}))}{2}\Big\}\Big)$$
$$=\beta\xi\left(\max\left\{d(x_{q(t)},x_{p(t)}),d(x_{q(t)},x_{q(t)+1}),d(x_{p(t)},x_{p(t)+1}),\frac{d(x_{p(t)},x_{q(t)+1})+d(x_{q(t)},x_{p(t)+1})}{2}\right\}\right).$$

Taking limit as $t\to\infty$ and using (2.5), (2.11) and (2.14)-(2.16), we get $\xi(\varepsilon)\le\beta\xi(\varepsilon)$, which is not possible by definition of ξ and choice of β. Hence $\{x_n\}$ is a Cauchy sequence in X. Similarly

$\{y_n\}$ and $\{z_n\}$ are also Cauchy sequence in X. But X is given to be complete. So there exist such that

$$x, y, z \in X \ x_n \to x, y_n \to y, z_n \to z \text{ as } n \to \infty. \tag{2.17}$$

Then by hypothesis (iii) $x_n \leq x$, $y_n \geq y$ and $z_n \leq z$ for all n.

So using hypothesis (iv), we get

$$\xi(d(x_{n+1}, F(x, y, x))) = \xi(d(F(x_n, y_n, z_n), F(x, y, z)))$$

$$\leq \beta\xi(\max\{d(x_n, x), d(x_n, F(x_n, y_n, z_n)), d(x, F(x, y, z)),$$

$$\left.\left.\frac{d(x, F(x_n, y_n, z_n)) + d(x_n, F(x, y, z)}{2}\right\}\right)$$

$= \beta\xi\left(\max\left\{d(x_n, x), d(x_n, x_{n+1}), d(x, F(x, y, z)), \frac{d(x, x_{n+1}) + d(x_n, F(x, y, z))}{2}\right\}\right)$. Now taking limit as $n \to \infty$ and using (2.5) and (2.17), we get

$$\xi(d(x, F(x, y, z))) \leq \beta\xi(d(x, F(x, y, z)))$$

$$\Rightarrow \quad d(x, F(x, y, z)) = 0$$

$$\Rightarrow \quad F(x, y, z) = x. \tag{2.18}$$

Since $y_n \geq y$ and $x_n \leq x$, so again using hypothesis (iv), we have

$$\xi(d(y_{n+1}, F(y, x, y))) = \xi(d(F(y_n, x_n, y_n), F(y, x, y)))$$

$$\leq \beta\xi(\max\{d(y_n, y), d(y_n, F(y_n, x_n, y_n)), d(x, F(y, x, y)),$$

$$\left.\left.\frac{d(z, F(z_n, y_n, x_n)) + d(z_n, F(z, y, x)}{2}\right\}\right)$$

$$= \beta\xi\left(\max\left\{d(z_n, z), d(z_n, z_{n+1}), d(z, F(z, y, x)), \frac{d(z, z_{n+1}) + d(z_n, F(z, y, x))}{2}\right\}\right).$$

$n \to \infty$ and using (2.6) and (2.17), we get

$$\xi(d(y, F(y, x, y))) \leq \beta\xi(d(y, F(y, x, y)))$$

$$\Rightarrow \quad d(y, F(y, x, y)) = 0$$

$$\Rightarrow \quad F(y, x, y) = y. \tag{2.19}$$

Finally, since $z_n \leq z$, $y_n \geq y$ and $x_n \leq x$ for all n, so again using hypothesis (iv), we get

$$\xi(d(z_{n+1}, F(z, y, x))) = \xi(d(F(z_n, y_n, x_n), F(z, y, x)))$$

$$\leq \beta\xi(\max\{d(z_n, z), d(z_n, F(z_n, y_n, x_n)), d(z, F(z, y, x)),$$

$$\left.\left.\frac{d(z, F(z_n, y_n, x_n)) + d(z_n, F(z, y, x)}{2}\right\}\right)$$

$$= \beta\xi\left(\max\left\{d(z_n, z), d(z_n, z_{n+1}), d(z, F(z, y, x)), \frac{d(z, z_{n+1}) + d(z_n, F(z, y, x))}{2}\right\}\right).$$

Now taking limit as $n \to \infty$ and using (2.7) and (2.17), we get

$$\xi(d(z, F(z, y, x))) \leq \beta\xi(d(z, F(z, y, x)))$$

$$\Rightarrow \quad d(z, F(z, y, x)) = 0$$

$$\Rightarrow \quad F(z, y, x) = z. \tag{2.20}$$

Thus from (2.18)-(2.20), we conclude that there exist $(x, y, z) \in X^3$ such that

$F(x, y, z) = x$, $F(y, x, y) = y$ and $F(z, y, x) = z$

which implies that (x, y, z) is a tripled fixed point of F.

In the following theorem we will establish the existence of tripled fixed point for a continuous mapping after the removal of hypothesis (iii) from Theorem 2.1.

Theorem 2.2. Let $(X,d,\leq)$ be a partially ordered complete metric space, $\xi:[0,\infty)\to[0,\infty)$ is an altering distance function and $\beta\in(0,1)$. Let $F:X^3\to X$ be a continuous mapping such that the following conditions are satisfied:

(i) there exist $(x_0,y_0,z_0)\in X^3$ such that

$$(x_0,y_0,z_0)\leq(F(x_0,y_0,z_0),F(y_0,x_0,y_0),F(z_0,y_0,x_0)),$$

(ii) for $(x_1,y_1,z_1),(x_2,y_2,z_2)\in X^3$,

$$(x_1,y_1,z_1)\leq(x_2,y_2,z_2)\Rightarrow F(x_1,y_1,z_1)\leq F(x_2,y_2,z_2),$$

$$F(y_1,x_1,y_1)\geq F(y_2,x_2,y_2) \text{ and } F(z_1,y_1,x_1)\leq F(z_2,y_2,x_2),$$

(iii) $$\xi(d(F(x_1,y_1,z_1),F(x_2,y_2,z_2))\leq\beta\xi\Big(\max\Big\{d(x_1,x_2), d(x_1,F(x_1,y_1,z_1),d(x_2,F(x_2,y_2,z_2)), \frac{d(x_2,F(x_1,y_1,z_1)+d(x_1,F(x_2,y_2,z_2))}{2}\Big\}\Big)$$

for all comparable $(x_1,y_1,z_1),(x_2,y_2,z_2)\in X^3$

Then F has a tripled fixed point.

Proof. From the proof of Theorem 2.1, we find sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ such that

$$\lim_{n\to\infty}x_n=x,\ \lim_{n\to\infty}y_n=y \text{ and } \lim_{n\to\infty}z_n=z.$$

Then continuity of F implies that

$$x=\lim_{n\to\infty}x_{n+1}=\lim_{n\to\infty}(x_n,y_n,z_n)=F(x,y,z),$$

$$y=\lim_{n\to\infty}y_{n+1}=\lim_{n\to\infty}(y_n,x_n,y_n)=F(y,x,y)$$

and

$$z=\lim_{n\to\infty}z_{n+1}=\lim_{n\to\infty}(z_n,y_n,x_n)=F(z,y,x)$$

Thus we find $(x,y,z)\in X^3$ such that

$F(x,y,z)=x$, $F(y,x,y)=y$ and $F(z,y,x)=z$

$\Rightarrow$ (x,y,z) is a tripled fixed point of F.

Theorem 2.3. Let $(X,d,\leq)$ be a partially ordered complete metric space and $\xi:[0,\infty)\to[0,\infty)$ is an altering distance function. Let: $F:X^3\to X$ be a mapping such that the following conditions are satisfied:

(i) there exist $(x_0,y_0,z_0)\in X^3$ such that

$$(x_0,y_0,z_0)\leq(F(x_0,y_0,z_0),F(y_0,x_0,y_0),F(z_0,y_0,x_0)),$$

(ii) for $(x_1,y_1,z_1),(x_2,y_2,z_2)\in X^2$,

$$(x_1,y_1,z_1)\leq(x_2,y_2,z_2)\Rightarrow F(x_1,y_1,z_1)\leq F(x_2,y_2,z_2),$$

$$F(y_1,x_1,y_1)\geq F(y_2,x_2,y_2),\ F(z_1,y_1,x_1)\leq F(z_2,y_2,x_2),$$

(iii) If $(x_n,y_n,z_n)\to(x,y,z)$ is monotone non-decreasing in first and third coordinates and non-increasing in second co-ordinate, then

$$x_n\leq x,\ y_n\geq y \text{ and } z_n\leq z \text{ for all } n.$$

(iv) $\xi(d(F(x_1,y_1,z_1),F(x_2,y_2,z_2))\leq\beta\xi(\max\{d(x_1,x_2),$

$$d(x_1, F(x_1, y_1, z_1)), d(x_2, F(x_2, y_2, z_2)),$$
$$\left.\left.\frac{d(x_2, F(x_1, y_1, z_1)) + d(x_1, F(x_2, y_2, z_2))}{2}\right\}\right)$$
$$-\eta(\max\{d(x_1, x_2), d(x_2, F(x_2, y_2, z_2))\})$$

for all comparable $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$,

where $\eta : [0, \infty) \to [0, \infty)$ is any continuous function with $\eta(t) = 0$ iff $t = 0$. Then F has a tripled fixed point.

Proof. We will construct the same sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ as in Theorem 2.1.

Now if there exist a positive integer l such that $x_l = x_{l+1}$, $y_l = y_{l+1}$ and $z_l = z_{l+1}$ then (x_l, y_l, z_l) will be a tripled fixed point of F.

Hence we assume that either $x_n \neq x_{n+1}$ or $y_n \neq y_{n+1}$ or $z_n \neq z_{n+1}$ for all n.

First we assume that $x_n \neq x_{n+1}$ for all n.

Now since $x_n \le x_{n+1}$, $y_n \ge y_{n+1}$ and $z_n \le z_{n+1}$, so using hypothesis (iv), we have

$$\xi(d(x_{n+1}, x_{n+2})) = \xi(d(F(x_n, y_n, z_n), F(x_{n+1}, y_{n+1}, z_{n+1})))$$
$$\le \xi(\max\{d(x_n, x_{n+1}), d(x_n, F(x_n, y_n, z_n)), d(x_{n+1}, F(x_{n+1}, y_{n+1}, z_{n+1})),$$
$$\left.\left.\frac{d(x_{n+1}, F(x_n, y_n, z_n)) + d(x_n, F(x_{n+1}, y_{n+1}, z_{n+1}))}{2}\right\}\right)$$
$$-\eta(\max\{d(x_n, x_{n+1}), d(x_{n+1}, F(x_{n+1}, y_{n+1}, z_{n+1}))\})$$

$$= \xi\left(\max\left\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{d(x_{n+1}, x_{n+1}) + d(x_n, x_{n+2})}{2}\right\}\right)$$
$$-\eta(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\})$$
$$\le \xi\left(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}\right) - \eta\left(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}\right)$$
$$\Rightarrow \quad \xi(d(x_{n+1}, x_{n+2})) \le \xi(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\})$$
$$-\eta(\max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}). \tag{2.21}$$

Suppose $d(x_n, x_{n+1}) \le d(x_{n+!}, x_{n+2})$ for some n.

Then (2.21) implies

$$\xi(d(x_{n+1}, x_{n+2})) \le \xi(d(x_{n+1}, x_{n+2})) - \eta(d(x_{n+1}, x_{n+2}))$$
$$\Rightarrow \quad \eta(d(x_{n+1}, x_{n+2})) \le 0$$
$$\Rightarrow \quad d(x_{n+1}, x_{n+2}) = 0$$

$\Rightarrow$ $x_{n+1} = x_{n+2}$ which is contradiction to our assumption

that $\Rightarrow x_n \neq x_{n+1}$ for all n.

$d(x_{n+1}, x_{n+2}) < (x_n, x_{n+1})$ for all n (2.22)

and $\{d(x_n, x_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers

$\Rightarrow$ there exist a $p \ge 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = p. \tag{2.23}$$

Now using (2.21) and (2.22) we have

$$\xi(d(x_{n+1}, x_{n+2})) \le \xi(d(x_n, x_{n+1})) - \eta(d(x_n, x_{n+1})).$$

Taking limit as $n \to \infty$ and using (2.23), we get

$\xi(p) \le \xi(p) - \eta(p)$ (using continuity of ξ and η)

which will give a contradiction unless $p = 0$.

So

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0\,. \tag{2.24}$$

Similarly if we assume $y_n \neq y_{n+1}$ and $z_n \neq z_{n+1}$, then we will arrive at contradiction and get

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0 \tag{2.25}$$

and

$$\lim_{n\to\infty} d(z_n, z_{n+1}) = 0 \tag{2.26}$$

in the respective cases.

Now we will show that $\{x_n\}$ is a Cauchy sequence in X. If we assume that $\{x_n\}$ is not a Cauchy sequence in X then following the arguments of Theorem 2.1, we have two sequences of positive integers $\{p(t)\}$ and $\{q(t)\}$ such that $p(t) > q(t) > t$ for all t

$$\lim_{t\to\infty} d(x_{p(t)}, x_{q(t)}) = \varepsilon\,, \tag{2.27}$$

$$\lim_{t\to\infty} d(x_{p(t)+1}, x_{q(t)+1}) = \varepsilon\,, \tag{2.28}$$

$$\lim_{t\to\infty} d(x_{p(t)}, x_{q(t)+1}) = \varepsilon\,, \tag{2.29}$$

$$\lim_{t\to\infty} d(x_{q(t)}, x_{p(t)+1}) = \varepsilon\,. \tag{2.30}$$

Now $p(t) > q(t) \Rightarrow x_{p(t)} \geq x_{q(t)}, y_{p(t)} \leq y_{q(t)}, z_{p(t)} \geq z_{q(t)}$ for all $t \in N$.

So using hypothesis (vi), we get

$$\xi(d(x_{q(t)+1}, x_{p(t)+1})) = \xi(d(F(x_{q(t)}, y_{q(t)}, z_{q(t)}), F(x_{p(t)}, y_{p(t)}, z_{p(t)})))$$

$$\leq \xi(\max\{d(x_{q(t)}, x_{p(t)}), d(x_{q(t)}, F(x_{q(t)}, y_{q(t)}, z_{q(t)})), d(x_{p(t)}, F(x_{p(t)}, y_{p(t)}, z_{p(t)})),$$

$$-\eta(\max\{d(x_{q(t)}, x_{p(t)}), d(x_{p(t)}, F(x_{p(t)}, y_{p(t)}, z_{p(t)}))\})$$

$$= \xi\left(\max\left\{d(x_{q(t)}, x_{p(t)}), d(x_{q(t)}, x_{q(t)+1}), d(x_{p(t)}, x_{p(t)+1}), \frac{d(x_{p(t)}, x_{q(t)+1}) + d(x_{q(t)}, x_{p(t)+1})}{2}\right\}\right)$$

$$-\eta(\max\{d(x_{q(t)}, x_{p(t)}), d(x_{p(t)}, x_{p(t)+1})\})\,.$$

Taking limit as $t \to \infty$ and using (2.24) and (2.27)-(2.30), we have $\xi(\varepsilon) \leq \xi(\varepsilon) - \eta(\varepsilon)$, which is not possible by definition of η.

Hence $\{x_n\}$ is a Cauchy sequence in X.

Similarly $\{y_n\}$ and $\{z_n\}$ are also Cauchy sequences in X.

But X is complete so there exists $x, y, z \in X$ such that

$$x_n \to x\,,\; y_n \to y \text{ and } z_n \to z \text{ as } n \to \infty\,. \tag{2.31}$$

Then by hypothesis (iii) $x_n \leq x$, $y_n \geq y$ and $z_n \leq z$ for all n.

So we can use condition (iv) and get

$$\xi(d(x_{n+1}, F(x,y,z))) = \xi(d(F(x_n, y_n, z_n), F(x,y,z)))$$

$$\leq \xi\left(\max\left\{d(x_n, x), d(x_n, F(x_n, y_n, z_n)), d(x, F(x,y,z)), \frac{d(x_n, F(x,y,z)) + d(x, F(x_n, y_n, z_n))}{2}\right\}\right)$$

$$-\eta(\max\{d(x_n, x), d(x_n, F(x,y,z))\})\,.$$

Taking limit as $n \to \infty$ and using equations (2.24) and (2.31), we get

$$\xi(d(x, F(x,y,z))) \leq \xi(d(x, F(x,y,z))) - \eta(d(x, F(x,y,z)))$$

($\because \xi, \eta$ are continuous)

which leads to a contradiction unless $d(x, F(x, y, z)) = 0$.

$$\Rightarrow \quad F(x, y, z) = x. \tag{2.32}$$

Similarly

$$F(y, x, y) = y \text{ and } F(z, y, x) = z. \tag{2.39}$$

Combining equations (2.32) and (2.33) we get (x, y, z) is tripled fixed point of F.

In the following theorem we will show existence of tripled fixed point for a continuous mapping after the removal of hypothesis (iii) of Theorem 2.3.

Theorem 2.4. Let $(X, d, \leq)$ be a partially ordered complete metric space and $\xi : [0, \infty) \to [0, \infty)$ be an altering distance function. $F : X^3 \to X$ be a continuous mapping such that the following. Let conditions are satisfied:

(i) there exist $(x_0, y_0, z_0) \in X^3$ such that $(x_0, y_0, z_0) \leq (F(x_0, y_0, z_0), F(y_0, x_0, y_0), F(z_0, y_0, x_0))$,

(ii) for $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^2$,

$$(x_1, y_1, z_1) \leq (x_2, y_2, z_2) \Rightarrow F(x_1, y_1, z_1) \leq F(x_2, y_2, z_2),$$
$$F(y_1, x_1, y_1) \geq F(y_2, x_2, y_2) \text{ and } F(z_1, y_1, x_1) \leq F(z_2, y_2, x_2),$$

(iii)
$$\xi(d(F(x_1, y_1, z_1), F(x_2, y_2, z_2)) \leq \beta\xi\Big(\max\Big\{d(x_1, x_2), d(x_1, F(x_1, y_1, z_1)), d(x_2, F(x_2, y_2, z_2)), \frac{d(x_2, F(x_1, y_1, z_1)) + d(x_1, F(x_2, y_2, z_2))}{2}\Big\}\Big) - \eta(\max\{d(x_1, x_2), d(x_2, F(x_2, y_2, z_2))\})$$

for all comparable $(x_1, y_1, z_1), (x_2, y_2, z_2) \in X^3$,

where $\eta : [0, \infty) \to [0, \infty)$ is any continuous function with $\eta(t) = 0$ iff $t = 0$. Then F has a tripled fixed point.

Proof. From the proof of Theorem 2.1, we find sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ such that

$$\lim_{n\to\infty} x_n \to x, \ \lim_{n\to\infty} y_n \to y \text{ and } \lim_{n\to\infty} z_n \to z.$$

Then continuity of F implies that

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n, y_n, z_n) = F(x, y, z),$$

$$y = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} F(y_n, x_n, y_n) = F(y, x, y),$$

and

$$z = \lim_{n\to\infty} z_{n+1} = \lim_{n\to\infty} F(z_n, y_n, x_n) = F(z, y, x)$$

that is $\exists\ (x, y, z) \in X^3$ such that

$$F(x, y, z) = x, \ F(y, x, y) = y \text{ and } F(z, y, x) = z.$$

$\Rightarrow \quad (x, y, z)$ is tripled fixed point of F.

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